WORKED SOLUTIONS

HIGHER LEVEL

interactive eBook inside

Mathematics
Applications and Interpretation
for the IB Diploma

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Pearson

Exercise 1.1

1. **(a)** 26

(b) 1

(c) 1201

(d) 83

2. **(a)** 27.05

(b) 800.01

(c) 3.14

(d) 0.00

3. (Answers will vary)

(a) height

(b) swimming competition times

(c) distance between cities

(d) photo file size

(e) conversion of 0.5 inches to cm

(f) grocery purchase

(g) cold day in Singapore

(h) current in amps

4. **1.** **(a)** $\frac{26-25.8}{25.8} = 0.00775 \approx 0.78\%$

(b) $\frac{1-0.61}{0.61} = 0.6393 \approx 63.93\%$

(c) 0.02%

(d) 0.56%

2. **(a)** 0.01%

(b) 0.00%

(c) 0.05%

(d) 100%

5. **(a)** 3

(b) 4

(c) 2

(d) 5

(e) 4

(f) 5

(g) 1

(h) 2

6. **(a)** 5630

(b) 3100

(c) 4 760 000

(d) 3.14

(e) 0.000 207

(f) 100

(g) 0.020 1

(h) 0.020 0

Exercise 1.2

1. $3^{4+2} = 3^6$

2. $3^{4-2} = 3^2$

3. $3^{6-2} = 3^4$

4. $3^{-6-2} = 3^{-8}$

5. $3^{4\times 2} = 3^8$

6. 3^3

7. $9^4 \cdot 3^2 = 3^{8+2} = 3^{10}$

8. $9^4 \cdot 81^2 = 3^{8+8} = 3^{16}$

9. $3^{4-4} = 3^0$

10. $\sqrt{3^3} = 3^{\frac{3}{2}}$

11. $3^{\frac{3}{2}} \cdot 3^2 \cdot 3^{\frac{1}{2}} = 3^4$

12. $3^{\frac{2}{3}} \cdot 3 \cdot 3^{\frac{2}{3}} = 3^{\frac{7}{3}}$

13. $3^{\frac{1}{2}} \cdot 3^{\frac{1}{3}} = 3^{\frac{5}{6}}$

14. $3^{\frac{1}{2}} \cdot 3^{\frac{2}{4}} = 3$

15. $3^{-4} \cdot \left(3 \cdot 3^{\frac{3}{2}}\right)^2 = 3^{-4+2+3} = 3$

16. 3^0

17. $3^4 \cdot 3^{\frac{1}{2}} = 3^{\frac{9}{2}}$

18. $3^{\frac{1}{2}} \cdot 3^{\frac{4}{3}} = 3^{\frac{11}{6}}$

19. $3^{\frac{1}{2}} \cdot 3^{\frac{2}{3}} = 3^{\frac{7}{6}}$

20. $3^{\frac{3}{2}} \cdot 3^{\frac{3}{2}} = 3^3$

21. $3^6 \cdot 3^{-\frac{1}{2}} = 3^{\frac{11}{2}}$

22. $3^2 \cdot 3^{-\frac{2}{3}} = 3^{\frac{4}{3}}$

23. $3^{\frac{2}{3}} \cdot 3^{-\frac{2}{3}} = 3^0$

24. $3^{-1} \cdot 3^{-\frac{1}{3}} = 3^{-\frac{4}{3}}$

Exercise 1.3

1. **(a)** 1.203×10^3

(b) 7×10^9

(c) 3.01×10^{-4}

(d) 2.001×10^1

(e) 2×10^3

(f) 7.0×10^{-4}

(g) 1.203×10^1

(h) 1.0006×10^4

(i) 1.0001×10^{1}

(j) 1×10^{100}

2. (a) $1.68\times10^{6}\approx2\times10^{6}$

(b) $3.6\times10^{-2}\approx4\times10^{-2}$

(c) $1.84\times10^{12}\approx2\times10^{12}$

(d) $1.84\times10^{1}\approx2\times10^{1}$

3. (a) $1073741824=1.07\times10^{9}$

(b) $2147483647=2.15\times10^{9}$

(c) $23.14=2.31\times10$

(d) $22.5=2.25\times10$

Exercise 1.4

1. (a) $3=\log_{10}1000=\log1000$

(b) $3=\log_{4}64$

(c) $\frac{3}{2}=\log_{100}1000$

(d) $\frac{1}{2}=\log_{9}3$

(e) $\frac{1}{2}=\log_{8}2\sqrt{2}$

(f) $0=\log_{10}1=\log1$

(g) $0=\log_{e}1=\ln1$

(h) $-2=\log_{6}\frac{1}{36}$

(i) $-2=\log_{\sqrt{2}}\frac{1}{2}$

(j) $-\frac{1}{2}=\log_{3}\frac{1}{\sqrt{3}}$

(k) $-3=\log_{\frac{1}{2}}8$

(l) $-\frac{1}{2}=\log_8 \frac{\sqrt{2}}{4}$

(m) the base of a logarithmic function cannot be negative; thus, this situation is not possible.

(n) $-1=\log_{0.01} 100$

(o) $3=\log_{\frac{\sqrt{2}}{2}} \frac{\sqrt{2}}{4}$

2. **(a)** $x=\log_2 y$

(b) $x=\log y$

(c) $x=\ln y$

(d) $3x=\log_2 y \Rightarrow x=\frac{1}{3}\log_2 y$

(e) $\frac{y}{3}=2^x \Rightarrow x=\log_2 \frac{y}{3}$

(d) $5-y=2^x \Rightarrow x=\log_2 (5-y)$

(g) $2x=\log_3 y \Rightarrow x=\frac{1}{2}\log_3 y$

(h) $\frac{x}{2}=\log_3 y \Rightarrow x=2\log_3 y$

(i) $2x=\ln y \Rightarrow x=\frac{1}{2}\ln y$

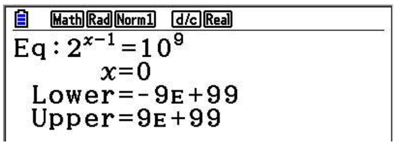
(j) $x-3=\log_2 y \Rightarrow x=\log_2 y+3$

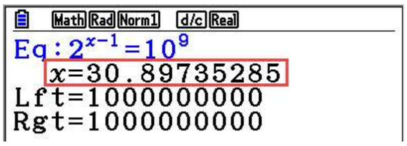
(k) $\frac{x}{2}=\ln y \Rightarrow x=2\ln y$

(l) $2x=\ln 2y \Rightarrow x=\frac{1}{2}\ln 2y$

3. To find out in which square $N = 2^{n-1} > 10^9$, we solve the equation $2^{x-1} = 10^9$ and round up the answer to the nearest natural number:

$2^{x-1} = 10^9 \Rightarrow (x-1)\log 2 = 9 \Rightarrow x = \dfrac{9}{\log 2} + 1 = 30.9 \Rightarrow 31^{\text{st}}$ square. GDC solver can be used instead:





4. **(a)** $10 = \dfrac{10^R}{10^{5.2}} \Rightarrow 10 \times 10^{5.2} = 10^R \Rightarrow R = 6.2$

(b) $2 = \dfrac{10^R}{10^{5.2}} \Rightarrow 2 \times 10^{5.2} = 10^R \Rightarrow R = \log 2 + 5.2 \approx 5.5$

Exercise 1.5

1. **(a)** $x = \log_2 16 \Rightarrow 2^x = 2^4 \Rightarrow x = 4$

(b) $x = \log_{16} 2 \Rightarrow 16^x = 2 \Rightarrow 2^{4x} = 2^1 \Rightarrow x = \dfrac{1}{4}$

(c) $x = \log_{\sqrt{2}} 16 \Rightarrow \sqrt{2}^x = 2^4 \Rightarrow 2^{\frac{x}{2}} = 2^4 \Rightarrow x = 8$

(d) $x = \log_2 \sqrt{2} \Rightarrow 2^x = \sqrt{2} \Rightarrow x = \dfrac{1}{2}$.

(e) $x = \log_2(-16) \Rightarrow 2^x = -16$, but $2^x > 0$, thus, there is no solution.

(f) $x = \log_2 2\sqrt{2} \Rightarrow 2^x = 2^{\frac{3}{2}} \Rightarrow x = \dfrac{3}{2}$

(g) $x = \log_{\sqrt{2}} 2\sqrt{2} \Rightarrow \sqrt{2}^x = 2^{\frac{3}{2}} \Rightarrow 2^{\frac{x}{2}} = 2^{\frac{3}{2}} \Rightarrow x = 3$

(h) $x = \log_{2\sqrt{2}} 2 \Rightarrow \left(2\sqrt{2}\right)^x = 2 \Rightarrow 2^{\frac{3x}{2}} = 2^1 \Rightarrow x = \dfrac{2}{3}$

(i) $\log 4 + \log 25 = \log(4 \times 25) = \log 100 = 2$

(j) $\log 30 - \log 300 = \log\left(\dfrac{30}{300}\right) = \log 10^{-1} = -1$

(k) $\ln\left(\frac{1}{e^2}\right) = \ln 1 - \ln e^2 = 0 - 2 = -2$

(l) $\ln\sqrt{e} = \frac{1}{2}\ln e = \frac{1}{2}$

2. **(a)** $\log_a a^3 = 3\log_a a = 3$

(b) $\log_a \sqrt{a} = \log_a a^{\frac{1}{2}} = \frac{1}{2}$

(c) $\log_{\sqrt{a}} a\sqrt{a} = \log_{\sqrt{a}} \left(\sqrt{a}\right)^3 = 3$

(d) $\log_{\sqrt{a}} \sqrt[3]{a} = \log_{\sqrt{a}} \left(\sqrt{a}\right)^{\frac{2}{3}} = \frac{2}{3}$

(e) $\log_{a^2} a^3 = \log_{a^2} \left(a^2\right)^{\frac{3}{2}} = \frac{3}{2}$

(f) $\log_{a^2} \sqrt{a} = \log_{a^2} \left(a^2\right)^{\frac{1}{4}} = \frac{1}{4}$

(g) $\log_{a^2} \sqrt[3]{a} = \log_{a^2} \left(a^2\right)^{\frac{1}{6}} = \frac{1}{6}$

(h) $\log_{a^2} a\sqrt{a} = \log_{a^2} \left(a^2\right)^{\frac{3}{4}} = \frac{3}{4}$

(i) $\log_a a^{-3} + \log_a a^4 = -3 + 4 = 1$

(j) $\log_{a^2} a^{-3} + \log_{a^2} a^4 = \log_{a^2} \left(a^2\right)^{-\frac{3}{2}} + \log_{a^2} \left(a^2\right)^2 = -\frac{3}{2} + 2 = \frac{1}{2}$

(k) $\log_a a^3 - \log_a a^2 = 3 - 2 = 1$

(l) $\log_a a^3 - \log_a \sqrt{a} = 3 - \frac{1}{2} = \frac{5}{2}$

3. **(a)** $\log_3 x = -2 \Rightarrow x = 3^{-2} = \frac{1}{9}$

(b) $\log_2 (x-3) = 5 \Rightarrow x - 3 = 2^5 \Rightarrow x = 35$

(c) $\log_x 3 = -2 \Rightarrow 3 = x^{-2} \Rightarrow x = \frac{1}{\sqrt{3}}$

(d) $\log_3 \left(x^2 + 2x + 1\right) = 0 \Rightarrow x^2 + 2x + 1 = 1 \Rightarrow x^2 + 2x = 0 \Rightarrow x = 0$ or $x = -2$

Chapter 1 practice questions

1. **(a)** 4

(b) 2

(c) 1

(d) 5

(e) 5

(f) 1

(g) 3

2. **(a)** 58 300

(b) 6110

(c) 124 000

(d) 1.62

(e) 0.00305

(f) 400

3. **(a)** 2^8

(b) 2^6

(c) 2^6

(d) 2^{-9}

(e) 2^6

(f) 2^6

(g) 2^{10}

(h) 2^{24}

(i) 2^6

(j) $2^{\frac{3}{2}}$

(k) 2^4

(l) 2^3

(m) $2^{\frac{5}{6}}$

(n) $2^{\frac{5}{4}}$

(o) 2^{-1}

(p) $2^{\frac{3}{2}}$

(q) $2^{\frac{9}{2}}$

(r) $2^{\frac{11}{6}}$

(s) 2^{3}

(t) 2

(u) $2^{\frac{1}{2}}$

(v) $2^{-\frac{5}{3}}$

(w) 2^{0}

(x) 2^{-1}

4. **(a)** 5.227×10^{4}

(b) 1.31401×10^{1}

(c) 6.04×10^{-5}

(d) 9×10^{-4}

(e) 9×10^{-3}

(f) 3.2001×10^{1}

(g) 5.00003×10^{5}

(h) 1.0000×10^{2}

(i) 1×10^{-6} m

5. **(a)** 1.00×10^{18}

(b) 1.52×10^{1}

(c) 1.00×10^{-9} s

(d) 1.62×10^{0}

6. **(a)** $5=\log_3 243$

(b) $8=\log_2 256$

(c) $\frac{1}{2}=\log_{100} 10$

(d) $\frac{1}{6} = \log_{64} 2$

(e) $\frac{5}{2} = \log_3 9\sqrt{3}$

(f) $-3 = \log 0.001$

(g) $0 = \ln 1$

(h) $-3 = \log_5\left(\frac{1}{125}\right)$

(i) $-2 = \log_{3\sqrt{3}}\left(\frac{1}{27}\right)$

(j) $-\frac{1}{2} = \log_8\left(\frac{1}{2\sqrt{2}}\right)$

(k) $-3 = \log_{\frac{1}{4}} 64$

(l) $-\frac{1}{2} = \log_{27}\left(\frac{\sqrt{3}}{9}\right)$

(m) Base of a logarithm cannot be negative – not possible.

(n) $-2 = \log_{0.1} 100$

(o) $3 = \log_{\frac{\sqrt{3}}{3}}\left(\frac{\sqrt{3}}{9}\right)$

(p) $-3 = \log_{\frac{1}{\sqrt{2}}} 2\sqrt{2}$

7. **(a)** $x = \log_5 y$

(b) $x = \log y$

(c) $x = \ln y$

(d) $2x = \log_2 y \Rightarrow x = \frac{1}{2}\log_2 y$

(e) $\log_3 3 + x = \log_3 y \Rightarrow x = \log_3 y - 1$

(f) $y - 7 = 3^x \Rightarrow x = \log_3(y-7)$

(g) $-2x = \log_2 y \Rightarrow x = -\frac{1}{2}\log_2 y$

(h) $\frac{x}{3} = \log_2 y \Rightarrow x = 3\log_2 y$

(i) $\frac{x}{2} = \ln y \Rightarrow x = 2\ln y$

(j) $x + 3 = \log_5 y \Rightarrow x = \log_5 y - 3$

(k) $x - 1 = \ln y \Rightarrow x = \ln y + 1$

(l) $-2x = \ln y \Rightarrow x = -\frac{1}{2}\ln y$

8. **(a)** $x = \log_3 243 \Rightarrow 3^x = 3^5 \Rightarrow x = 5$

(b) $x = \log_{243} 3 \Rightarrow 243^x = 3^1 \Rightarrow 3^{5x} = 5 \Rightarrow x = \frac{1}{5}$

(c) $x = \log_{\frac{1}{2}} 16 \Rightarrow \left(\frac{1}{2}\right)^x = 2^4 \Rightarrow x = -4$

(d) $x = \log_3 3\sqrt{3} \Rightarrow 3^x = 3^{\frac{3}{2}} \Rightarrow x = \frac{3}{2}$

(e) We can calculate logarithms of positive numbers only. This one is not defined.

(f) $x = \log_4 2\sqrt{2} \Rightarrow 4^x = 2^{\frac{3}{2}} \Rightarrow 2^{2x} = 2^{\frac{3}{2}} \Rightarrow x = \frac{3}{4}$

(g) $\log 50 + \log 20 = \log 1000 = 3$

(h) $\log 4000 - \log 4 = \log \frac{4000}{4} = \log 1000 = 3$

(i) $\ln e^{-2} = -2\ln e = -2$

(j) $\ln\left(\frac{1}{\sqrt{e}}\right) = \ln e^{-\frac{1}{2}} = -\frac{1}{2}$

9. **(a)** $2\log_{64} 8 = 2\log_{64}\sqrt{64} = 2\log_{64} 64^{\frac{1}{2}} = 2 \times \frac{1}{2} = 1$

(b) $\log_8 8^{\frac{1}{2}} = \frac{1}{2}$

(c) $\log_{\sqrt{3}}\left(\sqrt{3}\right)^3 = 3$

(d) $\log_{\sqrt{16}} \sqrt[3]{16} = \log_{\sqrt{16}} \left(\sqrt{16}\right)^{\frac{2}{3}} = \frac{2}{3}$

(e) $\log_{\sqrt{8}} 8^3 = \log_{\sqrt{8}} \left(\sqrt{8}\right)^6 = 6$

(f) $\log_{5^2} \sqrt{5} = \log_{5^2} \left(5^2\right)^{\frac{1}{4}} = \frac{1}{4}$

(g) $\log_9 9^{-3} + \log_3 9^4 = \log_9 9^{-3} + \log_3 3^8 = -3 + 8 = 5$

(h) $\log_{\sqrt{8}} 2^{-3} + \log_8 4^4 = \log_{\sqrt{8}} \left(\sqrt{8}^{\frac{2}{3}}\right)^{-3} + \log_8 \left(8^{\frac{2}{3}}\right)^4 = -2 + \frac{8}{3} = \frac{2}{3}$

(i) $\log_{2\sqrt{2}} 4^3 - \log_{\sqrt{2}} 2 = \log_{2\sqrt{2}} \left(\left(2\sqrt{2}\right)^{\frac{4}{3}}\right)^3 - \log_{\sqrt{2}} \left(\sqrt{2}\right)^2 = 4 - 2 = 2$

(j) $\log_{\frac{1}{3}} 3 - \log_3 \sqrt{3} = \log_{\frac{1}{3}} \left(\frac{1}{3}\right)^{-1} - \log_3 3^{\frac{1}{2}} = -1 - \frac{1}{2} = -\frac{3}{2}$

10. **(a)** $\log_5 x = -3 \Rightarrow x = 5^{-3} = \frac{1}{125}$

(b) $\log_x \frac{1}{4} = -2 \Rightarrow x^{-2} = \frac{1}{4} \Rightarrow x = 2$

(c) $\log_3 \left(x^2 - 2x - 5\right) = 1 \Rightarrow x^2 - 2x - 5 = 3 \Rightarrow x = 4$ or $x = -2$

11. **(a)** $m = 2, n = 4.$

(b) $8^{2x+1} = 16^{2x-3} \Rightarrow 2^{6x+3} = 2^{8x-12} \Rightarrow 6x + 3 = 8x - 12 \Rightarrow x = \frac{15}{2}$

12. Using the fact that $\log_a x = \frac{\ln x}{\ln a}$ or any other base, we have:

$$a = \frac{\ln 3}{\ln 2} \times \frac{\ln 4}{\ln 3} \times \frac{\ln 5}{\ln 4} \times \cdots \times \frac{\ln 32}{\ln 31} = \frac{\ln 32}{\ln 2} = \frac{\ln 2^5}{\ln 2} = \frac{5 \ln 2}{\ln 2} = 5$$

13. $\log_x y = 4 \log_y x \Rightarrow \frac{\ln y}{\ln x} = \frac{4 \ln x}{\ln y} \Rightarrow (\ln y)^2 = 4(\ln x)^2$

$$\Rightarrow \ln y = \pm 2 \ln x = \pm \ln x^2 \Rightarrow y = x^2 \text{ or } y = \frac{1}{x^2}$$

Exercise 2.1

1. **(a)** I (origin, gradient, $m > 1$), Yes, it is a function, because it passes the vertical-line test.

(b) J (horizontal line), Yes.

(c) H ($y = x - 2$, so $m = 1$ and y-intercept = -2), Yes.

(d) K (circle), No, it is not a function, since it fails the vertical-line test.

(e) L ($m < 0$), Yes

(f) B (parabola, $a > 1$), Yes

(g) C ($y = \sqrt[3]{x}$), Yes. For every x there is only one y-value.

(h) G ($y = \frac{a}{x}$, $a > 0$), Yes, it is a hyperbola

(i) F ($y = 2 - x^2$, $a < 0$), Yes – a parabola.

2. **(a)** Since we can only take the square root of a number greater than or equal to zero, $x - 4 \geq 0$. So $x \geq 4$, $\therefore a = 4$

(b) **(i)** $h(29) = \sqrt{29-4} = \sqrt{25} = 5$

(ii) $\mathrm{h}(53) = \sqrt{53-4} = \sqrt{49} = 7$

(c) As $x \to \infty$, $h(x) \to \infty$, but when $h = 4$ (minimum possible value), $h(4) = \sqrt{4-4} = 0$, so the range is $y \geq 0$

3. Outer rectangle is $(12 + 2x)$ m by $(18 + 2x)$m, so the area is

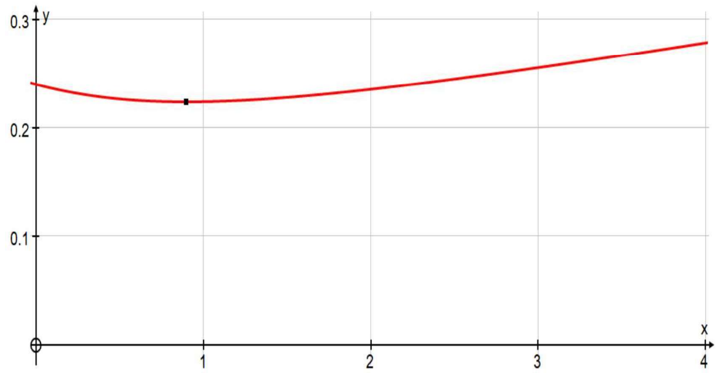
$$A = (12+2x)(18+2x) - 12\cdot 18 = 12\cdot 18 + 12\cdot 2x + 2x\cdot 18 + (2x)^2 - 12\cdot 18 = 4x^2 + 60x$$

4. **(a)** FP is the hypotenuse of a right-angled triangle, so

$$FP = \sqrt{x^2 + 1.2^2} = \sqrt{x^2 + 1.44}$$

$$\text{time} = \frac{\text{distance}}{\text{speed}} \quad \text{so } T(x) = T_{FP} + T_{PH} = \frac{\sqrt{x^2+1.44}}{15} + \frac{4-x}{25}$$

(b) Graphing $y = T(x)$ gives



The minimum of which is at $(0.9, 0.224)$, so the minimum time is approximately 0.224 hours, or 13 minutes and 26 seconds.

5. **(a)** Initial thickness is at 0 km driven, so $k = 0$, $\therefore t(0) = a\sqrt{120000} = 12$

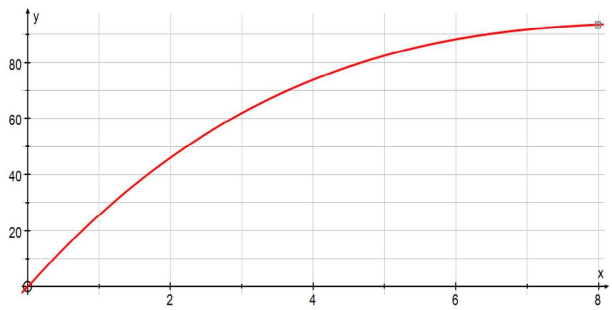
This gives $a = \dfrac{12}{\sqrt{120000}} = \dfrac{12}{\sqrt{12}\sqrt{10000}} = \dfrac{\sqrt{12}}{100} \approx 0.0346$ mm

(b)
$$t(90000) = \frac{\sqrt{12}}{100}\sqrt{120000 - 90000}$$
$$= \frac{\sqrt{12}}{100}\sqrt{30000}$$
$$= \frac{\sqrt{12}}{100}\sqrt{3}\cdot 100$$
$$= \sqrt{36}$$
$$= 6\text{ mm}$$

6. **(a)** 0.5 litres = 500 ml, so $a(t) = 500 - 2t$

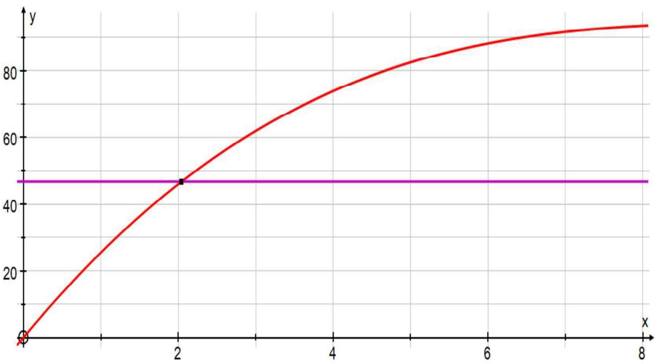
(b) $500 - 2t = 50$, $2t = 450$, $t = 225, \therefore$ the time is 225 min (or 3 h and 45 min).

7. **(a)** Graphing $v(h) = 3\pi\left(10 - \dfrac{(10-h)^3}{100}\right)$, $h \in [0, 8]$ gives:

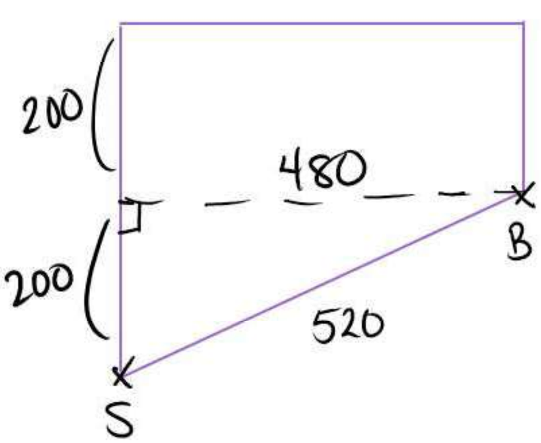


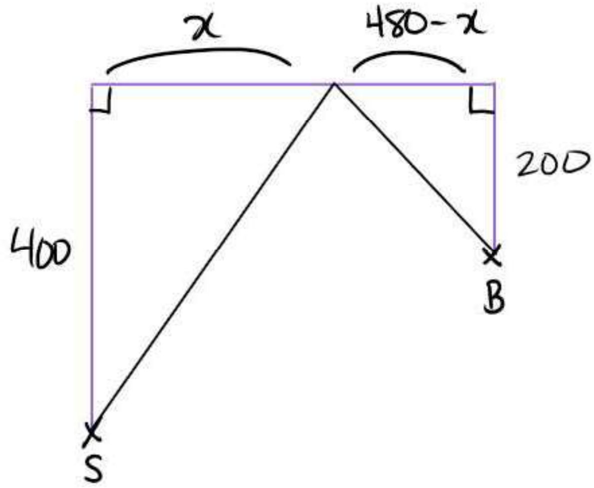
Greatest volume from graph, when $h = 8$, is approximately 93 cm^3.

(b) Solving for the intersection of $v(h)$ and $v = 46.75$ gives $h \approx 2.04$ cm (3 s.f.).



8. **(a)**





In the figure at left, we use Pythagoras to get the distance from Sarah's tent (S) to the burning one (B) parallel to the bank: $\sqrt{520^2 - 200^2} = 480$ m.

This gives us x and $480 - x$ in the next diagram, and using Pythagoras again will give us our function, but first we convert the speed units to metres and minutes:

$22\,\text{km h}^{-1} = \dfrac{22\,\text{km}}{1\,\text{h}} \cdot \dfrac{1\,\text{h}}{60\,\text{min}} \cdot \dfrac{1000\,\text{m}}{1\,\text{km}} = \dfrac{1100}{3}\,\text{m min}^{-1}$, and similarly

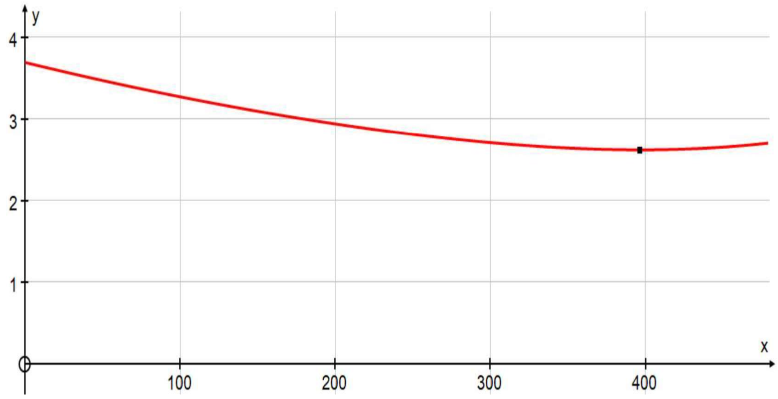
$12\,\text{km h}^{-1} = 200\ \text{m min}^{-1}$.

Since $\text{time} = \dfrac{\text{distance}}{\text{speed}}$, the total time she runs from s to the river (R) and then to B is

$$t(x) = t_{SR} + t_{RB} = \frac{\sqrt{x^2 + 400^2}}{\frac{1100}{3}} + \frac{\sqrt{(480 - x)^2 + 200^2}}{200} = \frac{3\sqrt{x^2 + 160000}}{1100} + \frac{\sqrt{x^2 - 960x + 270400}}{200}$$

(b) Domain is $0 \le x \le 480$

(c) The least time is found by locating the minimum point on the graph of $t(x)$ for the domain stated; here, (397, 2.62) to 3 s.f..



The least time is approximately 2.62 minutes (or 2 minutes and 37 seconds).

Exercise 2.2

1. Note: the following answers all have $a > 0$.

(a) $y = 3x - 4$ gives $3x - y - 4 = 0$

(b) Using point-slope form, $y - y_1 = m(x - x_1)$, where m is gradient and (x_1, y_1) is a known point on the line, we have
$y-(-1)=2(x-3)$, $y+1=2x-6$, $2x-y-7=0$.

(c) Gradient is $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{-10-5}{2-(-3)} = \dfrac{-15}{5} = -3$
$\therefore y-5=-3(x+3)$, $3x+y+4=0$

2. **(a)** Given Celsius, we want Fahrenheit, so C is the independent variable (like *x*), and F is the dependent (like *y*).

Thus, $m = \dfrac{F_2 - F_1}{C_2 - C_1} = \dfrac{212-32}{100-0} = \dfrac{9}{5}$ and $F - 32 = \dfrac{9}{5}(C-0)$, so $F = \dfrac{9}{5}C + 32$

(b) $F = \dfrac{9}{5} \cdot 37 + 32 = 98.6$

(c) $C = \dfrac{5}{9}(F-32) = \dfrac{5}{9}(9941-32) = 5505$

(d) Since the difference between freezing and boiling is 100, like Celsius, we can convert K to C by $C = K - 273$ then use our original formula, giving
$F = \dfrac{9}{5}(K-273)+32 = \dfrac{9}{5}K - 459.4$

(e) **(i)** $F = \frac{9}{5}(0 - 273) + 32 = -459.4$

(ii) $C = 0 - 273 = -273$

3. **(a)** The ratio of Philippine money to Thai is $\frac{P}{T} = \frac{400}{250}$, so $P = \frac{400}{250}T = \frac{8}{5}T$

(b) $P = \frac{8}{5} \cdot 520 = 832 \text{ PHP}$

(c) $B = \frac{5}{8}P = \frac{5}{8} \cdot 1280 = 800 \text{ THB}$

4. Let n = months and C = cost.

(a) $C_A = 400 + 50n$

(b) $C_B = 150 + 80n$

(c) $C_A = 400 + 50 \cdot 12 = 1000$, $C_B = 150 + 80 \cdot 12 = 1110$, so A is cheaper by \$110.

5. **(a)** $C = 0.8d + 4$

(b) $C = 0.8 \cdot 12 + 4 = 13.6$, so \$13.60

(c) Gradient represents cost per km, which is \$1.

(d) The intercept represents the fixed rate, essentially the cost of being picked up.

6. **(a)** $2x + 1 = ax + b$, $2 \cdot 3 + 1 = a \cdot 3 + 8$, $3a = -1$, $a = -\frac{1}{3}$

(b) $2 \cdot 3 + 1 = 4 \cdot 3 + b$, $7 = 12 + b$, $b = -5$

7. **(a)** $1 - x^2 = ax + b$, $1 - 2^2 = a \cdot 2 + 2$, $2a = -5$, $a = -\frac{5}{2}$

(b) $1 - 2^2 = 3 \cdot 2 + b$, $b = -9$

8. **(a)** $60 + 12 \cdot 5 = x + 9 \cdot 5$, $120 = x + 45$, $x = 75$

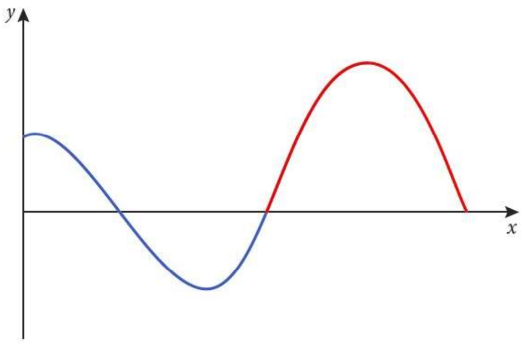
(b) $C(9) = 75 + 9 \cdot 9 = 156$

9. **(a)** $5^3 - 6 \cdot 5^2 + 3 \cdot 5 + 10 = -5 \cdot 5^2 + 70 \cdot 5 - 225$

$125 - 150 + 15 + 10 = -125 + 350 - 225$

$0 = 0$, hence when $x = 5$, both pieces of the function have a y-value of 0, so they meet at (5, 0), making the function continuous.

(b)

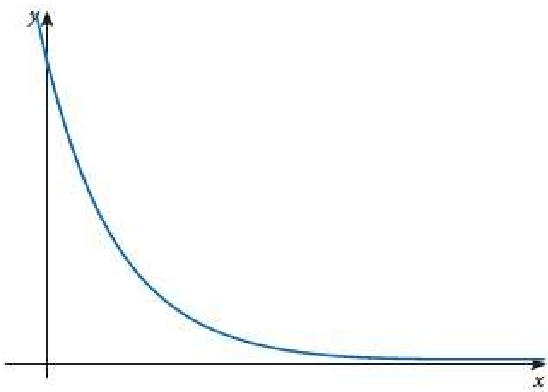


(c) Minimum value = -10.4 using the graph.

(d) Maximum value = 20

(e) Adding the line $y = 4$ to the graph and finding the intersections gives $x = 1.55,\ 5.21,\ 8.79$

10. **(a)**



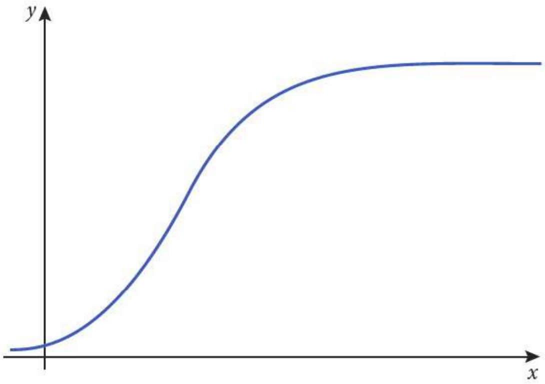
(b) $m(0) = \frac{30}{3^0} = \frac{30}{1} = 30$ becquerels

(c) $\frac{30}{3^t} = 15,\ 3^t = 2,\ t = \log_3 2 \approx 0.631$ days

(d) $m = 0$

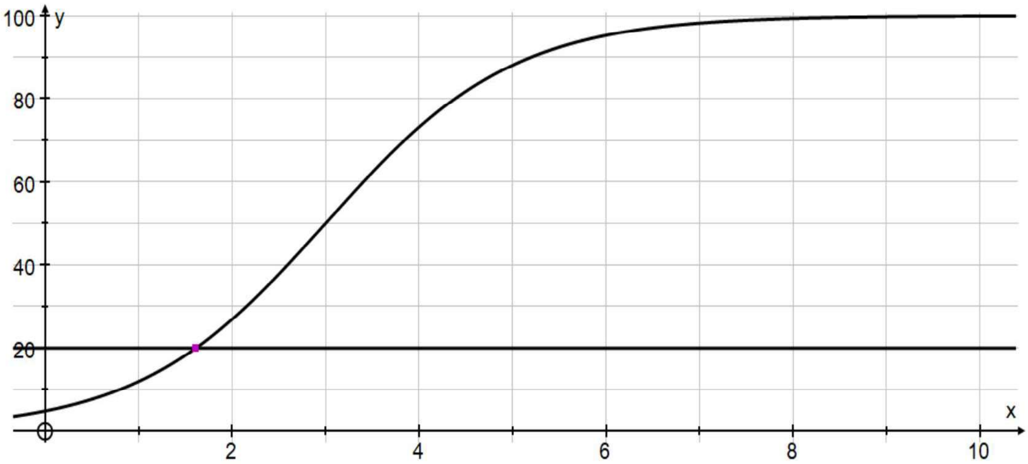
(e) It becomes less and less active, hence approaching 0 Bq.

11. **(a)**



(b) $P(0)=\frac{100}{1+e^{3-0}}\approx 4.74$, so we round this to 5 birds initially.

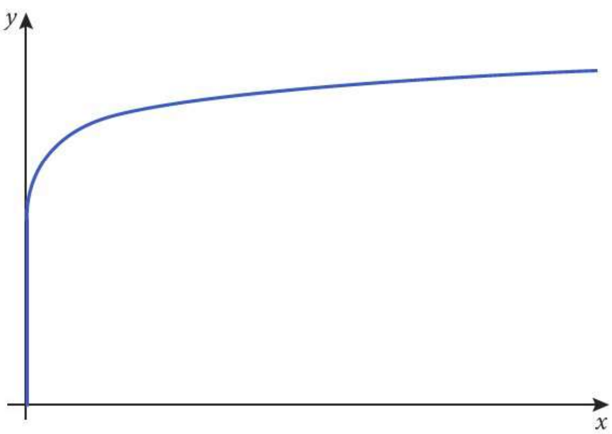
(c) Using the graph intersections, we find 1.61 years.



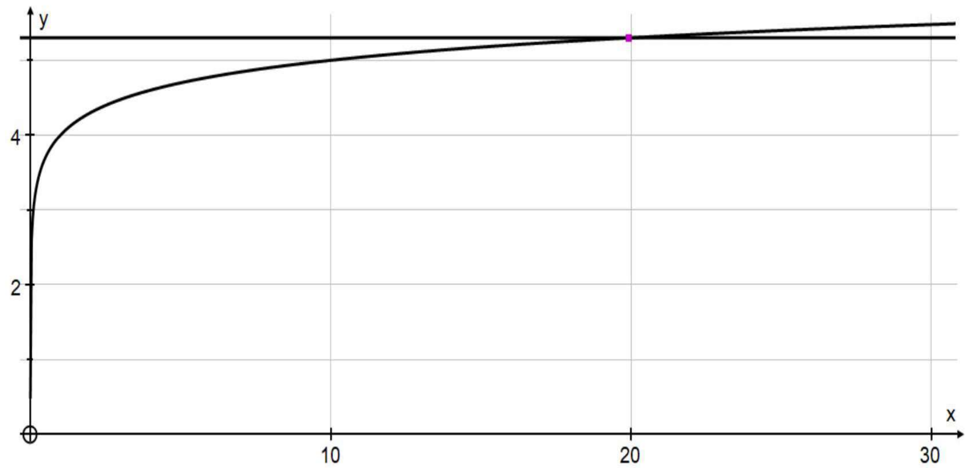
(d) $P=100$ is the value the graph approaches as $t\to\infty$

(e) The population keeps growing but more and more slowly, approaching 100 birds.

12. **(a)**



(b) The amplitude is approximately 20.0



(c) $M \approx \log_{10} 200 + 4 \approx 6.30$

(d) $\log A_1 + 4 = 4.9$, $\log A_1 = 0.9$, $A_1 = 10^{0.9}$. Similarly, $A_2 = 10^{0.3}$

The ratio of the A-values is

$$\frac{A_1}{A_2} = \frac{10^{0.9}}{10^{0.3}} = 10^{0.6} \approx 3.98$$

Exercise 2.3

1. **(a)** **(i)** $f\left(\frac{1}{5-3}\right) = f\left(\frac{1}{2}\right) = 2 \cdot \frac{1}{2} = 1$

(ii) $g(2 \cdot 5) = g(10) = \frac{1}{10-3} = \frac{1}{7}$

(b) **(i)** $f(g(x)) = f\left(\frac{1}{x-3}\right) = 2\left(\frac{1}{x-3}\right) = \frac{2}{x-3}$

(ii) $g(f(x)) = g(2x) = \frac{1}{2x-3}$

2. **(a)** **(i)** $f(g(0)) = f(2-0^2) = f(2) = 2 \cdot 2 - 3 = 1$

(ii) $g(f(0)) = g(2 \cdot 0 - 3) = g(-3) = 2-(-3)^2 = -7$

(iii) $f(f(4)) = f(2 \cdot 4 - 3) = f(5) = 2 \cdot 5 - 3 = 7$

(iv) $g(g(-3)) = g(-7) = 2-(-7)^2 = -47$

(v) $f(g(-1)) = f\left(2-(-1)^2\right) = f(1) = 2\cdot 1 - 3 = -1$

(vi) $g(f(-3)) = g(2(-3)-3) = g(-9) = 2-(-9)^2 = -79$

(b) (i) $f(g(x)) = f\left(2-x^2\right) = 2\left(2-x^2\right) - 3 = 1-2x^2$

(ii) $g(f(x)) = g(2x-3) = 2-(2x-3)^2 = -4x^2+12x-7$

(iii) $f(f(x)) = 2(2x-3)-3 = 4x-9$

(iv) $g(g(x)) = 2-\left(2-x^2\right)^2 = -x^4+4x^2-2$

3. **(a)** $4(2+3x)-1 = 12x+7,\ x\in\mathbb{R}$

(b) $(-2x)^2+1 = 4x^2+1,\ x\in\mathbb{R}$

(c) $1+\left(\sqrt{x+1}\right)^2 = x+2,\ x\geq -1$

(d) $\dfrac{2}{(x-1)+4} = \dfrac{2}{x+3}\ ,\ x\neq -3$

(e) $3\left(\dfrac{x-5}{3}\right)+5 = x,\ x\in\mathbb{R}$

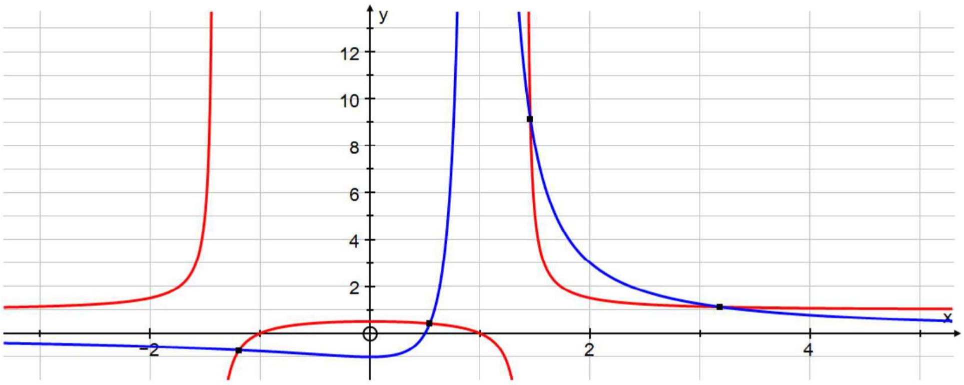
(f) $2-\left(\sqrt[3]{1-x^2}\right)^3 = x^2+1,\ x\in\mathbb{R}$

(g) $\dfrac{2\left(\dfrac{1}{x^2}\right)}{4-\dfrac{1}{x^2}} = \dfrac{2}{x^2}\cdot\dfrac{x^2}{4x^2-1} = \dfrac{2}{4x^2-1}\ ,\ x\neq 0, x\neq \pm\dfrac{1}{2}$

(h) $\dfrac{2}{\dfrac{5}{x-4}+3} = \dfrac{2(x-4)}{5+3(x-4)} = \dfrac{2x-8}{3x-7}\ ,\ x\neq 4, x\neq \dfrac{7}{3}$

4. Graphing $f\left(g(x)\right)=\dfrac{x^2-1}{x^2-1-1}$ and $g\left(f(x)\right)=\left(\dfrac{x}{x-1}\right)^2-1$

gives the solutions: −1.19, 0.543, 1.46 and 3.19



5. **(a)** Looking on the graph of g for the *y*-value when $x=-3$ gives $g(-3)=1$,

$\therefore g(g(-3)=f(1)$

Now we look on the graph of f for the *y*-value when $x=1$ to get 5.

(b) Similarly, $g\left(g(1)\right)=g(-1)=4$

(c) $g\left(f(-1)\right)=g(1)=-1$

6. **(a)** $f(x+3)=x+3+3=x+6$

(b) $f\left(f\circ f(x)\right)=f(x+6)=x+6+3=x+9$

(c) $f^{(4)}(x)=x+9+3=x+12$

(d) $f^{(n)}(x)=x+3n$

(e) $f^{(50)}(x)=x+3\cdot 50=x+150=20,\ x=-130$

7. **(a)** $f(2x)=2(2x)=4x$

(b) $f(4x)=2(4x)=8x$

(c) $f(8x)=2(8x)=16x$

(d) $f^{(n)}(x)=2^n x$

(e) $f^{(n)}(5)=2^n\cdot 5>1000000,\ 2^n>200000,\ n>\log_2 200000\approx 17.6,\therefore n=18$

Alternatively, find the intersections of the graphs of $y=2^x\cdot 5$ and $y=1000000$

8. **(a)** $f(3x-2)=3(3x-2)-2=9x-8$

(b) $f(9x-8)=3(9x-8)-2=27x-26$

(c) $f(27x-26)=3(27x-26)-2=81x-80$

(d) $f^{(n)}(x)=3^n x-(3^n-1)=3^n x-3^n+1$

(e) $f^{(5)}(x)=f^{(7)}(2)$, $3^5x-3^5+1=3^7\cdot2-3^7+1$, $3^5x-3^5=3^7\cdot2-3^7$, dividing by 3^5 gives $x-1=3^2\cdot2-3^2$, $x-1=9$, $x=10$

9. **(a)** The rate is 2 m/15 min = 48 m/2 h without maintenance, and 10 min cleaning is $\frac{1}{6}$ h, so we can say the actual rate is 48 m/$\frac{13}{6}$ h, which is $\frac{288}{13}$ m h^{-1} with cleaning. $\therefore L(t)=\frac{288}{13}t$

(b) $S=12L+10\sqrt{L}=12\cdot\frac{288}{13}t+10\sqrt{\frac{288}{13}t}=\frac{3456}{13}t+10\sqrt{\frac{288}{13}t}$

(c) 3 m = $\frac{3}{2}\cdot$ 2 m, so the machine produces $L_2(t)=\frac{3}{2}\cdot\frac{288}{13}t=\frac{432}{13}t$, and

$S_2(t)=12\cdot\frac{432}{13}t+10\sqrt{\frac{432}{13}t}=\frac{5184}{13}t+10\sqrt{\frac{432}{13}t}$

(d) $D(t)=S_2(t)-S(t)$

$$=\frac{5184}{13}t+10\sqrt{\frac{432}{13}t}-\left(\frac{3456}{13}t+10\sqrt{\frac{288}{13}t}\right)$$

$$=\frac{1728}{13}t+10\sqrt{\frac{432}{13}t}-10\sqrt{\frac{288}{13}t}$$

$$=\frac{1728}{13}t+10\left(\sqrt{\frac{432}{13}t}-\sqrt{\frac{288}{13}t}\right)$$

(e) 1 year = $365\cdot24\ h=8760\ h$. $D(8760)\approx1165396.23$, so the company is willing to invest a maximum of approximately \$1,165,396.

Exercise 2.4

1. For inverse functions, if $f(x)=y$, then $f^{-1}(y)=x$, giving the following results:

(i) 2 **(ii)** 6 **(iii)** -1 **(iv)** b

2. $3x-7=5,\ x=4=g^{-1}(5)$

3. $x^2-8x=-12,\ x^2-8x+12=0,\ (x-2)(x-6)=0,\ x-2=0$

or $x-6=0,\ x=2$ or $6.\ \because x\geq 4,\ x=6$

4. **(a)** $y=2x-3,\ x=2y-3,\ 2y=x+3,\ y=\dfrac{x+3}{2},\ x\in\mathbb{R}$

(b) $y=\dfrac{x+7}{4},\ x=\dfrac{y+7}{4},\ 4x=y+7,\ y=4x-7,\ x\in\mathbb{R}$

(c) $y=\sqrt{x},\ x=\sqrt{y},\ y=x^2,\ x\geq 0$

(d) $y=\dfrac{1}{x+2},\ x=\dfrac{1}{y+2},\ y+2=\dfrac{1}{x},\ y=\dfrac{1}{x}-2,\ x\neq 0$

(e) $y=4-x^2,\ x=4-y^2,\ y^2=4-x,\ y=\sqrt{4-x},\ x\leq 4$

(f) $y=\sqrt{x-5},\ x=\sqrt{y-5},\ x^2=y-5,\ y=x^2+5,\ x\geq 0$

(g) $y=ax+b,\ x=ay+b,\ x-b=ay,\ y=\dfrac{x-b}{a},\ x\in\mathbb{R}$

(h) $y=x^2+2x,\ x=y^2+2y,\ x=(y+1)^2-1,\ x+1=(y+1)^2,\ y+1=-\sqrt{x+1}$

(we use the negative root since the original function is the left half of the parabola, and its domain, $x\leq -1$, tells us the range of our inverse function should be $y\leq -1$), $y=-\sqrt{x+1}-1,\ x\geq -1$)

(i) $y = \frac{x^2 - 1}{x^2 + 1},\ x = \frac{y^2 - 1}{y^2 + 1}$

$xy^2 + x = y^2 - 1,\ xy^2 - y^2 = -x - 1$

$y^2(x-1) = -(x+1)$

$y^2 = -\frac{x+1}{x-1},\ y = -\sqrt{\frac{x+1}{1-x}}$

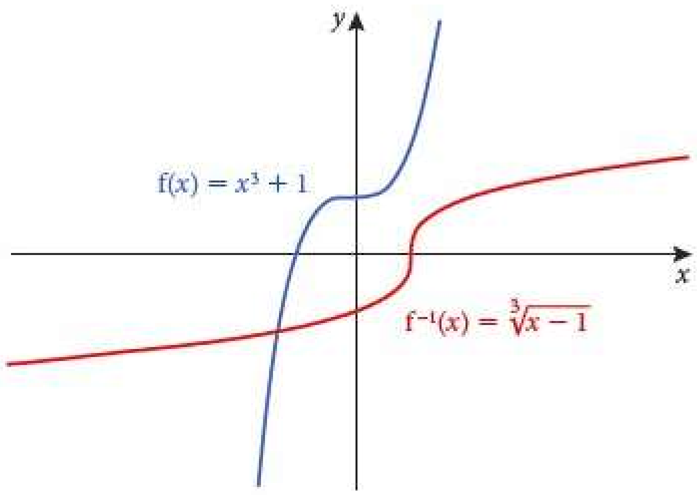
(again we need a negative root for a negative range as determined by the domain of the original function). Since we need the radicand to be positive or zero, considering the signs of the numerator and denominator is necessary.

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$x+1$	–	0	+	+	+
$1-x$	+	+	+	0	–
$\frac{x+1}{1-x}$	–	0	+	undefined	–

Thus, the domain is $-1 \le x < 1$

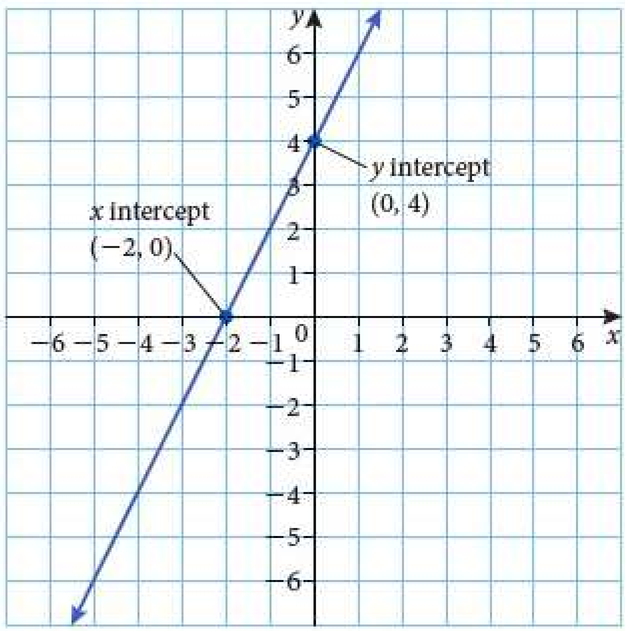
5. **(a)** $y = x^3 + 1,\ x = y^3 + 1,\ x - 1 = y^3,\ y = \sqrt[3]{x-1}$

(b)

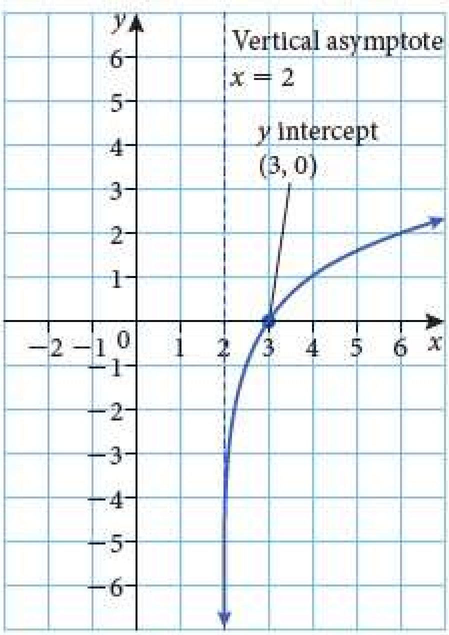


(c) $y = x$

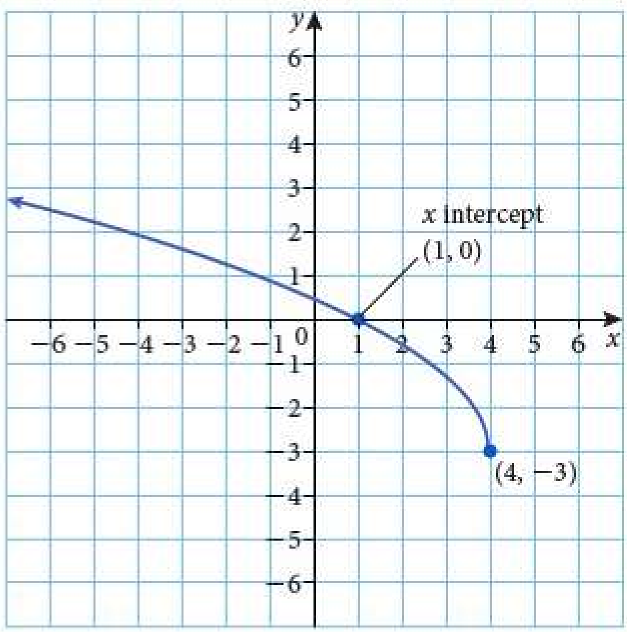
6. **(a)**


y
x
x intercept
(−2, 0)
y intercept
(0, 4)

(b)


y
x
Vertical asymptote
x = 2
y intercept
(3, 0)

(c)



7. **(a)** $g^{-1}(0) = x$, $\therefore g(x) = 0$. Using the graph, this x-coordinate is -1

(b) $f^{-1}(4) = x$, $\therefore f(x) = 4. x = -2$

(c) $f\left(g^{-1}(1)\right) = f(0)$, since $g(0) = 1$, and $f(0) = 3$

(d) $g^{-1}\left(f^{-1}(3)\right) = g^{-1}(0) = -1$

8. **(a)** $f\left(g(x)\right) = f(2x+1) = (2x+1)^2 + 3 = 4x^2 + 4x + 4$. $2x+1$ is a linear function, which can take on all values, but when squared, the minimum possible value is zero. So the minimum possible value of $(2x+1)^2 + 3$ is $0+3=3$. Hence, the range is $y \geq 3$

(b) $g^{-1}(x) = \frac{1}{2}(x-1)$, so $f(x) < 14g^{-1}(x)$ becomes $x^2 + 3 < 7(x-1)$

We could graph $y = x^2 + 3$ and $y = 7x - 7$ to solve this, by looking at where the parabola is below the line, or solve algebraically: $x^2 - 7x + 10 < 0$, $(x-2)(x-5) < 0$. The critical values, where the left side would equal zero, are $x = 2,5$. The product on the left is positive when $x < 2$ or $x > 5$ and negative when $2 < x < 5$. Hence $2 < x < 5$

9. **(a)** The domain of the log function is positive real numbers, so $x + 2 > 0$, or $x > -2$

Hence, the least possible value of a is -2.

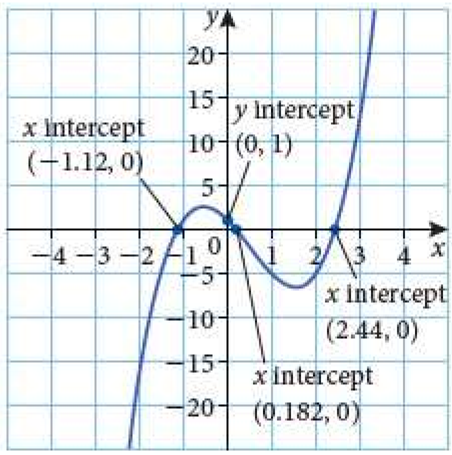
(b) $y = 5\log_3(x+2) - 4,\ x = 5\log_3(y+2) - 4$

$x + 4 = 5\log_3(y+2), \dfrac{x+4}{5} = \log_3(y+2)$

.

$y + 2 = 3^{(x+4)/5},\ y = 3^{(x+4)/5} - 2,\ y > -2$

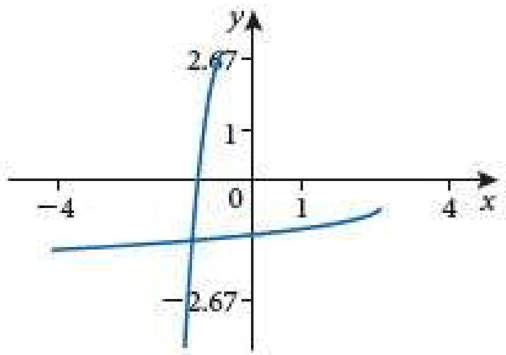
10. **(a)**



(b) It is not one-to-one, since it fails the horizontal-line test.

(c) The graph is increasing, and hence one-to-one, up until its relative maximum at $(-0.541,\ 2.51)$ (3 s.f.). So the greatest a is -0.541

(d)



11. **(a)** $\dfrac{150}{1150} \approx 0.1304$ or 13.0%.

(b) $C(x) = \dfrac{x}{1000+x}$

(c) $y = \dfrac{x}{1000+x},\ x = \dfrac{y}{1000+y}$

$1000x + xy = y,\ 1000x = y - xy$

$1000x = y(1-x),\ y = \dfrac{1000x}{1-x}$

(d) $y = \dfrac{1000 \cdot 0.4}{1-0.4} = \dfrac{400}{0.6} \approx 666.7$, so she should add 667 ml of lemon juice.

12. **(a)** Since he runs 20 km in 1 h, he runs 10 km in 0.5 h.

So the total distance is $10+30=40$ km and the total time is

$0.5 + \dfrac{30}{x} = \dfrac{0.5x+30}{x} = \dfrac{x+60}{2x}$

Thus the average speed is $40 \div \dfrac{x+60}{2x} = 40 \cdot \dfrac{2x}{x+60} = \dfrac{80x}{x+60}$

(b) $x = \dfrac{80y}{y+60}$, $xy+60x = 80y$, $60x = 80y - xy$, $60x = y(80-x)$, $y = \dfrac{60x}{80-x}$

(c) $y = \dfrac{60 \cdot 35}{80-35} \approx 46.7$ km h^{-1}.

13. **(a)** The distance run at the slower pace, after changing time to hours,

is speed × time $= 15 \cdot \dfrac{1}{3} = 5$ km, and at the quicker pace is $22 \cdot \dfrac{t}{60} = \dfrac{22t}{60}$ km

So the average speed is $\dfrac{5+\dfrac{22t}{60}}{\dfrac{1}{3}+\dfrac{t}{60}} = \dfrac{60 \cdot 5 + 60 \cdot \dfrac{22t}{60}}{60 \cdot \dfrac{1}{3} + 60 \cdot \dfrac{t}{60}} = \dfrac{300+22t}{20+t}$

(b) $t = \dfrac{300+22y}{20+y}$, $20t + yt = 300 + 22y$

$yt - 22y = 300 - 20t$, $y(t-22) = 300-20t$,

$y = \dfrac{300-20t}{t-22}$ or $\dfrac{20t-300}{22-t}$

(c) $y(20) = \dfrac{300-20 \cdot 20}{20-22} = \dfrac{-100}{-2} = 50$ min

14. **(a)** h is in kilometres, $d(8.848) = \sqrt{8.848(8.848+2 \cdot 6371)} \approx 336$ km

(b) $h = \sqrt{y(y+2r)}$, $h^2 = y(y+2r)$

$h^2 = y^2 + 2ry = (y+r)^2 - r^2$

$h^2 + r^2 = (y+r)^2$, $y + r = \sqrt{h^2+r^2}$

$y = \sqrt{h^2+r^2} - r$, $h > 0$

(c) $y(200) = \sqrt{200^2 + 6371^2} - 6371 \approx 3.14$ km

Exercise 2.5

1. For sketches, see answer section in textbook.

(a) vertical translation of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

(b) horizontal translation of $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

(c) reflection in x-axis

(d) translation of $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

(e) vertical stretch (x-axis invariant), stretch factor (s.f.) = 2

(f) horizontal stretch (y-axis invariant), s.f. = $\frac{1}{2}$

(g) reflection in y-axis

(h) horizontal translation of $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$

AND vertical stretch (x-axis invariant), s.f. = 2 AND reflection in x-axis

2. **(a)** $y = 4f(x) = 4(3x^2 - 2x) = 12x^2 - 8x$

(b) $y = f\left(\frac{1}{2}x\right) = 3\left(\frac{1}{2}x\right)^2 - 2\left(\frac{1}{2}x\right) = 3\left(\frac{1}{4}x^2\right) - x = \frac{3}{4}x^2 - x$

(c) $y = f(x+1) + 3 = 3(x+1)^2 - 2(x+1) + 3$

$= 3(x^2 + 2x + 1) - 2x - 2 + 3$

$= 3x^2 + 4x + 4$

(d) $y = f(x-2) - 1 = 3(x-2)^2 - 2(x-2) - 1$

$= 3(x^2 - 4x + 4) - 2x + 4 - 1 = 3x^2 - 14x + 15$

(e) $y = -f(x) = -3x^2 + 2x$

(f) $y = f(-x) = 3(-x)^2 - 2(-x) = 3x^2 + 2x$

3. $f(x) = x^3 - 4x,\ f_1(x) = f(-x) = (-x)^3 - 4(-x) = -x^3 + 4x,$

$f_2(x) = f_1(x-2) = -(x-2)^3 + 4(x-2) = -x^3 + 6x^2 - 8x$

4. $f_1(x) = -f(x) = -x^3 + 4x,\ f_2(x) = f_1(x) + 2 = -x^3 + 4x + 2$

5. **(a)** 1. Translation of $\begin{pmatrix} -4 \\ 0 \end{pmatrix}$ (since this is in brackets),

2. Stretch by s.f. = 2, x-axis invariant,

3. Reflection in x-axis

Note: Other orders are possible.

(b) Translation of $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(c) $f(-2x+4) = f(-2(x-2))$, so, this will be

1. Reflection in y-axis.

2. Stretch by s.f. = $\frac{1}{2}$, y-axis invariant,

3. Translation of $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$,

(d) 1. Stretch by s.f. = 2, x-axis invariant,

2. Translation of $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$

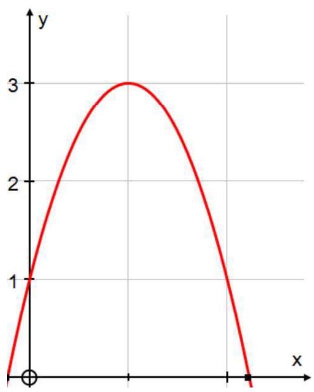
(e) 1. Stretch by s.f. = 3, x-axis invariant,

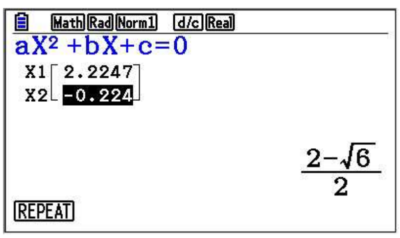
2. Reflection in x-axis,

3. Translation of $\begin{pmatrix} 0 \\ 6 \end{pmatrix}$

6. **(a)** Ground implies $h = 0$. Graphing gives $d \approx 2.22$ m.

Alternatively, use the equation solver, or the quadratic formula.



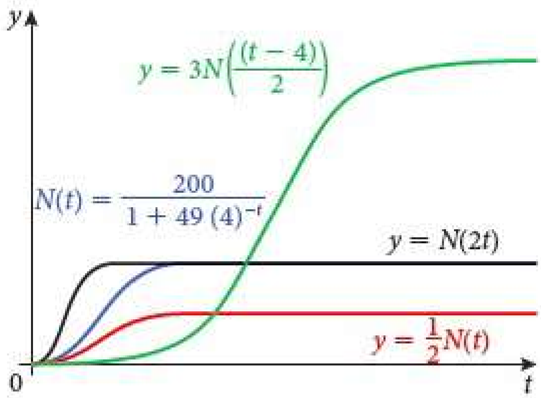


(b) On the graph we can see the maximum height of 3 m, which we could also find using the equation of the axis of symmetry, $x = -\frac{b}{2a} = -\frac{4}{2(-2)} = 1$, and substituting this for d to find the corresponding h-value.

(c) $h(d) = h_{\text{old}}\left(\frac{1}{2}d\right) = 1 + 4\left(\frac{1}{2}d\right) - 2\left(\frac{1}{2}d\right)^2 = 1 + 2d - 2\left(\frac{1}{4}d^2\right) = 1 + 2d - \frac{1}{2}d^2$

7. **(a)** $N(0) = \frac{200}{1 + 49 \cdot 4^0} = \frac{200}{50} = 4$ people infected

(b)



(c) $N = 200$

(d) See **(a)**.

(e) Horizontal asymptote is only affected by:
vertical stretches, reflections and translations.

(i) $y = \frac{1}{2} \cdot N(0) = \frac{1}{2} \cdot 4 = 2$; asymptote $y = \frac{1}{2} \cdot 200 = 100$

(ii) $y = N(2 \cdot 0) = N(0) = 4$; asymptote $y = 200$

(iii) $y = 3N\left(\frac{0-4}{2}\right) = 3N(-2) = 3 \cdot \frac{200}{1 + 49 \cdot 4^2} \approx 0.764$;

asymptote $y = 3 \cdot 200 = 600$

(f) $3N\left(\frac{t-4}{2}\right) = 3N\left(\frac{1}{2}(t-4)\right) = 3\left(\frac{200}{1 + 49 \cdot 4^{-(1/2(t-4))}}\right)$

In the denominator we have $4^{-\left(\frac{1}{2}(t-4)\right)}$ which simplifies to

$\left(4^{-\frac{1}{2}}\right)^{t-4} = \frac{1}{2}^{t-4} = \left(\frac{1}{2}\right)^t \cdot \left(\frac{1}{2}\right)^{-4} = 16\left(\frac{1}{2}\right)^t = 16(2)^{-t}$, giving

$\frac{600}{1 + 49 \cdot 16(2)^{-t}} = \frac{600}{1 + 784(2)^{-t}}$

8. **(a)** **(i)** $\frac{(8000-6000)^2}{40000}=\frac{2000^2}{40000}=\frac{4000000}{40000}=\$100$

(ii) $\frac{20000}{10}-600=2000-600=\$1400$

(b) $T_1(x)=1.2\left(T(x)+200\right)=1.2T(x)+240$

$$T_1(x)=\begin{cases}1.2\cdot 0+240=240, & x\le 6000\\ \frac{1.2(x-6000)^2}{40000}+240=\frac{3(x-6000)^2}{100000}+240, & 6000<x\le 10000\\ \frac{1.2x}{10}-1.2\cdot 600+240=\frac{3x}{25}-480, & 10000<x\end{cases}$$

$T_2(x)=T(x+1000)$, so $6000<x+1000\le 10000$ becomes $5000<x\le 9000$

$$T_2(x)=\begin{cases}0, & x\le 5000\\ \frac{(x+1000-6000)^2}{40000}=\frac{(x-5000)^2}{40000}, & 5000<x\le 9000\\ \frac{x+1000}{10}-600=\frac{x}{10}+100-600=\frac{x}{10}-500, & 9000<x\end{cases}$$

$T_3(x)=T(2x+1000)$,

so $6000<2x+1000\le 10000$ becomes $2500<x\le 4500$.

Then

$$T_3(x)=\begin{cases}0, & x\le 2500\\ \frac{(2x+1000-6000)^2}{40000}=\frac{(2x-5000)^2}{40000}=\frac{2^2(x-2500)^2}{40000}=\frac{(x-2500)^2}{10000}, & 2500<x\le 4500\\ \frac{2x+1000}{10}-600=\frac{x}{5}+100-600=\frac{x}{5}-500, & 4500<x\end{cases}$$

(c) **(i)** $T_1(8000)=\frac{3(8000-6000)^2}{100000}+240=\frac{12000000}{100000}+240=360$

$T_2(8000)=\frac{(8000-5000)^2}{40000}=\frac{9000000}{40000}=225$

$T_3(8000)=\frac{8000}{5}-500=1600-500=1100$

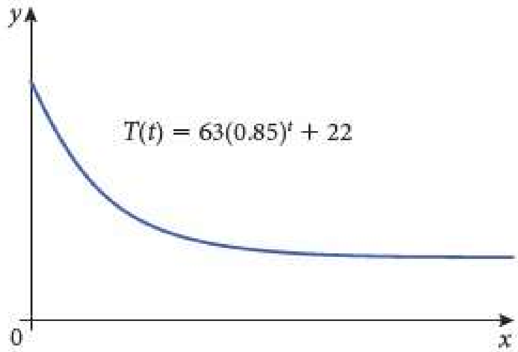
(ii) $T_1(20000) = \frac{3 \cdot 20000}{25} - 480 = 2400 - 480 = 1920$

$T_2(20000) = \frac{20000}{10} - 500 = 2000 - 500 = 1500$

$T_3(20000) = \frac{20000}{5} - 500 = 4000 - 500 = 3500$

9. **(a)** $T(0) = 63 + 22 = 85$

(b)



(c) $T(12) = 63(0.85)^{12} + 22 \approx 31.0$

(d) As $t \to \infty$, $(0.85)^t \to 0$, so $T(t) \to 22$. $T = 22$ is the asymptote of the graph, so the most accurate reading we can get for the temperature is 22.0 °C.

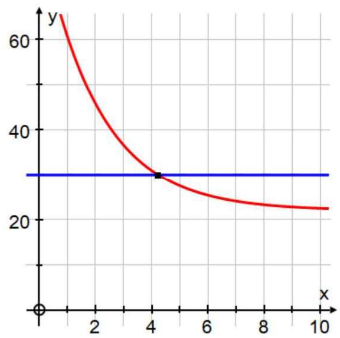
(e) This graph must be stretched horizontally by a scale factor of 2, so

$$T_2(t) = T\left(\frac{1}{2}t\right) = 63(0.85)^{\frac{1}{2}t} + 22 = 63((0.85)^{\frac{1}{2}})^t + 22 \approx 63(0.922)^t + 22$$

(f) Clearly, $T_2(24) = T\left(\frac{1}{2} \cdot 24\right) = T(12)$

(g) $T_3(t) = T(3t) = 63(0.85)^{3t} + 22 = 63((0.85)^3)^t + 22 \approx 63(0.614)^t + 22$

(h) Using the graph, we find the intersection of $T_3(t) = 63(0.614)^t + 22$ and $T = 30$ when $t = 4.23$ minutes.



Chapter 2 Practice questions

1. **(a)** $\frac{M}{E}=\frac{2850}{600}, M=\frac{2850}{600}E=\frac{19}{4}E$

(b) $M=\frac{19}{4}\cdot 170=807.50$ ringgit

(c) $E=\frac{4}{19}M=\frac{4}{19}\cdot 3500\approx 736.84$ euros

2. $x=\frac{2y-5}{3y-1}$, $3xy-x=2y-5$, $3xy-2y=x-5$, $y(3x-2)=x-5$, $y=\frac{x-5}{3x-2}$, $x\neq\frac{2}{3}$

3. **(a)** $15\,\text{min}=\frac{1}{4}\,\text{h}$, $\frac{1}{4}\cdot 12=3\,\text{km}$.

$t=\frac{d}{s}$, average speed$=\frac{\text{total distance}}{\text{total time}}$

$$=\frac{3+x}{\frac{1}{4}+\frac{x}{18}}$$

$$=\frac{3+x}{\frac{9+2x}{36}}$$

$$=(3+x)\cdot\frac{36}{9+2x}$$

$$=\frac{36x+108}{2x+9}$$

(b) $x=\frac{36y+108}{2y+9}$

$$2xy+9x=36y+108$$

$$2xy-36y=108-9x$$

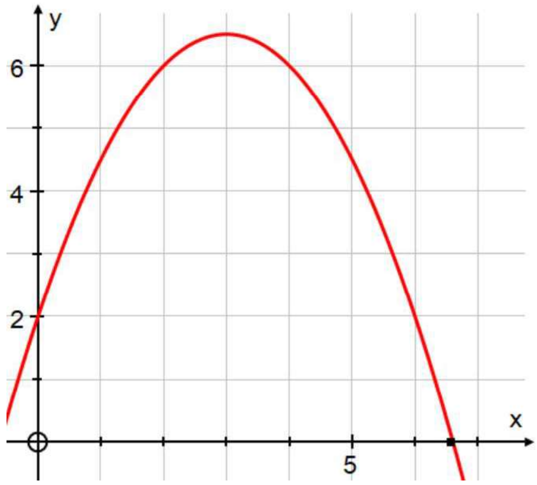
$$y(2x-36)=108-9x$$

$$y=\frac{108-9x}{2x-36}$$

(c) $\frac{108-9\cdot 16}{2\cdot 16-36}=\frac{-36}{-4}=9$

4. **(a)** Using the quadratic formula with $a=-\frac{1}{2}, b=3, c=2$ gives

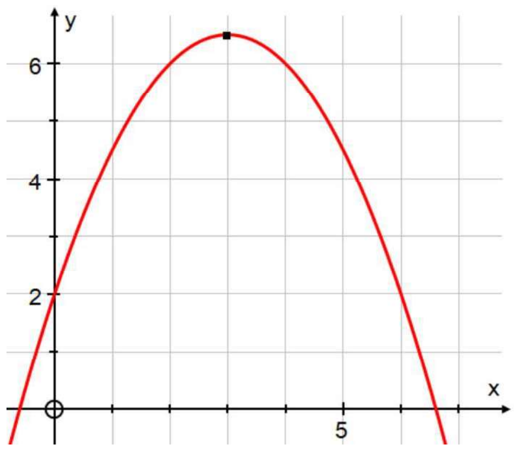
$$d=\frac{-3\pm\sqrt{3^2-4\cdot\left(-\frac{1}{2}\right)\cdot 2}}{2\cdot\left(-\frac{1}{2}\right)}\approx 6.61$$ metres, as does graphing $h=2+3d-\frac{1}{2}d^2$



(b) We can find the x-coordinate of the vertex using the formula for the axis of symmetry,

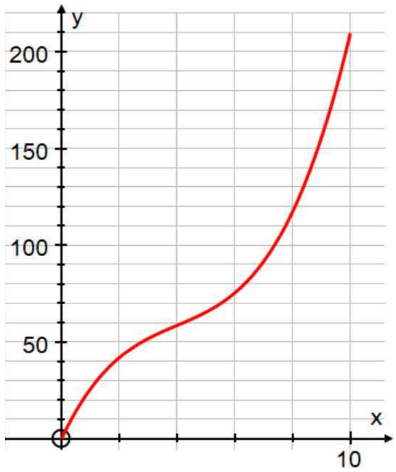
$x=-\frac{b}{2a}=-\frac{3}{2\left(-\frac{1}{2}\right)}=3$, and then substitute this into $h(d)$ or use the graph

to find the maximum height of 6.5 metres.



(c) $h_2(d)=h(2d)=2+3(2d)-\frac{1}{2}(2d)^2=2+6d-2d^2$

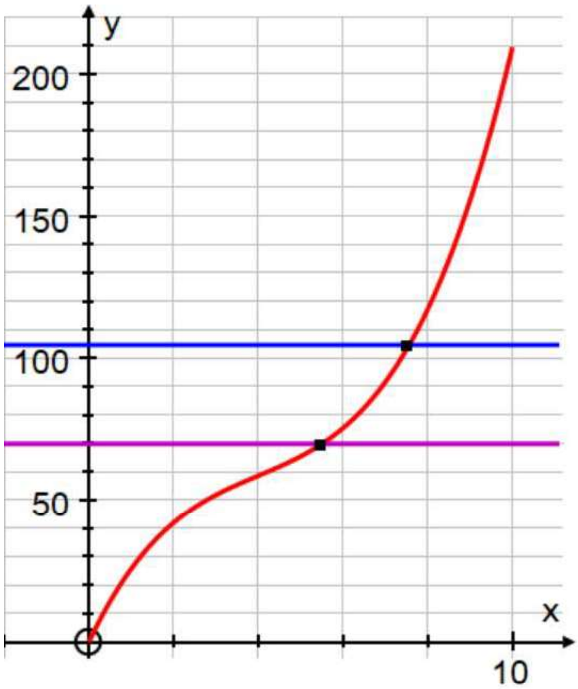
5. **(a)**



(b) From the graph, we see that the maximum is at the endpoint when $h = 10$

Finding $v(10)$ or using the graph gives approximately 209 cm^3.

(c) From the graph we have **(i)** 7.56 cm and **(ii)** 5.50 cm.



(d) From the graph, when the height is 3 cm, the volume is approximately 51.8 cm^3, so this is how much orange juice is in the glass.

The concentration is then $C(h) = \dfrac{51.8}{v(h)}$

(e) $C = \dfrac{51.8}{200} = 0.259$, or 25.9%.

6. **(a)** **(i)** $u \circ f = f(x) - 3$, so the range is $[-1-3, 1-3] = [-4, -2]$ or $-4 \le y \le -2$

(ii) $u \circ v \circ f = 2f(x) - 3$,

so the range is $[2(-1)-3, 2(1)-3] = [-5, -1]$ or $-5 \le y \le -1$

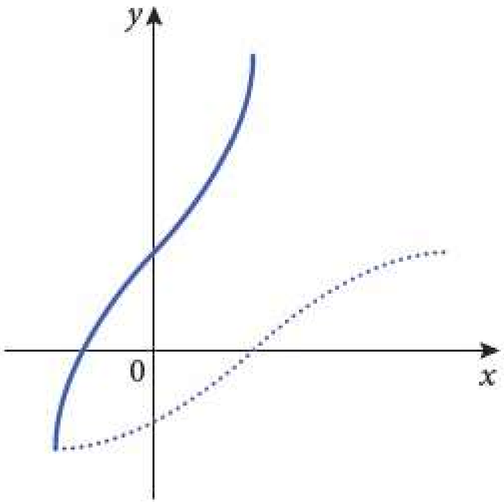
(iii) $f \circ v \circ u = f(2(x-3)) = f(2x-6)$, so $-3 \le 2x-6 \le 5$,

which gives $3 \le 2x \le 11$, so $\frac{3}{2} \le x \le \frac{11}{2}$

(b) **(i)** f fails the horizontal line test, hence it is not one-to-one.

(ii) From the minimum to the maximum we have $-1 \le x \le 3$

(iii) The x- and y-coordinates of the points on the original graph are swapped, so the y-intercept is at $(0,1)$ and the endpoints are at: $(-1,\ -1), (1,\ 3)$.



(c) **(i)** $x = \frac{2y-5}{y+d}$, $xy + dx = 2y - 5$

$xy - 2y = -dx - 5$, $y(x-2) = -(dx+5)$

$y = -\frac{dx+5}{x-2}$, $y = \frac{5+dx}{2-x}$

(ii) For $h^{-1}(x) = \frac{5+dx}{2-x} = \frac{2x-5}{x+d} = h(x)$, $d = -2$, since

$$\frac{5+(-2)x}{2-x} = \frac{2x-5}{x-2} = \frac{2x-5}{x+(-2)}$$

(iii) $h(k(x)) = \frac{2k(x)-5}{k(x)-2} = \frac{2x}{x+1}$

Cross-multiplying gives

$2x \cdot k(x) - 4x = 2x \cdot k(x) - 5x + 2k(x) - 5,$

$-4x = -5x + 2k(x) - 5$

$x + 5 = 2k(x)$

$k(x) = \frac{1}{2}x + \frac{5}{2} = \frac{x+5}{2}$

7. **(a)** **(i)** From the point $(0,-3)$, we get $f^{-1}(-3) = 0$

(ii) $f^{-1}(g(1)) = f^{-1}(5) = 6,$ from (1, 5) on g and (6, 5) on f.

(iii) $f^{-1}(f^{-1}(-4)) = f^{-1}(-3) = 0$, from $(-3,-4)$ on f and from **(i)**.

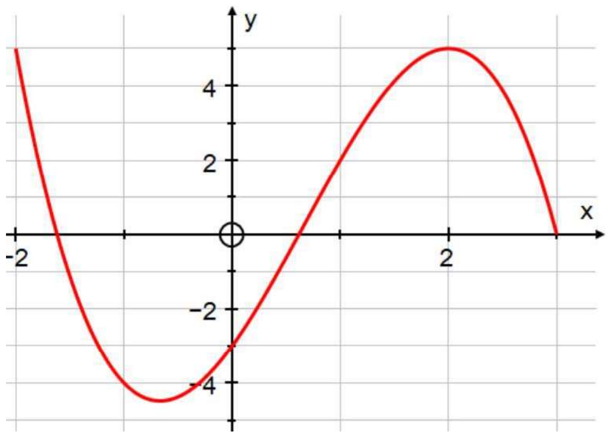
(b) It fails the horizontal line test, hence it is not one-to-one.

(c) $g(g(3)) = g(5) = -3$

To solve $g(x) = -3,$ we find points with a *y*-coordinate of -3,

which are at $x = -1$ or 5

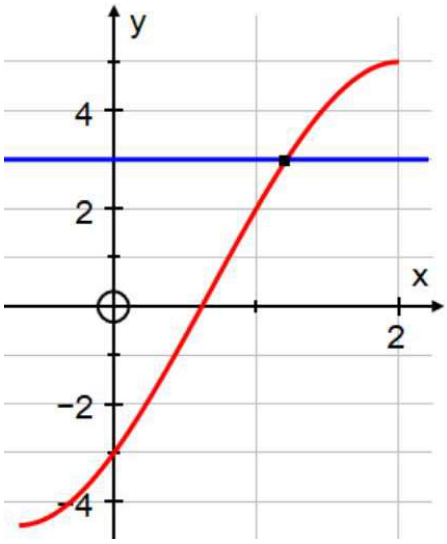
8. **(a)** *y*-intercept at $(0,-3)$, *x*-intercepts at $-1.62,\ 0.618,\ 3$



(b) It fails the horizontal line test, hence it is not one-to-one.

(c) This is the *x*-value of the minimum, $x \approx -0.667$

(d) Drawing the horizontal line $y = 3$ shows the x-coordinate of $x \approx 1.21$



9. **(a)** Since $\log_1 x = y$ implies that $1^y = x$, if $a = 1$,

this function would only be defined for $x = 1$ not all $x \in \mathbb{R}^+$.

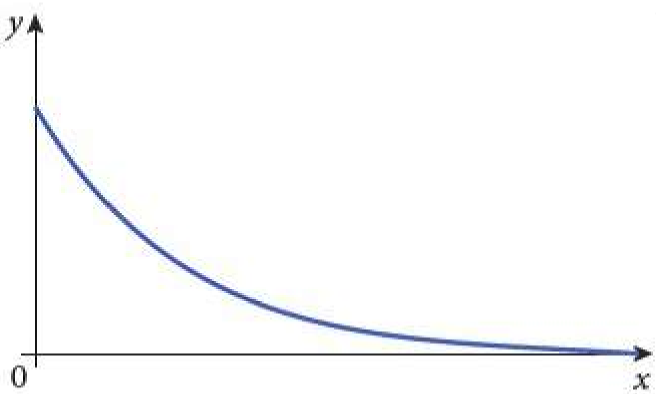
Hence $a > 0,\ a \neq 1$.

(b) Using the base change rule, we have

$\dfrac{\log y}{\log x} = \dfrac{9\log x}{\log y}$, $(\log y)^2 = 9(\log x)^2$, $\log y = \pm 3\log x$, $\log y = \log x^3$ or $\log x^{-3}$,

so $y = x^3$ or $y = \dfrac{1}{x^3}$

10. **(a)**



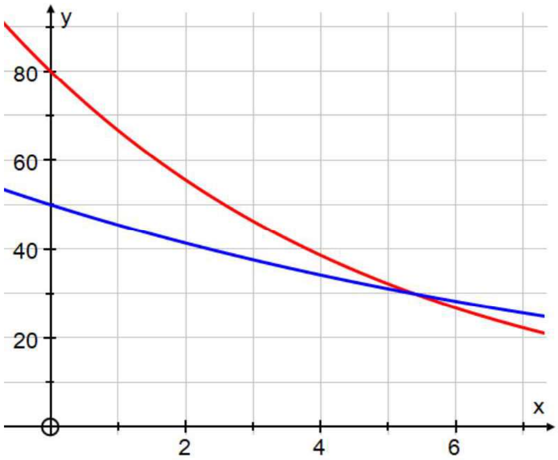
(b) $A(0) = \dfrac{80}{1.2^0} = 80$

(c) $25 = \dfrac{80}{1.2^t}$, $1.2^t = \dfrac{80}{25}$, $t = \log_{1.2}\dfrac{80}{25} \approx 6.38$

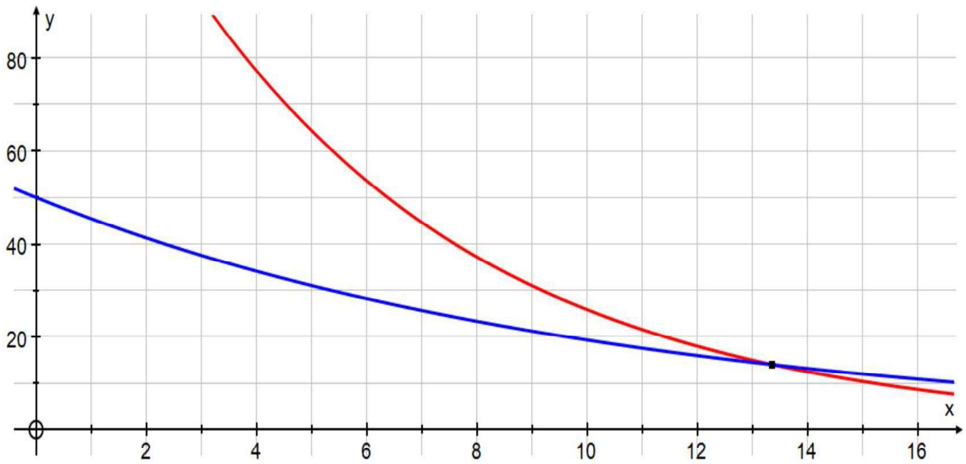
(d) $A = 0$, since as $t \to \infty$, $1.2^t \to \infty$ and so $\dfrac{80}{1.2^t} \to 0$

(e) The activity slows, meaning it approaches zero.

(f) The point of intersection of the two graphs is at $t \approx 5.40$



(g) $B = 2A = \dfrac{160}{1.2^t}$ when $t \approx 13.4$



11. (a) $1 - 2\cdot 2 = -3 = \frac{3}{4}(2-2)^2 - 3$, hence the function is continuous.

(b) $1 - 2(-x) = 1 + 2x,\ 1 + 2(x-2) = 1 + 2x - 4 = 2x - 3$

$\frac{3}{4}(-x-2)^2 - 3 = \frac{3}{4}(-(x+2))^2 - 3 = \frac{3}{4}(x+2)^2 - 3,\ \frac{3}{4}(x-2+2)^2 - 3 = \frac{3}{4}x^2 - 3$

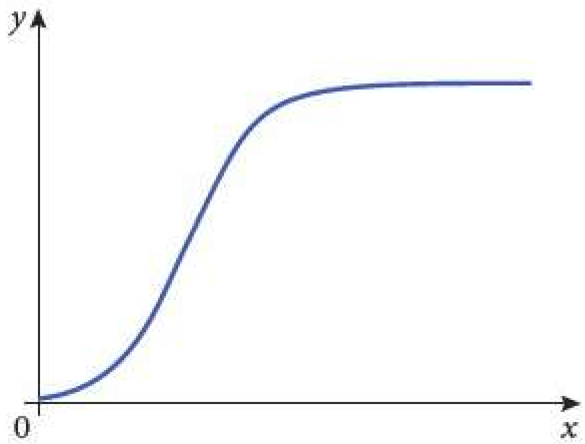
$x \le 2$ reflected is $x \ge -2$, then translated two units right, it is $x \ge 0$

$$\therefore g(x) = \begin{cases} 2x - 3, & x \ge 0 \\ \frac{3}{4}x^2 - 3, & x < 0 \end{cases}$$

12. **(a)** $3+0.8\cdot 10=11,\ x+0.6\cdot 10=11\rightarrow x=5$

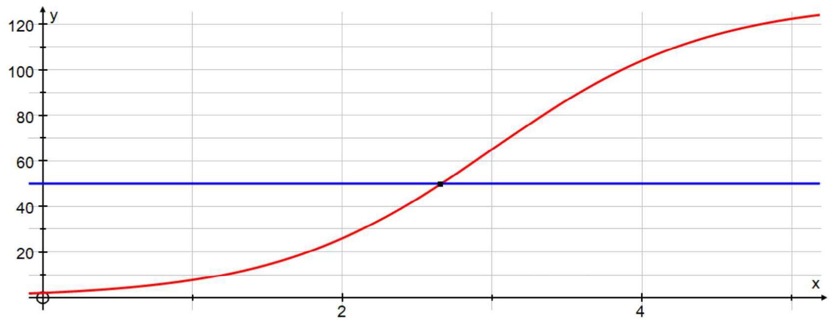
(b) $C(16)=5+0.6\cdot 16=5+9.6=14.6,\ \therefore \$14.60$

13. **(a)**



(b) $P(0)=\dfrac{130}{1+4^{3-0}}=\dfrac{130}{65}=2$

(c) Finding the intersection of the graphs of $P(t)$ and $P=50$ gives 2.66 years.

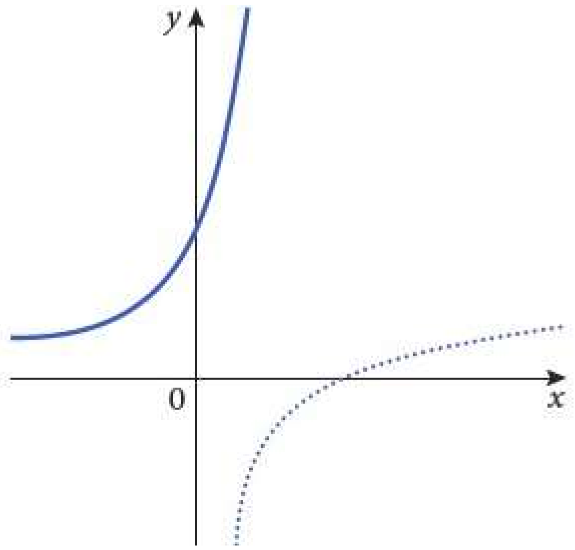


(d) As $t\rightarrow\infty,\ 4^{3-t}\rightarrow 0,$ so $\dfrac{130}{1+4^{3-t}}\rightarrow\dfrac{130}{1+0}=130$

Hence the asymptote is $P=130$

(e) The population grows rapidly, then more slowly, before levelling off at approximately 130 birds after a longer period of time.

14. **(a)**



(b) The domain of f is $x > 1$, as seen from the asymptote, so the range of f^{-1} is $y > 1$

(c) $f(4) = \ln(4a + b) = 0,\ 4a + b = e^0 = 1,\ b = 1 - 4a$

Also, the domain of the logarithmic function is positive real numbers, so $ax + b > 0,\ ax + 1 - 4a > 0,\ ax > 4a - 1,\ x > \frac{4a-1}{a}$, but we know $x > 1$, so $\frac{4a-1}{a} = 1$ This gives $4a - 1 = a,\ 3a = 1,\ a = \frac{1}{3}$. Then

$b = 1 - 4\left(\frac{1}{3}\right) = \frac{3}{3} - \frac{4}{3} = -\frac{1}{3}$

15. **(a)** f has a minimum at $x = 0$, since $x^2 \geq 0$, so the range is $y \geq f(0) = \frac{3}{75} = \frac{1}{25}$

g has a minimum value of 0, since the modulus function must be positive or zero, hence its range is $y \geq 0$

(b) $f\left(\frac{|3x-4|}{10}\right) = \frac{1}{75}\left(2\left(\frac{|3x-4|}{10}\right)^2 + 3\right) = \frac{1}{75}\left(\frac{2}{100}(9x^2 - 24x + 16) + 3\right)$

$= \frac{1}{75}\left(\frac{1}{50}(9x^2 - 24x + 16 + 150)\right) = \frac{9x^2 - 24x + 166}{3750}$

(c) **(i)** $x = \frac{2}{75}y^2 + \frac{3}{75},\ \frac{2}{75}y^2 = x - \frac{3}{75},\ 2y^2 = 75x - 3,\ y^2 = \frac{75x-3}{2},\ y = \sqrt{\frac{75x-3}{2}}$

(ii) $x \geq \frac{1}{25}$ (from range of f in **(a)**), $y \geq 0$ (from domain of f in **(a)**)

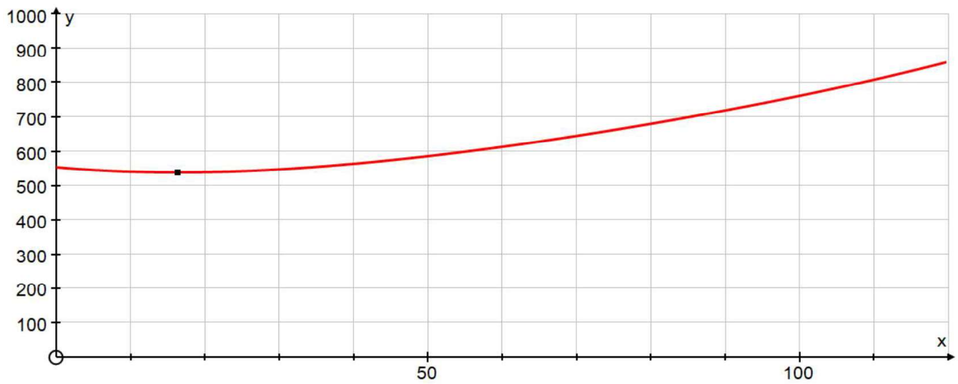
16. **(a)** $AP = \sqrt{x^2 + 50^2} = \sqrt{x^2 + 2500}$ and

$BP = \sqrt{(120 - x)^2 + 40^2} = \sqrt{x^2 - 240x + 14400 + 1600} = \sqrt{x^2 - 240x + 16000}$

Hence the cost, in millions of dollars, is

$C(x) = 6\sqrt{x^2 + 2500} + 2\sqrt{x^2 - 240x + 16000}$

(b) Graphing to find the minimum gives approximately \$538 million.



17. $g(x) = -(\ln(x-3) - 2) = -(\ln(x-3) - \ln e^2) = -\left(\ln\dfrac{x-3}{e^2}\right) = \ln\left(\dfrac{x-3}{e^2}\right)^{-1} = \ln\dfrac{e^2}{x-3}$

18. reflection: $-2x^2 + 3x - 1$.

translation: $-2(x-1)^2 + 3(x-1) - 1 + 2 = -2(x^2 - 2x + 1) + 3x - 3 + 1$

$= -2x^2 + 4x - 2 + 3x - 2 = -2x^2 + 7x - 4$

stretch: $-2\left(\dfrac{1}{2}x\right)^2 + 7\left(\dfrac{1}{2}x\right) - 4 = -2\left(\dfrac{1}{4}x^2\right) + \dfrac{7}{2}x - 4 = -\dfrac{1}{2}x^2 + \dfrac{7}{2}x - 4$

Exercise 3.1

1. In questions a – c, substitute n (or k) = 1, 2,…, 5 into the given formula.

(a) $s(n) = 2n - 3 \quad \Rightarrow s(1) = -1,\ s(2) = 1,\ s(3) = 3,\ s(4) = 5,\ s(5) = 7$

(b) $g(k) = 2^k - 3 \quad \Rightarrow g(1) = -1,\ g(2) = 1,\ g(3) = 5,\ g(4) = 13,\ g(5) = 29$

(c) $f(k) = 3 \times 2^{-n} \quad \Rightarrow f(1) = \frac{3}{2},\ f(2) = \frac{3}{4},\ f(3) = \frac{3}{8},\ f(4) = \frac{3}{16},\ f(5) = \frac{3}{32}$

(d) $a_n = (-1)^n(2^n) + 3 \quad \Rightarrow a_1 = 1,\ a_2 = 7,\ a_3 = -5,\ a_4 = 19,\ a_5 = -29$

(e) $\begin{cases} a_1 = 5 \\ a_n = a_{n-1} + 3 \end{cases} \Rightarrow a_1 = 5,\ a_2 = 8,\ a_3 = 11,\ a_4 = 14,\ a_5 = 17$

(f) $\begin{cases} b_1 = 3 \\ b_n = b_{n-1} + 2n \end{cases} \Rightarrow b_1 = 3,\ b_2 = 7,\ b_3 = 13,\ b_4 = 21,\ b_5 = 31$

2. In parts **(a) – (d)**, simply substitute n = 1, 2, …, 5 and n = 50 into the formula.

(a) $-1, 1, 3, 5, 7 \qquad a_{50} = 97$

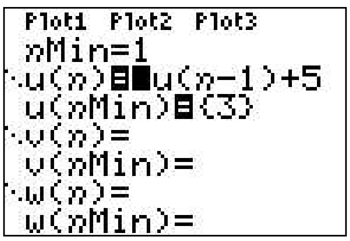
(b) $2, 6, 18, 54, 162 \qquad b_{50} = 2 \cdot 3^{49} = 4.786 \times 10^{23}$

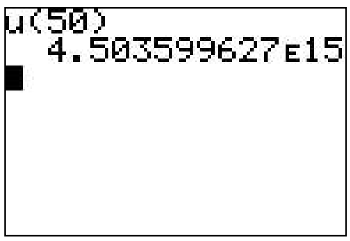
(c) $\frac{2}{3}, -\frac{2}{3}, \frac{6}{11}, -\frac{4}{9}, \frac{10}{27} \qquad u_{50} = -\frac{100}{2502} = -\frac{50}{1251}$

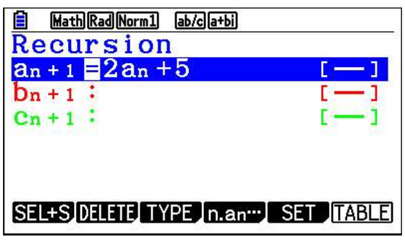
(d) $1, 2, 9, 64, 625 \qquad a_{50} = 50^{49} = 1.776 \times 10^{83}$

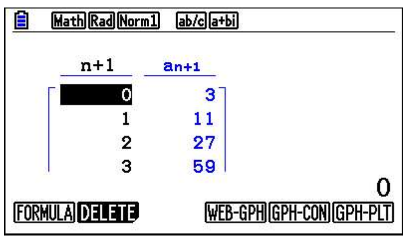
In parts **(e)–(h)** with the first term and substitute it in the given formula to find the second term, and so on. To find the 50th term, we will use a GDC in Sequential mode. Be careful that some GDCs start with $u(n + 1)$ rather than $u(n)$ as shown in the second set of screen shots. In this case you start with n = 0 and end with n = 49.

(e) $3, 11, 27, 59, 123 \qquad a_{50} = 4.50 \times 10^{15}$

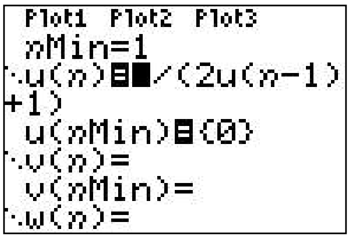


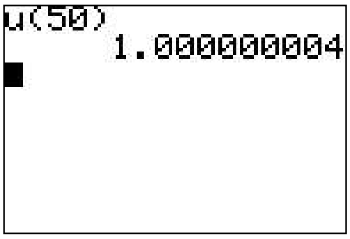




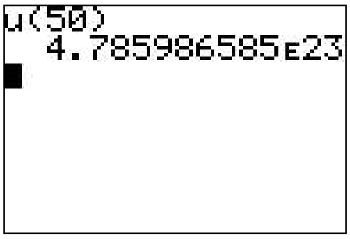


(f) $0,\ 3,\ \frac{3}{7},\ \frac{3}{2\cdot\frac{3}{7}+1}=\frac{21}{13},\ \frac{39}{55}$ $\qquad u_{50} \approx 1.00$

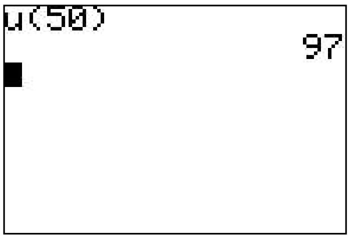




(g) 2, 6, 18, 54, 162 $\qquad b_{50} \approx 4.786 \times 10^{23}$



(h) −1, 1, 3, 5, 7 $\qquad a_{50} = 97$



3. In this question you need to observe and spot the pattern. Perhaps trial and error!

(a) $u_n = \frac{1}{4}u_{n-1},\quad u_1 = \frac{1}{3}$

(b) $u_n = \frac{4a^2}{3}u_{n-1},\quad u_1 = \frac{1}{2}a$

(c) $u_n = u_{n-1} + a + k,\quad u_1 = a - 5k$

4. In this question too you need to observe and spot the pattern. Perhaps trial and error!

(a) $u_n = n^2 + 3$

(b) $u_n = 3n - 1$

(c) $u_n = \frac{2n-1}{n^2}$

(d) $u_n = \frac{2n-1}{n+3}$

5. **(a)** Here is a part of the spreadsheet we used.

n	*Fn*	*an*
1	1	1
2	1	2
3	2	1.5
4	3	1.666666667
⇓	⇓	⇓
29	514229	1.618033989
30	832040	1.618033989

(b) $\lim_{n\to\infty} a_n \approx 1.61803$

6. **(a)** Here is a part of the spreadsheet we used.

n	*Gn*	*bn*
1	1	1
2	1	3
3	3	2.333333
4	7	2.428571
⇓	⇓	⇓
29	26102926097	2.414214
30	63018038201	2.414214

(b) $\lim_{n\to\infty} b_n \approx 2.41421$

Exercise 3.2

1. **(a)** Arithmetic: $a_{n+1} - a_n = [2(n+1)-3] - (2n-3) = 2 \Rightarrow d = 2$

$$\Rightarrow a_{50} = a_1 + 49d = -1 + 49 \cdot 2 = 97$$

(b) Arithmetic: $b_{n+1} - b_n = (n+1+2) - (n+2) = 1 \Rightarrow d = 1$

$$\Rightarrow b_{50} = b_1 + 49d = 3 + 49 \cdot 1 = 52$$

(c) Arithmetic: $c_1 = -1,\ c_2 = 1,\ c_3 = 3 \Rightarrow d = 2 \Rightarrow c_{50} = c_1 + 49d = -1 + 49 \cdot 2 = 97$

(d) $e_2 - e_1 = 5 - 2 = 3,\ e_3 - e_2 = 7 - 5 = 2$

There is no constant common difference, so the sequence is not arithmetic.

(e) Arithmetic: $f_2 - f_1 = f_3 - f_2 = f_4 - f_3 = -7 \Rightarrow d = -7$

$$\Rightarrow f_{50} = f_1 + 49d = 2 + 49 \cdot (-7) = -341$$

2. **(a)** **(i)** $a_1 = -2,\ d = 4$: $a_8 = a_1 + (8-1)d = -2 + 7\cdot 4 = 26$

(ii) $a_n = -2 + (n-1)\cdot 4 = 4n - 6$

(iii) $a_n = a_{n-1} + 4,\ a_1 = -2$

(b) **(i)** $a_1 = 10.07,\ d = -0.12$: $a_8 = a_1 + 7d = 10.07 + 7\cdot(-0.12) = 9.23$

(ii) $a_n = 10.07 + (n-1)\cdot(-0.12) = -0.12n + 10.19$

(iii) $a_n = a_{n-1} - 0.12,\ a_1 = 10.07$

(c) **(i)** $a_1 = 100,\ d = -3$: $a_8 = a_1 + 7d = 100 + 7\cdot(-3) = 79$

(ii) $a_n = 100 + (n-1)\cdot(-3) = -3n + 103$

(iii) $a_n = a_{n-1} - 3,\ a_1 = 100$

(d) **(i)** $a_1 = 2,\ d = -\frac{5}{4}$: $a_8 = a_1 + 7d = 2 + 7\cdot(-\frac{5}{4}) = -\frac{27}{4}$

(ii) $a_n = 2 + (n-1)\cdot(-\frac{5}{4}) = -\frac{5}{4}n + \frac{13}{4}$

(iii) $a_n = a_{n-1} - \frac{5}{4},\ a_1 = 2$

3. We need to find the first term and the common difference

$a_5 = 6, a_{14} = 42 \Rightarrow a_1 + 4d = 6$, and

$a_{14} = 42 \Rightarrow a_1 + 13d = 42$, and solving the system

$d = 4,\ a_1 = -10 \Rightarrow a_n = -10 + (n-1)\cdot 4 = 4n - 14$

4. Similar to Q3:

$$\left.\begin{matrix} a_3 = -40 \\ a_9 = -18 \end{matrix}\right\} \Rightarrow \begin{cases} a_1 + 2d = -40 \\ a_1 + 8d = -18 \end{cases}$$

$$\Rightarrow d = \frac{11}{3},\ a_1 = -\frac{142}{3} \Rightarrow a_n = -\frac{142}{3} + (n-1)\cdot\frac{11}{3} = \frac{11}{3}n - 51$$

5. In all the exercises, we need to use the *n*th term formula

(a)
$a_1 = 3, d = 6, a_n = 525$
$a_n = a_1 + (n-1)d \quad \Rightarrow 525 = 3 + (n-1)\cdot 6 \quad \Rightarrow n = 88$

(b)
$a_1 = 9, d = -6, a_n = -201$
$a_n = a_1 + (n-1)d \quad \Rightarrow -201 = 9 + (n-1)\cdot(-6) \quad \Rightarrow n = 36$

(c)
$a_1 = \frac{1}{3}, d = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}, a_n = 2\frac{5}{6}$
$a_n = a_1 + (n-1)d \quad \Rightarrow \frac{17}{6} = \frac{1}{3} + (n-1)\cdot\frac{1}{6} \quad \Rightarrow n = 16$

(d)
$a_1 = 1 - k, d = (1+k) - (1-k) = 2k, a_n = 1 + 19k$
$a_n = a_1 + (n-1)d \quad \Rightarrow 1 + 19k = 1 - k + (n-1)\cdot 2k \quad \Rightarrow 20k = (n-1)\cdot 2k$
$\Rightarrow n = 11$

6.
$a_{30} = 147, d = 4$
$a_{30} = a_1 + 29d \quad \Rightarrow 147 = a_1 + 29\cdot 4 \quad \Rightarrow a_1 = 31$
$a_n = a_1 + (n-1)d = 31 + (n-1)\cdot 4 = 4n + 27$

7.
$a_1 = -7, d = 3, a_n = 9803$
$a_n = a_1 + (n-1)d \quad \Rightarrow 9803 = -7 + (n-1)\cdot 3 \quad \Rightarrow n = 3271$

Yes, 9803 is the 3271th term of the sequence.

8.
$a_1 = 9689, a_{100} = 8996$
$a_n = a_1 + (n-1)d \quad \Rightarrow 8996 = 9689 + 99d \quad \Rightarrow d = -7$
$a_{110} = a_1 + 109d = 9689 + 109\cdot(-7) = 8926$

$a_n = 1 \quad \Rightarrow 9689 + (n-1)\cdot(-7) = 1 \quad \Rightarrow n = 1385$

Yes, 1 is the 1385th term of the sequence.

9.
$a_1 = 2, a_{30} = 147,$
$a_n = a_1 + (n-1)d \quad \Rightarrow 147 = 2 + 29d \quad \Rightarrow d = 5$

$a_n = 995 \quad \Rightarrow 2 + (n-1)\cdot 5 = 995 \quad \Rightarrow n = \frac{998}{5}$

As a fractional result is not possible for *n,* we conclude that 995 is not a term of this sequence.

10. **(a)** Use GDC/spreadsheet to calculate differences and then find the average

$$d \approx \frac{4.3+4.5+4.2+4.3+4.4+4.2+4.4}{7} \approx 4.329$$

(b) $u_n = u_1 + (n-1)d \Rightarrow u_n = 4.4 + 4.33(n-1) = 0.07 + 4.33n$

(c) **(i)** $u_7 = 0.07 + 4.33 \times 7 \approx 30.38$

(ii) $\frac{30.38-30.3}{30.3} \approx 0.264\%$. However, if we use 3 sf figures only then

$$\frac{30.4-30.3}{30.3} \approx 0.330\%$$

11. **(a)** Use GDC/spreadsheet to calculate differences and then find the average

$$d \approx \frac{-25-20\cdots-30}{5} \approx -27$$

(b) $u_n = u_1 + (n-1)d \Rightarrow u_n = 280 - 27(n-1) = 307 - 27n$

(c) **(i)** $u_n = 307 - 27n \Rightarrow u_0 = 307$

(ii) An overestimate. The remaining mass includes the mass of the tub, so the amount of ice cream will likely be less than 307 g

(d) **(i)** $u_n = 307 - 27n \Rightarrow 0 = 307 - 27n \Rightarrow n = 11.37 \approx 12$

(ii) An underestimate. When there is only a small amount of ice cream left, it will be difficult to fill the spoon. So, the final few spoonfuls will likely be smaller, meaning more spoonfuls will be required.

12. **(a)** Each year the interest is $500 \times 0.032 = \$16$ and in 5 years this will be \$80. Thus the amount at the end of 5 years is \$580.

(b) $500 + 16n \geq 2000 \Rightarrow n \geq 93.75 = 94$ years.

13. **(a)** $v(n) = 16500 - 1650n \Rightarrow v(4) = \$9900$

(b) $9900 - 0.5 \times 16500 = \$1650$.

14. Amount after 7 years can be usually found using

$$u_7 = 450 + 7 \times 450r \Rightarrow r = \frac{560-450}{7 \times 450} \approx 0.0349 \approx 3.5\%.$$

15. Let t be the number of years, then the amount outstanding after t years using simple interest is $16000(1+rt) = 16000(1+0.08t)$

Nanako pays \$3000 per year, and thus, \$3000t in t years.

So, $3000t = 16000(1+0.08t) \Rightarrow 1.72t = 16 \Rightarrow t = 9.30$ years

Exercise 3.3

1. **(i)** $3, 3^{a+1}, 3^{2a+1}, 3^{3a+1}, \ldots\ldots$ The sequence is geometric.

(ii) $r = \dfrac{u_n}{u_{n-1}} = \dfrac{3^{a+1}}{3} = \cdots = \dfrac{3^{3a+1}}{3^{2a+1}} = 3^a$

(iii) $u_{10} = u_1 r^9 = 3 \cdot \left(3^a\right)^9 = 3^{9a+1}$

2. **(i)** $0, 3, 6, 9\ldots$. The sequence is arithmetic.

(ii) $d = a_n - a_{n-1} = (3n-3) - \left[3(n-1) - 3\right] = 3$

(iii) $a_{10} = a_1 + 9d = 0 + 9 \cdot 3 = 27$

3. **(i)** $8, 16, 32, 64, \ldots$. The sequence is geometric.

(ii) $r = \dfrac{b_n}{b_{n-1}} = \dfrac{16}{8} = \cdots = 2$

(iii) $b_{10} = b_1 r^9 = 8 \cdot 2^9 = 4096.$

4. **(i)** $-1, -4, -10, -22\ldots, c_{10} = -1534$

The sequence is neither arithmetic nor geometric.

5. **(i)** $4, 12, 36, 108, \ldots$ The sequence is geometric.

(ii) $r = \dfrac{u_n}{u_{n-1}} = \dfrac{12}{4} = \cdots = 3$

(iii) $u_{10} = u_1 r^9 = 4 \cdot 3^9 = 78732$

6. **(i)** $2, 5, 12.5, 31.25, 78.125\ldots$. The sequence is geometric.

(ii) $r = \dfrac{u_n}{u_{n-1}} = \dfrac{5}{2} = \dfrac{12.5}{5} = \cdots = 2.5$

(iii) $u_{10} = u_1 r^9 = 2 \cdot 2.5^9 \approx 7629.39$

7. **(i)** $2, -5, 12.5, -31.25, 78.125\ldots.$ The sequence is geometric.

(ii) $r = \dfrac{-5}{2} = \dfrac{12.5}{-5} = \cdots = -2$

(iii) $u_{10} = u_1 r^9 = 2 \cdot (-2.5)^9 \approx -7629.39$

8. **(i)** $2, 2.75, 3.5, 4.25, 5\ldots.$ The sequence is arithmetic.

(ii) $d = 2.75 - 2 = 3.5 - 2.75 = 4.25 - 3.5 = 5 - 4.25 = 0.75$

(iii) $u_{10} = u_1 + 9d = 2 + 9 \cdot 0.75 = 8.75$

9. **(i)** $18, -12, 8, -\frac{16}{3}, \frac{32}{9}, \ldots$ The sequence is geometric.

(ii) $r = \frac{-12}{18} = \frac{8}{-12} = \cdots = -\frac{2}{3}$

(iii) $u_{10} = u_1 r^9 = 18 \cdot \left(-\frac{2}{3}\right)^9 = -\frac{1024}{2187} \approx -0.468$

10. **(i)** $52, 55, 58, 61, \cdots$. The sequence is arithmetic.

(ii) $d = 55 - 52 = \cdots = 61 - 58 = 3$

(iii) $u_{10} = u_1 + 9d = 52 + 9 \cdot 3 = 79$

11. **(i)** $-1, 3, -9, 27, -81, \cdots$. The sequence is geometric.

(ii) $r = \frac{3}{-1} = \frac{-9}{3} = \cdots = -3$

(iii) $u_{10} = u_1 r^9 = (-1) \cdot (-3)^9 = 19683$

12. **(i)** 12. $0.1, 0.2, 0.4, 0.8, 1.6, 3.2, \cdots$. The sequence is geometric.

(ii) $r = \frac{0.2}{0.1} = \cdots = \frac{3.2}{1.6} = 2$

(iii) $u_{10} = u_1 r^9 = 0.1 \cdot 2^9 = 51.2$

13. **(i)** $3, 6, 12, 18, 21, 27, \ldots$. The sequence is neither arithmetic nor geometric.

14. **(i)** $6, 14, 20, 28, 34, \ldots$. The sequence is neither arithmetic nor geometric.

15. **(i)** $2.4, 3.7, 5, 6.3, 7.6, \cdots$. The sequence is arithmetic.

(ii) $d = 3.7 - 2.4 = \cdots = 7.6 - 6.3 = 1.3$

(iii) $u_{10} = u_1 + 9d = 2.4 + 9 \cdot 1.3 = 14.1$

16. **(i)** Arithmetic: $d = 2 - (-3) = \cdots = 5 \Rightarrow a_8 = a_1 + 7d = -3 + 7 \cdot 5 = 32$

(ii) $a_n = -3 + (n-1) \cdot 5 = 5n - 8$

(iii) $a_1 = -3,\ a_n = a_{n-1} + 5$ for $n > 1$

17. **(i)** Arithmetic: $d = 15 - 19 = \cdots = -4 \Rightarrow a_8 = a_1 + 7d = 19 + 7 \cdot (-4) = -9$

(ii) $a_n = 19 + (n-1) \cdot (-4) = 23 - 4n$

(iii) $a_1 = 19,\ a_n = a_{n-1} - 4$ for $n > 1$

18. **(i)** Arithmetic: $d = 3 - (-8) = \cdots = 11 \Rightarrow a_8 = a_1 + 7d = -8 + 7 \cdot 11 = 69$

(ii) $a_n = -8 + (n-1) \cdot 11 = 11n - 19$

(iii) $a_1 = -8,\ a_n = a_{n-1} + 11$ for $n > 1$

19. **(i)** Arithmetic: $d = 9.95 - 10.05 = \cdots = -0.1 \Rightarrow a_8 = a_1 + 7d = 10.05 + 7 \cdot (-0.1) = 9.35$

(ii) $a_n = 10.05 + (n-1) \cdot (-0.1) = 10.15 - 0.1n$

(iii) $a_1 = 10.05,\ a_n = a_{n-1} - 0.1$ for $n > 1$

20. **(i)** Arithmetic: $d = 99 - 100 = \cdots = -1 \Rightarrow a_8 = a_1 + 7d = 100 + 7 \cdot (-1) = 93$

(ii) $a_n = 100 + (n-1) \cdot (-1) = 101 - n$

(iii) $a_1 = 100,\ a_n = a_{n-1} - 1$ for $n > 1$

21. **(i)** Arithmetic: $d = \frac{1}{2} - 2 = \cdots = -\frac{3}{2} \Rightarrow a_8 = a_1 + 7d = 2 + 7 \cdot \left(-\frac{3}{2}\right) = -\frac{17}{2}$

(ii) $a_n = 2 + (n-1) \cdot \left(-\frac{3}{2}\right) = \frac{7-3n}{2}$

(iii) $a_1 = 2,\ a_n = a_{n-1} - \frac{3}{2}$ for $n > 1$

22. **(i)** Geometric: $r = \frac{6}{3} = \cdots = 2 \Rightarrow a_8 = a_1 \cdot r^7 = 3 \cdot 2^7 = 384$

(ii) $a_n = 3 \cdot 2^{n-1}$

(iii) $a_1 = 3,\ a_n = 2a_{n-1}$ for $n > 1$

23. **(i)** Geometric: $r = \frac{12}{4} = \cdots = 3 \Rightarrow a_8 = a_1 \cdot r^7 = 4 \cdot 3^7 = 8748$

(ii) $a_n = 4 \cdot 3^{n-1}$

(iii) $a_1 = 4,\ a_n = 3a_{n-1}$ for $n > 1$

24. **(i)** Geometric: $r = \frac{-5}{5} = \frac{5}{-5} = -1 \Rightarrow a_8 = a_1 \cdot r^7 = 5 \cdot (-1)^7 = -5$

(ii) $a_n = 5 \cdot (-1)^{n-1}$

(iii) $a_1 = 5,\ a_n = -a_{n-1}$ for $n > 1$

25. **(i)** Geometric: $r = \frac{-6}{3} = \frac{12}{-6} = \cdots = -2 \Rightarrow a_8 = a_1 \cdot r^7 = 3 \cdot (-2)^7 = -384$

(ii) $a_n = 3 \cdot (-2)^{n-1}$

(iii) $a_1 = 3,\ a_n = -2a_{n-1}$ for $n > 1$

26. The sequence is neither arithmetic nor geometric.

27. **(i)** Geometric: $r = \frac{3}{-2} = \cdots = -\frac{3}{2} \Rightarrow a_8 = a_1 \cdot r^7 = (-2) \cdot \left(-\frac{3}{2}\right)^7 = \frac{2187}{64}$

(ii) $a_n = -2 \cdot \left(-\frac{3}{2}\right)^{n-1} = \frac{3^{n-1}}{(-2)^{n-2}}$

(iii) $a_1 = -2,\ a_n = \left(-\frac{3}{2}\right) a_{n-1}$ for $n > 1$

28. **(i)** Geometric: $r = \frac{25}{35} = \cdots = \frac{5}{7} \Rightarrow a_8 = a_1 \cdot r^7 = 35 \cdot \left(\frac{5}{7}\right)^7 \approx 3.32$

(ii) $a_n = 35 \cdot \left(\frac{5}{7}\right)^{n-1} = \frac{5^n}{7^{n-2}}$

(iii) $a_1 = 35,\ a_n = \frac{5}{7} a_{n-1}$ for $n > 1$

29. **(i)** Geometric: $r = \frac{-3}{-6} = \cdots = \frac{1}{2} \Rightarrow a_8 = a_1 \cdot r^7 = (-6) \cdot \left(\frac{1}{2}\right)^7 = -\frac{3}{64}$

(ii) $a_n = -6 \cdot \left(\frac{1}{2}\right)^{n-1} = -\frac{3}{2^{n-2}}$

(iii) $a_1 = -6,\ a_n = \frac{1}{2} a_{n-1}$ for $n > 1$

30. **(i)** Geometric: $r = \frac{19}{9.5} = \cdots = 2 \Rightarrow a_8 = a_1 \cdot r^7 = 9.5 \cdot 2^7 = 1216$

(ii) $a_n = 9.5 \cdot (2)^{n-1}$

(iii) $a_1 = 9.5,\ a_n = 2a_{n-1}$ for $n > 1$

31. **(i)** Geometric: $r = \frac{95}{100} = \cdots = 0.95 \Rightarrow a_8 = a_1 \cdot r^7 = 100 \cdot 0.95^7 \approx 69.83$

(ii) $a_n = 100 \cdot (0.95)^{n-1}$

(iii) $a_1 = 100,\ a_n = 0.95a_{n-1}$ for $n > 1$

32. **(i)** Geometric: $r = \frac{\frac{3}{4}}{2} = \cdots = \frac{3}{8} \Rightarrow a_8 = a_1 \cdot r^7 = 2 \cdot \left(\frac{3}{8}\right)^7$

(ii) $a_n = 2 \cdot \left(\frac{3}{8}\right)^{n-1}$

(iii) $a_1 = 2,\ a_n = \frac{3}{8}a_{n-1}$ for $n > 1$

33. $u_8 = 3 \times 5^7 = 234375$

34. It is best if we go backwards, the ratio will be 3, and the first (third backwards) will be $7 \times 3^2 = 63$.

35. Using the nth term, we have

$$\left.\begin{aligned} 8 &= u_1 r^2 \\ 27 &= u_1 r^5 \end{aligned}\right\} \frac{27}{8} = r^3 \Rightarrow r = \frac{3}{2} \Rightarrow u_1 = \frac{32}{9},\ \text{thus, } u_{10} = \frac{32}{9} \times \left(\frac{3}{2}\right)^9 = \frac{2187}{16}$$

Alternatively, 27 is the 4th term in a subsequence starting at 8, i.e.,

$27 = 8r^3 \Rightarrow r = \frac{3}{2}$.

Also, u_{10} is the fifth term of a subsequence starting at 27, i.e.,

$u_{10} = v_5 = 27\left(\frac{3}{2}\right)^4 = \frac{2187}{16}$.

36. 8.64 is the third term in a subsequence starting at 6 $\Rightarrow 8.64 = 6r^2 \Rightarrow r = \pm 1.2$.

Now, $6 = u_5 = u_1 r^4 = u_2 r^3 \Rightarrow u_2 = \frac{6}{r^3} \Rightarrow u_2 = \pm \frac{125}{36}$

37. $\frac{243}{512}$ is the 6th term of a sequence starting at 2 $\Rightarrow \frac{243}{512} = 2r^5 \Rightarrow r = \frac{3}{4}$

$\frac{19683}{131072}$ is the $(n-1)$th term of a sequence starting at 2 $\Rightarrow \frac{19683}{131072} = 2 \times \left(\frac{3}{4}\right)^{n-2}$

Now, by trial and error, using your GDC solver, or logarithms, $n = 11$.

38. Value = $12000\times(1-0.12)^{11}=\$2940.97$

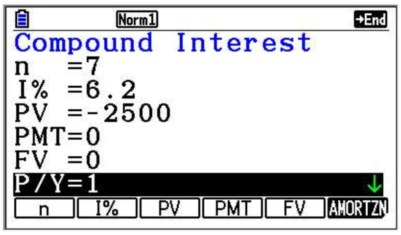
39. $2300=14900(1-r)^{8}$, using your GDC solver, or direct calculation will give

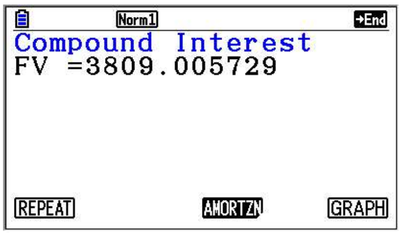
$$1-r=\sqrt[8]{\frac{2300}{14900}}\Rightarrow r\approx 0.208=20.8\%$$

40. **(a)** $483(1-0.062)^{14}\approx 197.15$ million tonnes

(b) $\frac{483-197.15}{483}\approx 0.5918\approx 59.2\%$

41. **(a)** $A(7)=2500(1+0.062)^{7}\approx \$3809.01$, or use your GDC TVM





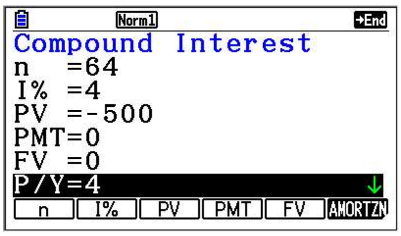
(b) $5000=2500(1+0.062)^{n}\Rightarrow n\approx 11.52\Rightarrow n\geq 12$

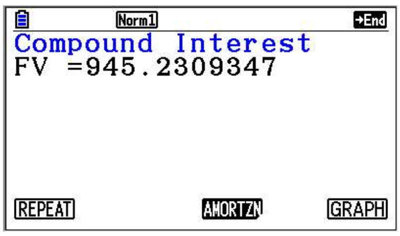
42. Compound interest formula or GDC:

$P=500,\ r=0.04,\ n=4,\ t=16$

$$A=P\left(1+\frac{r}{n}\right)^{nt}=500\left(1+\frac{0.04}{4}\right)^{4\times 16}\approx 945.23$$

Jane will have £945.23 on her 16th birthday.

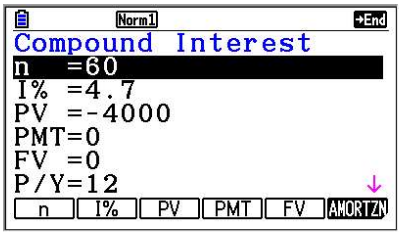


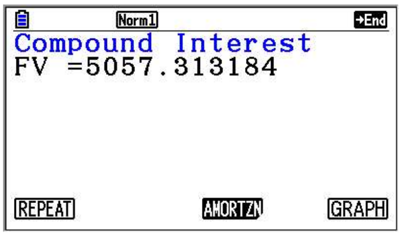


43. **(a)** Compound interest formula or GDC:

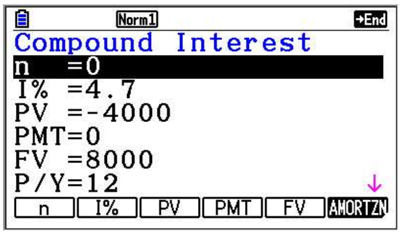
$P = 4000,\ r = 0.047,\ n = 12,\ t = 12 \times 5 = 60$

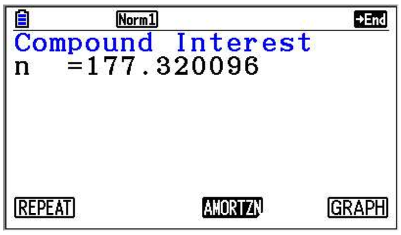
$$A = P\left(1+\frac{r}{n}\right)^{nt} = 4000\left(1+\frac{0.047}{12}\right)^{60} \approx 5057.31$$





(b) $8000 = 4000\left(1+\frac{0.047}{12}\right)^{t} \Rightarrow t = 178$ months





44. **(a)** $1.324 \times 1.012^4 \approx 1.389$ billion people in India

(b) $1.379 \times 1.005^4 \approx 1.407$ billion people in China

(c) We need to solve the following for *t*:

$1.324 \times 1.012^t \geq 1.379 \times 1.005^t \Rightarrow t \approx 5.86$ years since 2016.

45. Since there is one compounding period per year, then the real rate of return is simply the difference between the nominal and inflation rates: 4% − 1.7% = 2.3%.

46. The annual rate of return is $\left(1+\frac{0.035}{12}\right)^{12} - 1 \approx 0.0356 \approx 3.56\%$, then the real rate of return is the difference between the nominal and inflation rates: 3.56% − 2.1% = 1.46%.

Exercise 3.4

1. Firstly, we need to determine the number of terms in the series.

$a_1 = 11,\ d = 6, a_n = 365$

$a_n = a_1 + (n-1)d \Rightarrow 365 = 11 + (n-1)\cdot 6 \Rightarrow n = 60$

The sum of the sequence is $S_{60} = \frac{60}{2}(11+365) = 11280.$

2. This is an arithmetic series with first term = 9, common difference of 4 and *n*th term 85.

$\Rightarrow u_n = u_1 + (n-1)d \Rightarrow 85 = 9 + 4(n-1) \Rightarrow n = 20$

$\text{Sum} = \frac{n}{2}(u_1 + u_n) = \frac{20}{2}(9+85) = 940$

3. This is an arithmetic series with first term = 8, common difference of 6 and *n*th term 278.

$\Rightarrow u_n = u_1 + (n-1)d \Rightarrow 278 = 8 + 6(n-1) \Rightarrow n = 46$

$\text{Sum} = \frac{n}{2}(u_1 + u_n) = \frac{46}{2}(8+278) = 6578$

4. This is an arithmetic series with first term = 155, common difference of 36 and *n*th term 527.

$\Rightarrow u_n = u_1 + (n-1)d \Rightarrow 527 = 155 + 3(n-1) \Rightarrow n = 125$

$\text{Sum} = \frac{n}{2}(u_1 + u_n) = \frac{125}{2}(155+527) = 42625$

5. This is a geometric series with first term 120 and common ratio
and *n*th term $\frac{24}{78125}$.

$$\frac{24}{78125} = 120 \times \left(\frac{1}{5}\right)^{n-1} \Rightarrow \left(\frac{1}{5}\right)^{n-1} = \frac{1}{390625} \Rightarrow n = 9$$

$$= 120\left(\frac{1-\left(\frac{1}{5}\right)^9}{1-\frac{1}{5}}\right)\frac{11718744}{78125} \approx 150$$

6. The series is geometric. Firstly, we need to determine the number of terms in the series.

$$a_1 = 2,\ r = -\frac{3}{2}, a_n = -\frac{177147}{1024}$$

$$a_n = a_1 r^{n-1} \Rightarrow -\frac{177147}{1024} = 2\cdot\left(-\frac{3}{2}\right)^{n-1} \Rightarrow n = 12$$

The sum of the series is $S_{12} = \dfrac{2\left(\left(-\frac{3}{2}\right)^{12} - 1\right)}{-\frac{3}{2} - 1} = -\dfrac{105469}{1024} \approx -103.$

7. $\displaystyle\sum_{k=0}^{13}(2 - 0.3k) = 2 + 1.7 + 1.4 + \ldots + (-1.9)$

The series is arithmetic with 14 terms, $a_1 = 2$, and $d = -0.3$.

The sum of the sequence is $S_{14} = \dfrac{14}{2}\left(2 + (-1.9)\right) = 0.7 = \dfrac{7}{10}.$

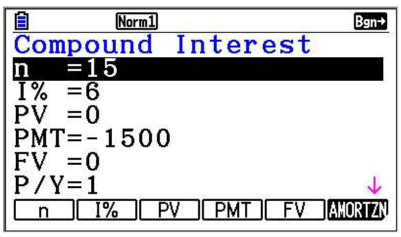
8. $2 - \dfrac{4}{5} + \dfrac{8}{25} - \dfrac{16}{125} + \ldots$ is an infinite geometric series with $a_1 = 2$ and $r = -\dfrac{2}{5}$.

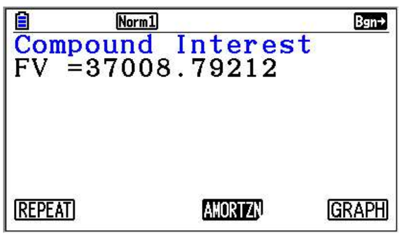
The sum is $S_\infty = \dfrac{2}{1 - \left(-\frac{2}{5}\right)} = \dfrac{10}{7}.$

9. $\dfrac{1}{3} + \dfrac{\sqrt{3}}{12} + \dfrac{1}{16} + \dfrac{\sqrt{3}}{64} + \dfrac{3}{256} + \ldots$ is an infinite geometric series with $a_1 = \dfrac{1}{3}$ and $r = \dfrac{\sqrt{3}}{4}$.

The sum is $S_\infty = \dfrac{\frac{1}{3}}{1 - \frac{\sqrt{3}}{4}} = \dfrac{4}{3(4-\sqrt{3})}\cdot\dfrac{4+\sqrt{3}}{4+\sqrt{3}} = \dfrac{16 + 4\sqrt{3}}{39}.$

10. This is an annuity due! Use your GDC's TVM solver. Make sure you set the payments up at "Begin" of the payment period. This differs among GDCs!





There will be 37008.79 in the account after 15 years.

11. $a_k = 2+3k \Rightarrow a_1 = 2+3\cdot 1 = 5,\ a_n = 2+3n$

$\Rightarrow S_n = \frac{n}{2}[5+2+3n] = \frac{n(3n+7)}{2}$

12. For the arithmetic series $17+20+23...$ we have:

$a_1 = 17,\ d = 3 \Rightarrow S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}[34+(n-1)\cdot 3] = \frac{n(3n+31)}{2}$

$S_n > 678 \quad \Rightarrow \frac{n(3n+31)}{2} > 678 \quad \Rightarrow 3n^2 + 31n - 1356 > 0$

The solutions of the quadratic equation $3n^2 + 31n - 1356 = 0$ are 16.71 and – 27.05, so the solutions of the inequality are $n > 16.71$ or $n < -27.05$. Since $n \in \mathbb{N}$, we conclude that we need to add 17 terms to exceed 678.

13. For the arithmetic series $-18-11-4...$ we have:

$a_1 = -18,\ d = 7 \Rightarrow S_n = \frac{n}{2}[2a_1 + (n-1)d] = \frac{n}{2}[-36+(n-1)\cdot 7] = \frac{n(7n-43)}{2}$

$S_n > 2335 \quad \Rightarrow \frac{n(7n-43)}{2} > 2335 \quad \Rightarrow 7n^2 - 43n - 4670 > 0$

The solutions of the quadratic equation $7n^2 - 43n - 4670 = 0$ are 29.08 and –22.94, so the solutions of the inequality are $n > 29.08$ or $n < -22.94$. Since $n \in \mathbb{N}$, we conclude that we need to add 30 terms to exceed 2335.

14. **(a)** For the arithmetic sequence $3, 7, 11, ..., 999$ we have:

$a_1 = 3,\ d = 4, a_n = 999$

$a_n = a_1 + (n-1)d \quad \Rightarrow 999 = 3 + (n-1)\cdot 4 \quad \Rightarrow n = 250$

$S_{250} = \frac{250}{2}(3+999) = 125250$

(b) The removed terms, $11, 23, 35, ..., 995,$ form an arithmetic sequence with 83 terms

and $b_1 = 11$ and $d = 12 \Rightarrow S_{83} = \frac{83}{2}[2\cdot 11 + 82\cdot 12] = 41749$

The sum of the remaining terms is then $125250 - 41749 = 83501$.

15. We have the following system of simultaneous equations that can be solved by any method of your choice:

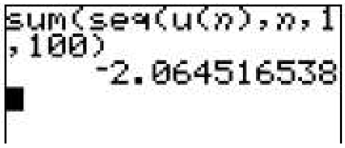
$$\begin{cases} a+(a+d)+(a+2d)+...+(a+9d) = 235 \\ (a+10d)+(a+11d)+...+(a+19d) = 735 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{10}{2}[a+(a+9d)] = 235 \\ \frac{10}{2}[(a+10d)+(a+19d)] = 735 \end{cases} \Rightarrow \begin{cases} 2a+9d = 47 \\ 2a+29d = 147 \end{cases} \Rightarrow d = 5,\ a = 1$$

16. **(a)** For $\sum_{k=1}^{20}(k^2+1)$, using a GDC in Sequential mode:

(b) For $\sum_{i=3}^{17}\frac{1}{i^2+3}$:

(c) For $\sum_{n=1}^{100}(-1)^n\frac{3}{n}$:



17. The heights that the ball reaches after each bounce form an infinite geometric sequence: $16\cdot 0.81, 16\cdot 0.81^2, \ldots.$

(a) After the 10th bounce: $16\cdot 0.81^{10}\approx 1.945\,\text{m}$

(b) $16+2\cdot\left(16\cdot 0.81+16\cdot 0.81^2+16\cdot 0.81^3+\ldots\right)=16+2\cdot\frac{12.96}{1-0.81}\approx 152.42\,\text{m}$

18. **(a)** The first shaded area is $4\cdot 2-2\cdot 1=6$.

The second shaded area is $1\cdot\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{4}=\frac{3}{8}$.

The third shaded area is $\frac{1}{4}\cdot\frac{1}{8}-\frac{1}{8}\cdot\frac{1}{16}=\frac{3}{128}$.

Total shaded area is $6+\frac{3}{8}+\frac{3}{128}=\frac{819}{128}$.

(b) If the process is repeated indefinitely, the total shaded area forms an infinite geometric sequence with $a_1=6,\ r=\frac{1}{16}$:

$$S_\infty=\frac{6}{1-\frac{1}{16}}=\frac{32}{5}$$

19. **(a)** The shaded area in the first square is made up of two right triangles of 8 cm on each side. When the two triangles are joined at their hypotenuse, they make a square whose side is 8. Thus, the shaded area is $8^2 = 64$.

In each successive square, each of the shaded triangles is half the previous ones, thus the area of the new shaded area is one half of the shaded area of the previous square, so in the second square the shaded area is $\frac{1}{2} \cdot 64 = 32$, in the third 16, etc.

Total shaded area forms a geometric series with $a_1 = 64,\ r = \frac{1}{2}$.

$$S_{10} = 64 \cdot \frac{1-\left(\frac{1}{2}\right)^{10}}{1-\frac{1}{2}} = \frac{1023}{8} = 127.875$$

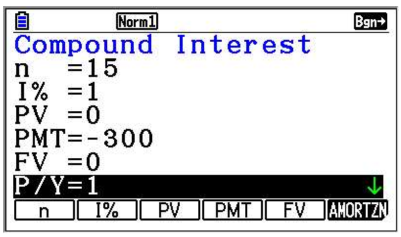
(b) $S_\infty = \dfrac{64}{1-\frac{1}{2}} = 128$

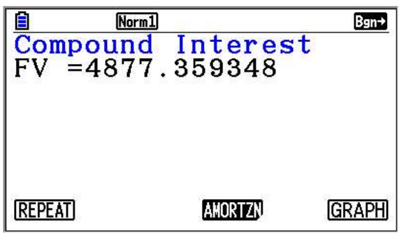
Exercise 3.5

In the following solutions, the use of GDCs' TVM solver is very important. You are not expected to know/use the formulas involved. If you have not done so yet, go ahead and learn how the financial solver in your GDC works. Answers here may differ slightly from end of book answers due to rounding.

1. **(a)** $A = 300 \times 1.01\left(\dfrac{1.01^{15}-1}{1.01-1}\right) \approx 4877.36$

(b) Remember to set up TVM solver at "Begin".





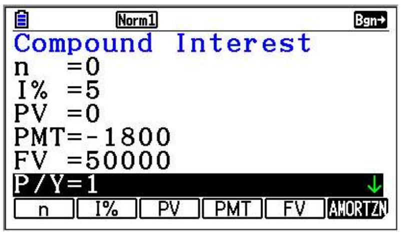
2. **(a)** \$60 is invested at the beginning of every year for 30 years at 1.5% interest, compounded annually. Solving for the future value gives \$2286.11.

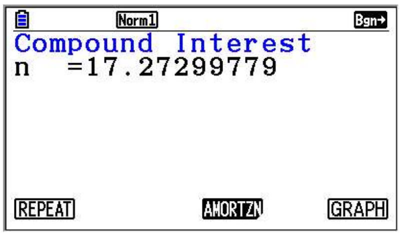
(b) A regular payment is invested at the beginning of every year for 15 years at 3% interest, compounded annually. The future value is \$1000. Solving for the annual payment gives \$52.20.

(c) \$100 is invested at the beginning of every year in an account earning 1.2% interest, compounded annually. The future value is \$2500. Solving for the number of years gives 21.76.

(d) \$50 is invested at the beginning of every month for 30 months at 4% interest, compounded semi-annually. Solving for the future value gives \$1579.38.

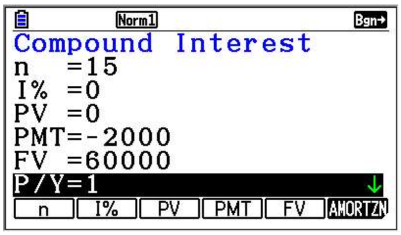
3. This is an annuity due (beginning of period). We are given the future value and we need to calculate the time. TVM solver will give us the result.

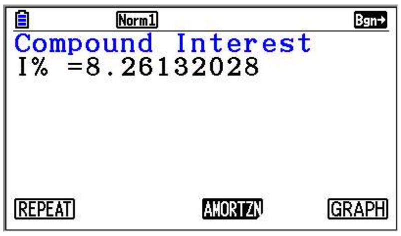




It will take 17.3 years for the annuity to accumulate to \$50 000.

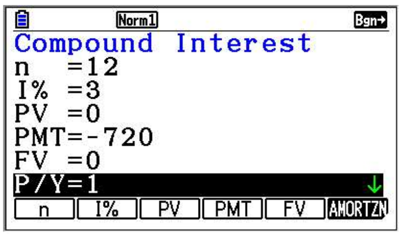
4. This is an annuity due with all the data except for interest. Here is a TVM output

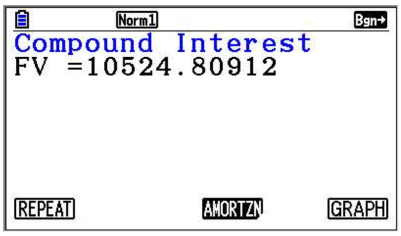




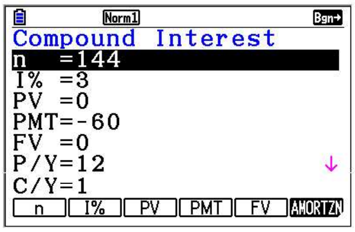
Interest rate is 8.26%.

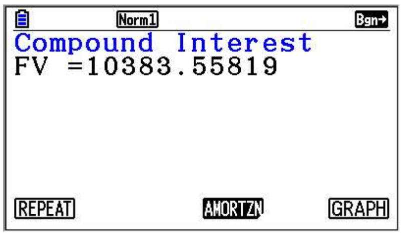
5. **(a)** The future value of this investment is \$10524.81. TVM output below





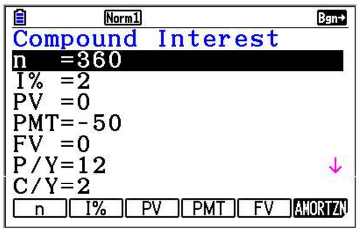
(b) The annual amount paid but the future value of this investment is \$10383.56.

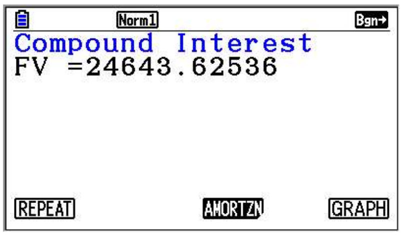




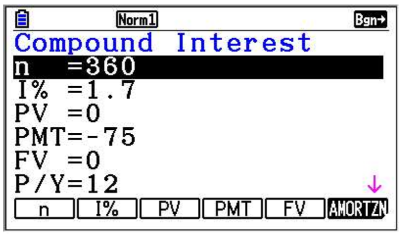
Notice that in both cases the annual investment in **(b)** is the same as in **(a)** because $60 \times 12 = 720$, however, the future value is greater in part **(a)** because the investment earns interest for longer.

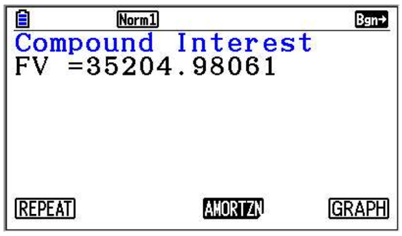
6. **(a)** \$50 are paid for 360 months. The investment will be \$24643.63 at the end of 30 years.





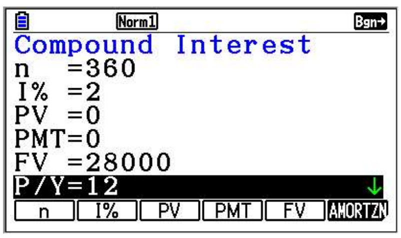
(b) The value of the investment will be \$35204.98

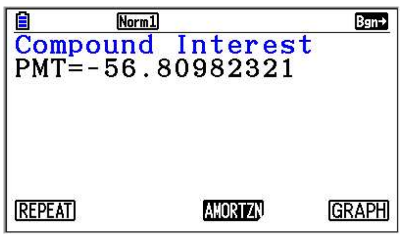




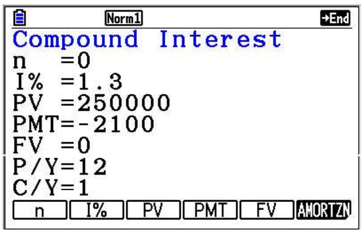
Even though the interest rate is lower than in **(a)**, but the final amount is larger.

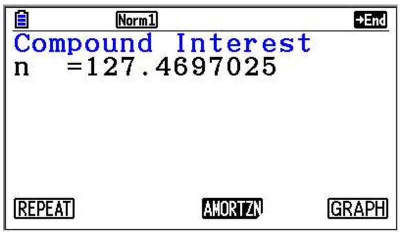
(c) In this part, we have all data except for the monthly payment.





7. **(a)** This is an ordinary annuity as they are paying at the end of each month not at the beginning.





It will take them 10 years and 7 or 8 months to pay back the loan. This depends on their ability to pay the last instalment. If they can pay more than \$2100, then they can finish with 10 years and 7 months.

(b) The approximate amount paid back is $2100 \times 127.5 \approx \$267\,750$

(c) Doing the same calculations as above with a payment of \$2300 will take the family 115.66 years, i.e., 9 years 7.7 months to pay the loan back.

They will be paying $115.66 \times 2300 = \$266018$ back, which is less than earlier. If they can afford the extra payment, it may be a better arrangement.

Chapter 3 practice questions

1. **(a)** Denote the terms of the geometric sequence by g_1, g_2 and g_3. Then

$$\frac{g_2}{g_1} = \frac{g_3}{g_2} \Rightarrow \frac{a+2d}{a+6d} = \frac{a}{a+2d} \Rightarrow 2d(2d-a) = 0$$

and since $d \neq 0 \Rightarrow d = \frac{a}{2}$.

(b) $a + 6d = 3 \Rightarrow a + 6 \times \frac{a}{2} = 3 \Rightarrow a = \frac{3}{4}, d = \frac{3}{8}$

For the arithmetic sequence

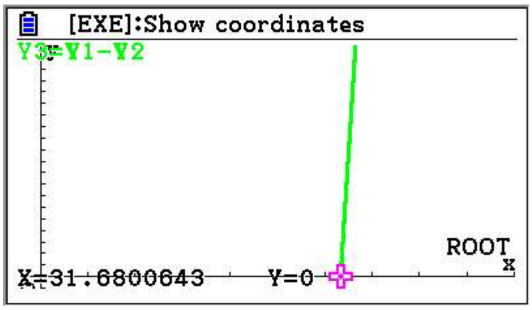
$$S_n = \frac{n}{2}\left(2a + (n-1)d\right) = \frac{n}{2}\left(2 \times \frac{3}{4} + \frac{3}{8}(n-1)\right) = \frac{n}{2}\left(\frac{9}{8} + \frac{3}{8}n\right)$$

For the geometric sequence

$$g_1 = 3, r = \frac{a+2d}{a+6d} = \frac{2a}{4a} = \frac{1}{2} \Rightarrow S_n = 3 \cdot \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} = 6\left(1-\left(\frac{1}{2}\right)^n\right)$$

Now, we either use the equation solver on a GDC or graph two functions

$f(x) = \frac{x}{2}\left(\frac{9}{8} + \frac{3}{8}x\right)$ and $g(x) = 6\left(1-\left(\frac{1}{2}\right)^x\right) + 200$ and look for the point of intersection as shown below. The first n where the sum of the arithmetic sequence exceeds that of the geometric sequence by 200 happens at $x = 31.68$, implying that $n = 32$.



2. **(a)** One of several approaches:

This is an arithmetic sequence with first term of 14 and last term 196. This allows to find the number of terms:

$$196 = 14 + 7(n-1) \Rightarrow n = 27 \Rightarrow S_{27} = \frac{27}{2}(14+196) = 2835$$

(b) $\sum_{n=1}^{27} 14 + 7(n-1) = \sum_{n=1}^{27} 7 + 7n$

(c) $S_n = \frac{n}{2}(2000 - 6(n-1)) = n(1003 - 3n)$

Using equation solver or graph we can see that this sum is positive for $n = 334$, and it moves to the negative side at $n = 335$. So, $n = 335$.

3. **(a)** $a + ar = 10 \Rightarrow a + ar + ar^2 + ar^3 = a + ar + r^2(a + ar) = 10 + 10r^2$

Thus, $10 + 10r^2 = 30 \Rightarrow r^2 = 2$.

(b) **(i)** $r = \sqrt{2} \Rightarrow a + a\sqrt{2} = 10 \Rightarrow a = \frac{10}{1+\sqrt{2}}$

(ii) $S_{10} = \frac{10}{1+\sqrt{2}} \cdot \frac{1-(\sqrt{2})^{10}}{1-\sqrt{2}} = 310$

4. **(a)** The two conditions given will induce a system of 2 equations

$$\left.\begin{matrix} 34 = u_1 + 3d \\ 76 = u_1 + 9d \end{matrix}\right\} \Rightarrow d = 7 \Rightarrow u_1 = 3$$

(b) $S_n = \frac{n}{2}(26 + 7(n-1)) > 5000 \Rightarrow n \geq 40$.

5.

$$76 = \frac{a}{1-r};\ 36 = \frac{a}{1-r^3}$$

$$\Rightarrow \frac{76}{36} = \frac{a}{1-r} \cdot \frac{1-r^3}{a} = 1 + r + r^2$$

$$\Rightarrow 9r^2 + 9r - 10 = 0 \Rightarrow r = \frac{2}{3}$$

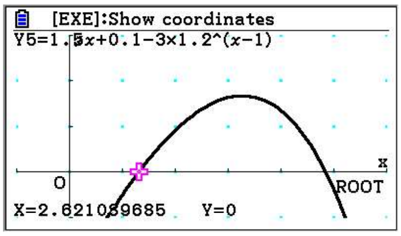
Note that there is another answer to the equation, namely, $r = -\frac{5}{3}$, but it is rejected since the condition for a sum to infinity is violated, i.e., $|r| = \frac{5}{3} > 1$.

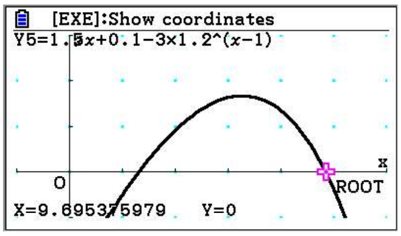
6. **(a)** $u_n = 1.6 + 1.5(n-1) = 1.5n + 0.1; v_n = 3\times 1.2^{n-1}$

$u_n - v_n = 1.5n + 0.1 - 3\times 1.2^{n-1}.$

A GDC is needed for the next 2 parts.

(b) $u_n > v_n \Rightarrow u_n - v_n = 1.5n + 0.1 - 3\times 1.2^{n-1} > 0$





Since n must be a natural number, then $3 \le n \le 9$.

(c) The graph gives us a maximum value of 1.67 at $x = 6.53$. However, n must be a natural number and so, the maximum must be at a natural number. Thus, we evaluate the expression at $n = 6$, and at $n = 7$. The maximum is at (7, 1.642).

7. Let the smallest piece be a. Since the last piece is the 10^{th} term of the sequence, then

$8a = ar^9 \Rightarrow r = \sqrt[9]{8}$. The sum of all pieces (terms) must be 1 m, thus,

$$a\frac{1-\left(\sqrt[9]{8}\right)^{10}}{1-\sqrt[9]{8}} = 1 \Rightarrow a = \frac{1-\sqrt[9]{8}}{1-\left(\sqrt[9]{8}\right)^{10}} \approx 0.02863$$

8. **(a)** The original height is the first term in a geometric sequence, 'after the fourth' bounce is the fifth term. Thus $h = 4\times 0.95^4 = 3.258$

(b) We must find n such that $4\times 0.95^n \le 1 \Rightarrow 0.95^n \le \frac{1}{4}$

By using logarithms or GDC, $n = 28$

For example, $n \le \frac{-\ln 4}{\ln 0.95} = 27.02$

(c) Every vertical distance, except the first is travelled twice – once up and once down. So, the total distance is twice the sum of the geometric sequence minus one time the original height:

$$h = 2\left(\frac{4}{1-0.95}\right) - 4 = 156\text{ m}$$

9. **(a)** $\frac{81}{2} = \frac{27}{1-r} \Rightarrow r = 1 - \frac{54}{81} = \frac{1}{3}$

(b) $v_2 = 27 \times \frac{1}{3} = 9; v_4 = 27 \times \frac{1}{27} = 1$

In the arithmetic sequence,
$v_4 = 9 = v_1 + 3d = v_1 + d + 2d = v_2 + 2d = 1 + 2d \Rightarrow d = -4$, and hence the first term is 13.

$$S_N = \frac{N}{2}(26 - 4(N-1)) = N(15 - 2N)$$

Due to the symmetry of the underlying graph, a downward concave parabola, which has an x-intercept of 7.5 after which it becomes negative. $N = 7$.

10. **(a)** $u_n = S_n - S_{n-1} = \frac{7^n - a^n}{7^n} - \frac{7^{n-1} - a^{n-1}}{7^{n-1}} = \frac{7^n - a^n - 7 \cdot 7^{n-1} + 7a^{n-1}}{7^n}$

The nth term of a geometric sequence is of the form $u_n = u_1 r^{n-1}$, thus, we can simplify the last result to reduce it to this form

$$u_n = \frac{7^n - a^n - 7 \cdot 7^{n-1} + 7a^{n-1}}{7^n} = \frac{(7-a)a^{n-1}}{7 \cdot 7^{n-1}} = \frac{(7-a)}{7}\left(\frac{a}{7}\right)^{n-1}$$

(b) It is apparent from the last expression that the first term is $u_1 = \frac{(7-a)}{7}$ and the common ratio is $\frac{a}{7}$.

(c) **(i)** The sum to infinity exists as long as $|r| < 1 \Rightarrow \frac{a}{7} < 1 \Rightarrow 0 < a < 7$

(ii) $S = \frac{u_1}{1-r} = \frac{\frac{(7-a)}{7}}{1 - \frac{a}{7}} = 1$

11. $u_1 = 1.5, u_n = 7.5$ and the sum of all terms is 81 m.

$S_n = \frac{n}{2}(u_1 + u_n) = \frac{n}{2}(1.5 + 7.5) = 81 \Rightarrow n = 18.$

$u_{18} = 7.5 = 1.5 + (18-1)d \Rightarrow d = \frac{6}{17}.$

12. **(a)** **(i)** The first month, P earns interest at the rate of I% which should be added to the amount owed to the bank. By that time, we pay \$R back, which should be subtracted from what we owe. Thus

$$S_1 = P + P\cdot\frac{I}{100} - R = P\left(1+\frac{I}{100}\right) - R$$

During the second month S_1 earns interest at the rate of I%, which should be added to the outstanding amount becoming $S_1 + S_1\cdot\frac{I}{100}$.

At the end of this month we pay \$$R$ into the account, and so,

$$S_2 = S_1 + S_1\cdot\frac{I}{100} - R = S_1\left(1+\frac{I}{100}\right) - R = \left(P\left(1+\frac{I}{100}\right) - R\right)\left(1+\frac{I}{100}\right) - R$$

$$= P\left(1+\frac{I}{100}\right)^2 - R\left(1+\left(1+\frac{I}{100}\right)\right)$$

(ii) During the following months, the same procedure is followed, i.e.,

$$S_3 = S_2\left(1+\frac{I}{100}\right) - R = \left(P\left(1+\frac{I}{100}\right)^2 - R\left(1+\left(1+\frac{I}{100}\right)\right)\right)\left(1+\frac{I}{100}\right) - R$$

$$= P\left(1+\frac{I}{100}\right)^3 - R\left(1+\left(1+\frac{I}{100}\right)+\left(1+\frac{I}{100}\right)^2\right)$$

Thus, we can generalise to the nth month

$$S_n = P\left(1+\frac{I}{100}\right)^n - R\left(1+\left(1+\frac{I}{100}\right)+\cdots+\left(1+\frac{I}{100}\right)^{n-1}\right)$$

Now, $1+\left(1+\frac{I}{100}\right)+\cdots+\left(1+\frac{I}{100}\right)^{n-1}$ is a geometric series with first term of 1 and common ratio $r=\left(1+\frac{I}{100}\right)$, and its partial sum is

$$u_1\frac{r^n-1}{r-1} = \frac{\left(1+\frac{I}{100}\right)^n - 1}{\left(1+\frac{I}{100}\right)-1} = \frac{\left(1+\frac{I}{100}\right)^n - 1}{\frac{I}{100}} = \frac{100}{I}\left(\left(1+\frac{I}{100}\right)^n - 1\right)$$

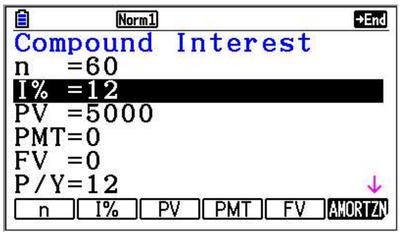
And therefore

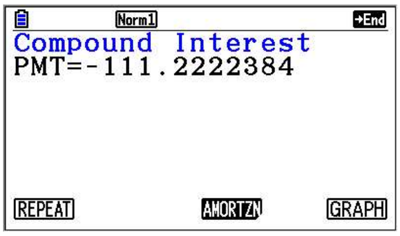
$$S_n = P\left(1+\frac{I}{100}\right)^n - R\left(1+\left(1+\frac{I}{100}\right)+\cdots+\left(1+\frac{I}{100}\right)^{n-1}\right)$$

$$= P\left(1+\frac{I}{100}\right)^n - \frac{100R}{I}\left(\left(1+\frac{I}{100}\right)^n - 1\right)$$

(b) **(i)** We will use a TVM solver for this part. Remember to set the solver for 'End' because the payments happen at the end of each month.

Notice here that for this TVM, we need to enter the nominal annual interest rate. The monthly payment is \$111.22



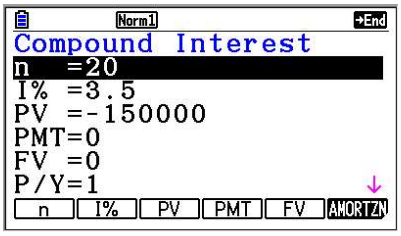


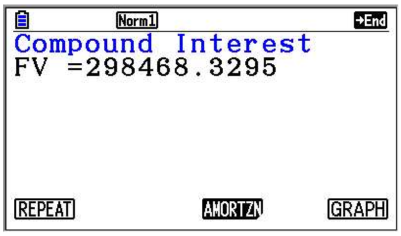
(ii) We need to know the outstanding amount at the end of 20 months. Thus we use the formula established in (a, ii)

$$S_{20} = P\left(1+\frac{1}{100}\right)^{20} - \frac{100R}{1}\left(\left(1+\frac{1}{100}\right)^{20} - 1\right) \approx \$3652$$

13. **(a)** This is a compound interest case with annual rate of 3.5% for 20 years.

Phil owes the bank \$298.468 (banks would prefer to round up instead - \$298.469).





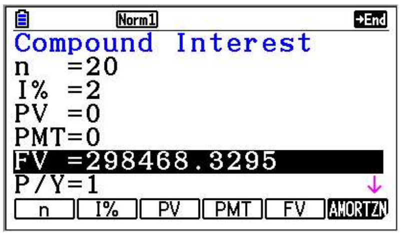
(b) The amount deposited at the end of the first year will earn interest of 2% for 10 years, and thus, will be worth $P \times 1.02^{19}$, the amount deposited the second year will earn interest for 18 years, and will be worth, $P \times 1.02^{18}$, and so on till the last deposit which is take out immediately. So, the total value is

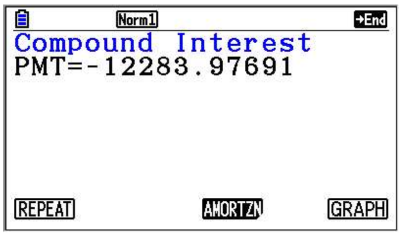
$$P + P\times 1.02 + \cdots + P\times 1.02^{18} + P\times 1.02^{19} = P\left(1 + 1.02 + \cdots + 1.02^{18} + 1.02^{19}\right)$$

The amount in brackets is a geometric series with 1 as first term and 1.02 as common ratio and 20 terms.

Future value at this bank is $P \cdot 1 \cdot \dfrac{1.02^{20} - 1}{1.02 - 1} = 50P\left(1.02^{20} - 1\right)$

(c) Using TVM, we know the future value needed from **(a)**, we need the periodic payment. Phil needs to deposit \$12284 annually in order to collect \$298,468 after 20 years.





Alternatively, you can solve the equation $50P\left(1.02^{20}-1\right)=298468$ for P.

(d) **(i)** David's investment of $\$Q$ will grow to $Q\times 1.028^n$

Withdrawing \$5000 each year will mean that this amount will be 'forgoing' potential interest relative to the number of years away from the last withdrawal:

Last withdrawal will not forgo any interest as it is withdrawn at the end. The withdrawal before last would forgo interest for one year, and so on

The value of withdrawals is therefore

$5000+5000\times 1.028+\cdots+5000\times 1.028^{n-1}$

The deposit will be enough, if the future value of $\$Q$ deposited is at least equal to the total withdrawn

$$Q\times 1.028^n=5000+5000\times 1.028+\cdots+5000\times 1.028^{n-1}$$

$$\Rightarrow Q=\frac{5000}{1.028^n}+\frac{5000}{1.028^{n-1}}+\cdots+\frac{5000}{1.028}$$

(ii) The amount needed can be written as

$$Q=5000\left(\frac{1}{1.028}+\cdots+\frac{1}{1.028^{n-1}}+\frac{1}{1.028^n}\right),$$

The amount inside the brackets is an infinite geometric series with fist term $\frac{1}{1.028}$ and common ratio $\frac{1}{1.028}$. Thus the indefinite sum is the sum to infinity of this sequence

$S=\dfrac{\frac{1}{1.028}}{1-\frac{1}{1.028}}=\dfrac{1}{0.028}\approx 35.7$, thus, David needs to invest at least

$5000\times 35.7\approx \$178572.$

Exercise 4.1

1. **(a)** $VW = \sqrt{(4+2)^2 + (-3-5)^2} = 10$

(b) $KL = \sqrt{(3+1)^2 + (-1-7)^2} = \sqrt{80} = 4\sqrt{5}$

(c) $TG = \sqrt{(-6-2)^2 + (7+8)^2} = \sqrt{289} = 17$

2. **(a)** $\left(\frac{4-2}{2}, \frac{-3+5}{2}\right) = (1,1)$

(b) $\left(\frac{3+1}{2}, \frac{-1+7}{2}\right) = (2,3)$

(c) $\left(\frac{-6+2}{2}, \frac{7-8}{2}\right) = \left(-2, \frac{-1}{2}\right)$

3. **(a)** gradient $= 4$; perpendicular has gradient $\frac{-1}{4}$ (negative reciprocal)

(b) gradient $= -2$; perpendicular has gradient $\frac{1}{2}$ (negative reciprocal)

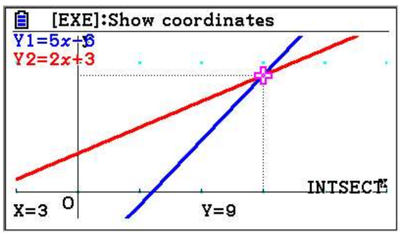
(c) gradient $= \frac{-2}{3}$; perpendicular has gradient $\frac{3}{2}$ (negative reciprocal)

4. **(a)** Using GDC: $(4,11)$, or solving algebraically

$3x - 1 = x + 7 \Rightarrow 2x = 8 \Rightarrow x = 4$

$y = 3 \cdot 4 - 1 = 11$

(b) Using GDC: $(3,9)$



(c) Using GDC: $(7,-2)$

(d) Using GDC: $\left(\frac{1}{3}, 3\right)$

(e) Using GDC: $(30, 23)$

(f) Using GDC: $(2,-3)$

5. Parallel so same gradient: $\frac{1}{2}$

Substitute $(2,-4)$ into $y=\frac{1}{2}x+c$ to obtain: $-4=\frac{1}{2}(2)+c$ and solve to obtain $c=-5$

Answer: $y=\frac{1}{2}x-5$

6. Parallel so same gradient: $\frac{-2}{3}$

Substitute $(-6,-1)$ into $y=\frac{-2}{3}x+c$ to obtain: $-1=\frac{-2}{3}(-6)+c$ and solve to obtain $c=-5$

Answer: $y=\frac{-2}{3}x-5$

7. Perpendicular so use the negative reciprocal to obtain gradient: $\frac{-1}{2}$

Substitute $(4,2)$ into $y=\frac{-1}{2}x+c$ to obtain: $2=\frac{-1}{2}(4)+c$ and solve to obtain $c=4$

Answer: $y=\frac{-1}{2}x+4$

8. Perpendicular so use the negative reciprocal to obtain gradient: $\frac{3}{2}$

Substitute $(4,5)$ into $y=\frac{3}{2}x+c$ to obtain: $5=\frac{3}{2}(4)+c$ and solve to obtain $c=-1$

Answer: $y=\frac{3}{2}x-1$

9. Gradient of segment $[PQ]$ = $\frac{6-12}{7+5}=\frac{-1}{2}$

"Perpendicular" so use the negative reciprocal to obtain gradient: 2

"Bisector", so through the mid-point: $\left(\frac{-5+7}{2},\frac{12+6}{2}\right)=(1,9)$

Substitute $(1,9)$ into $y=2x+c$ to obtain: $9=2(1)+c$ and solve to obtain $c=7$

Answer: $y=2x+7$

10. Segments $[AB]$ and $[CD]$ are the parallel so they have the same gradient. Let us use $[AB]$ to obtain $\frac{10-8}{1+3}=\frac{1}{2}$

"Perpendicular" so use the negative reciprocal to obtain gradient: -2

"Bisector", so through the mid-point of either $[AB]$ or $[CD]$. Let us use $[AB]$ to obtain the mid-point $\left(\frac{-3+1}{2},\frac{8+10}{2}\right)=(-1,9)$

Substitute $(-1,9)$ into $y=-2x+c$ to obtain: $9=-2(-1)+c$ and solve to obtain $c=7$

Answer: $y=-2x+7$

11. **(a)** Gradient of $[TX]$ is $\frac{6-3}{4-1}=1$

"Perpendicular" so use the negative reciprocal to obtain gradient: -1

"Bisector", so through the mid-point $[TX]$: $\left(\frac{1+4}{2},\frac{3+6}{2}\right)=\left(\frac{5}{2},\frac{9}{2}\right)$

Substitute $\left(\frac{5}{2},\frac{9}{2}\right)$ into $\left(\frac{5}{2}\right)=-1\left(\frac{9}{2}\right)+c$ to obtain: $9=\frac{2}{5}(-1)+c$

and solve to obtain $c=7$

Equation of the perpendicular bisector of $[TX]$: $y=-x+7$

$[CG]$ has gradient $\frac{8-4}{8-6}=2$

Substitute $(6,4)$ into $y=2x+c$ to obtain: $4=2(6)+c$ and solve to obtain $c=-8$

Equation of $[CG]$: $y=2x-8$

The intersection is found by GDC: $(5,2)$

(b) Regardless of where the point X has moved, the treasure will still lie somewhere along the line (GC). A trench of indeterminate length may have to be dug, however.

Exercise 4.2

1. **(a)** $\cos A = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{c}{b} = \dfrac{AB}{AC}$

We need AB. Using Pythagoras, $AC^2 = AB^2 + BC^2$ so $AB^2 = 25^2 - 7^2 = 576$ so $AB = 24$.

Therefore, $\cos A = \dfrac{24}{25}$

(b) $\sin C = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{c}{b} = \dfrac{AB}{AC} = \dfrac{24}{25}$

(c) $\tan A = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{a}{c} = \dfrac{BC}{AB} = \dfrac{7}{24}$

(d) $\arcsin\left(\dfrac{24}{25}\right) \approx 73.73979529 \approx 74^{\circ}$

(e) $\arctan\left(\dfrac{7}{24}\right) \approx 16.26020471 \approx 16^{\circ}$

(f) $\arccos\left(\dfrac{24}{25}\right) \approx 16.26020471 \approx 16^{\circ}$

2. **(a)** $\arcsin\left(\dfrac{4}{5}\right) \approx 53.1^{\circ}$ (to 3 significant figures)

(b) $\arccos\left(\dfrac{8}{17}\right) \approx 61.9^{\circ}$ (to 3 significant figures)

(c) $\arctan(1) = 45^{o}$ (exactly)

3. (draw a diagram) With trigonometry: $\theta = \arctan\left(\dfrac{1.62}{2}\right) \approx 39.0°$

Since the distance from the tip of the shadow to the base of the streetlight is 5 m, we have height of the streetlight $\approx 5 \cdot \tan 39.0° \approx 4.05\,\text{m}$.

Without trigonometry: Since there are two similar triangles, we have $\dfrac{1.62}{2} = \dfrac{\text{height}}{5}$ so

$\text{height} = \dfrac{1.62}{2} \cdot 5 = 4.05\,\text{m}$

4. $\sin(54) = \dfrac{\text{height}}{60}$ so $\text{height} = 60 \cdot \sin(54) \approx 48.541 \approx 49\,\text{m}$ (to the nearest metre)

5. $\cos(36) = \frac{\text{height}}{2}$ so $\text{height} = 2 \cdot \cos(36) \approx 1.618\,\text{m}$ (This is Φ, the Golden Ratio)

6. **(a)** **(i)** Let h be the height of the building

$\tan(39°) = \frac{h}{50}$ so $h = 50 \cdot \tan(39°) \approx 40\,\text{m}$

(ii) Let H be the height of the top of the antenna (building + antenna)

$\tan(50°) = \frac{H}{50}$ so $H = 50 \cdot \tan(50°) \approx 60\,\text{m}$

(iii) $H - h = 50 \cdot \tan(50°) - 50 \cdot \tan(39°) \approx 19.09847797 \approx 19.1\,\text{m}$

(b) The part of the diagram containing 11° is not a right-angled triangle

7. We do not need the second piece of information (the angle of depression) to determine the distance **(d)** between the buildings. It would be required, however, to calculate their heights.

$\tan(30°) = \frac{16}{d}$ so the distance $d = \frac{16}{\tan(30°)} \approx 27.7\,\text{m}$

8. Let h be the height of the hill, so that the top of the building has a height of $(h + 30)$ from the base of the hill. And let d be the distance from the observation point to the base of the hill, directly below the building.

We then have: $\tan(55°) = \frac{h+30}{d}$ and $\tan(50°) = \frac{h}{d}$

Re-arrange to isolate d: $d = \frac{h+30}{\tan(55°)}$ and $d = \frac{h}{\tan(50°)}$

Substitute and solve: $\frac{h+30}{\tan(55°)} = \frac{h}{\tan(50°)}$

$(h+30)\tan(50°) = h\tan(55°)$

$h\tan(50°) + 30\tan(50°) = h\tan(55°)$

$30\tan(50°) = h\tan(55°) - h\tan(50°)$

$30\tan(50°) = h(\tan(55°) - \tan(50°))$

$\frac{30\tan(50°)}{\tan(55°) - \tan(50°)} = h$ so $h \approx 151\,\text{m}$

9. We can use Pythagoras' theorem to get $d = \sqrt{30^2 - 18^2} = \sqrt{576} = 24\,\text{m}$

or, the ratio of the known opposite side to the hypotenuse $\sin\theta = \frac{18}{30} = \frac{3}{5}$, making this a familiar 3-4-5 right-angled triangle. The distance along the ground is equal to $4 \times 6 = 24\,\text{m}$

10. The original distance is $d_1 = \dfrac{50}{\tan 4°}$ and the second distance is $d_2 = \dfrac{50}{\tan 12°}$; hence,

the distance travelled in 5 minutes is $d_1 - d_2 = \dfrac{50}{\tan 4°} - \dfrac{50}{\tan 12°} \approx 479.8$

Its speed would then be approximately $\dfrac{\frac{479.8}{1000}}{\frac{5}{60}} \approx 5.7576\,\text{km.h}^{-1} \approx 3.11$ knots

Exercise 4.3

1. **(a)** Find side a with the cosine rule, the angle B with the sine rule, then angle C by subtraction from 180°.

$$\text{Area} = \frac{1}{2}b(c\sin A)$$

(b) Find angle B by subtraction from 180°, then the missing sides with the sine rule.

$$\text{Area} = \frac{1}{2}b(c\sin A)$$

(c) Find angle C by subtraction from 180°, then the 2 missing sides with the sine rule.

$$\text{Area} = \frac{1}{2}\left(\frac{c^2 \sin A \sin B}{\sin C}\right)$$

(d) Find angle A with the cosine rule, angle B with the sine rule, then angle C by subtraction from 180°.

$$\text{Area} = \frac{1}{2}b(c\sin A)$$

2. **(a)** $\hat{C} = 180 - (30 + 72) = 78°$

$$\frac{a}{\sin(30°)} = \frac{10}{\sin(78°)} \text{ so } a = \frac{10\sin(30°)}{\sin(78°)} \approx 5.11\,\text{cm}$$

$$\frac{b}{\sin(72°)} = \frac{10}{\sin(78°)} \text{ so } b = \frac{10\sin(72°)}{\sin(78°)} \approx 9.72\,\text{cm}$$

(b) $\hat{C} = 180 - (36 + 72) = 72°$

$$\frac{b}{\sin(72°)} = \frac{8}{\sin(36°)} \text{ so } b = \frac{8\sin(72°)}{\sin(36°)} \approx 12.9\,\text{cm}$$

Since we have two angles of 72°, we have an isosceles triangle and $c = b \approx 12.9\,\text{cm}$

(c) $c^2 = 6^2 + 10^2 - 2 \cdot 6 \cdot 10 \cdot \cos(60°)$ so $c = \sqrt{76} \approx 8.72\,\text{cm}$

Use the cosine rule again to calculate one missing angle (either), but take care not to use the rounded value 8.72. Use the calculator's unrounded value instead.

Substituting values: $6^2 = 10^2 + 8.717797887^2 - 2 \cdot 10 \cdot 8.717797887 \cdot \cos\left(A\right)$

and solving by isolating $\cos\left(\hat{A}\right)$:

$$2 \cdot 10 \cdot 8.717797887 \cdot \cos\left(A\right) = 10^2 + 8.717797887^2 - 6^2$$

$$\cos\left(A\right) = \frac{10^2 + 8.717797887^2 - 6^2}{2 \cdot 10 \cdot 8.717797887} \approx 0.8029550685$$

or substitute directly in the re-arranged form of the cosine rule:

$$\cos\left(A\right) = \frac{b^2 + c^2 - a^2}{2bc}$$

Finally, $A \approx \arccos\left(0.8029550685\right) \approx 36.6° \approx 37°$

And $B = 180 - \left(60 + 36.58677556\right) \approx 83°$

(d) Substituting in the re-arranged form of the cosine rule:

$$\cos\left(A\right) = \frac{15^2 + 18^2 - 12^2}{2 \cdot 15 \cdot 18} = \frac{3}{4} \quad \text{so} \quad A \approx \arccos\left(\frac{3}{4}\right) \approx 41°$$

Similarly:

$$\cos\left(B\right) = \frac{15^2 + 12^2 - 18^2}{2 \cdot 15 \cdot 12} = \frac{1}{8} \quad \text{so} \quad B \approx \arccos\left(\frac{1}{8}\right) \approx 83°$$

And $C = 180 - \left(83 + 41\right) \approx 56°$

3. **(a)** $\text{Area} = \frac{1}{2}\left(\frac{10^2 \sin 30° \sin 72°}{\sin 78°}\right) \approx 24.3\ \text{cm}^2$

alternatively, $Area = \frac{1}{2}(10)(9.723036846)\sin 30° \approx 24.3\ \text{cm}^2$

(b) $\text{Area} = \frac{1}{2}(8)(12.94427191)\sin 72° \approx 49.2\ \text{cm}^2$

(c) $\text{Area} = \frac{1}{2}(15)(18)\sin 41.40962211° \approx 89.3\ \text{cm}^2$

(d) $\text{Area} = \frac{1}{2}(10)(9.723036846)\sin 30° \approx 24.3\ \text{cm}^2$

4. **(a)** The height of triangle PQR is 7 cm. Since $p = 10$ cm and $7 < p < 14$, there are two triangles possible.

(b) $\frac{\sin(R)}{14} = \frac{\sin(30°)}{10}$ so $\sin(R) = \frac{14 \cdot \sin(30°)}{10} = 0.7$

so $R = \arcsin(0.7) \approx 44.4°$ or $180° - \arcsin(0.7) = 135.6°$

If $R \approx 44.4°$:

$Q \approx 180 - (30 + 44.4) \approx 105.6$

$\frac{q}{\sin(105.6)} = \frac{10}{\sin(30°)}$ so $q \approx 19.3$ cm

If $R \approx 135.6°$:

$Q \approx 180 - (30 + 135.6) \approx 14.4$

$\frac{q}{\sin(14.4)} = \frac{10}{\sin(30°)}$ so $q \approx 4.98$ cm

Careful: the use of rounded values in the previous calculation would lead to the incorrect value 4.97 cm

5. If $R \approx 44.4°$, Area $= \frac{1}{2}(14)(10)\sin 105.6° \approx 67.4$ cm^2

$R \approx 135.6°$, Area $= \frac{1}{2}(14)(10)\sin 14.4° \approx 17.4$ cm^2

Exercise 4.4

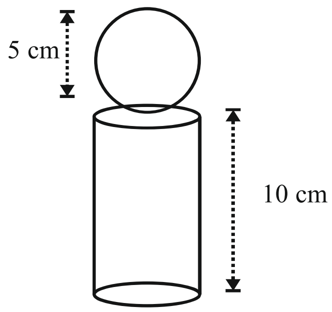
1. **(a)** $V = \frac{1}{3} \cdot 12 \cdot 20 \cdot 10 = 800$ cm^3

(b) The volume would be the same, since multiplication is commutative (switch the 10 and the 20 in the formula above).

2. Total surface of the ball = $4\pi \cdot 11^2 \approx 1520.53$

so the area of one panel = $\frac{1520.53}{32} \approx 47.5$

3. **(a)** Volume of the sphere + volume of the cylinder = $\frac{4}{3}\pi \cdot 2.5^3 + \pi \cdot 2.5^2 \cdot 10 \approx 262$ cm^3



(b) This is the volume of the whole cylinder minus the volume of the sphere.

$\pi \cdot 2.5^2 \cdot 15 - 262 \approx 32.7 \text{ cm}^3$ (or 11.1%)

4. Since the radius of the hemisphere is limited to the radius of the cylindrical blank, it can be no greater than 2.5 cm; the height of the inverted cone is therefore no greater than 7.5 cm.

total volume = volume of the hemisphere + volume of the cone

$$= \frac{1}{2} \cdot \frac{4}{3} \cdot \pi \cdot 2.5^3 + \frac{1}{3} \cdot \pi \cdot 2.5^2 \cdot 7.5 \approx 81.8 \text{ cm}^3$$

Percentage used = $\dfrac{81.8}{\pi \cdot 2.5^2 \cdot 10} \times 100 \approx 41.7\%$

(The height of this cylinder is 7.5 + 2.5 = 10)

5. Let $2x$ be the side of the cube. The radius of the sphere is therefore x cm.

$$\frac{\text{difference of volumes}}{\text{volume of the cube}} = \frac{(2x)^3 - \frac{4}{3}\pi \cdot x^3}{(2x)^3} = \frac{x^3\left(8 - \frac{4}{3}\pi\right)}{8x^3} = \frac{8 - \frac{4}{3}\pi}{8} \approx 0.476401$$

so 47.6%

6. **(a)** Flying along the edge of the grid from O to P = 10+20+20=50 metres

Directly back along the diagonal from P to O = $\sqrt{10^2 + 20^2 + 20^2} = 30$ metres

so 80 metres all together.

(b) 60 metres is 50 metres (along the edges) and 10 metres
(a third of the way back along the diagonal)

so position = $\frac{2}{3}$ of OP = $\frac{2}{3}(10, 20, 20) = \left(\frac{20}{3}, \frac{40}{3}, \frac{40}{3}\right)$

Chapter 4 practice questions

1. **(a)** $AB=\sqrt{(11+1)^2+(2+3)^2}=13$

(b) $CD=\sqrt{(9-3)^2+(-3-5)^2}=10$

(c) $EF=\sqrt{(5-9)^2+(3\sqrt{5}-\sqrt{5})^2}=6$

2. **(a)** $\left(\frac{6-4}{2},\frac{-1+5}{2}\right)=(1,2)$

(b) $\left(\frac{-7-1}{2},\frac{-2+8}{2}\right)=(-4,3)$

(c) $\left(\frac{1+8}{2},\frac{-4+0}{2}\right)=\left(\frac{9}{2},-2\right)$

3. **(a)** Equations re-arrange to $y=2x-1$ and $y=2x-2$ respectively, so parallel.

(b) Second equation re-arranges to $y=\frac{1}{2}x+\frac{3}{2}$ so intersecting because gradients are different.

(c) Second equation re-arranges to $y=2x-4$ so coincident.

4. **(a)** Using GDC, or substituting/eliminating: $\frac{3}{2}x-1=x+3$ which leads to $x=8$

then substituting again: $y=x+3=8+3=11$ to give $(8,11)$

(b) Using GDC, or
substituting/eliminating: $x-6=2x+4$ which leads to $x=-10$

then substituting again: $y=x-6=-10-6=-16$ to give $(-10,-16)$

(c) Using GDC, or

by elimination $\begin{cases} x+2y=3 \text{ (eq.1)} \\ 2x-3y=-8 \text{ (eq.2)} \end{cases}$

Twice (eq.1) minus (eq.2) gives: $7y=14$ so $y=2$ and (eq.1): $x=3-2\cdot 2=-1$

Solution: $(-1,2)$

5. Parallel so same gradient: $\frac{-2}{3}$

Substitute $(-3,0)$ into $y=\frac{-2}{3}x+c$ to obtain: $0=\frac{-2}{3}(-3)+c$ and solve to obtain $c=-2$

Answer: $y=\frac{-2}{3}x-2$

6. Perpendicular so use the negative reciprocal to obtain gradient: $\frac{-3}{2}$

Substitute $(-3,-1)$ into $y=\frac{-3}{2}x+c$ to obtain: $-1=\frac{-3}{2}(-3)+c$ and solve to obtain $c=\frac{-11}{2}$

Answer: $y=\frac{-3}{2}x-\frac{11}{2}$

7. Perpendicular so use the negative reciprocal to obtain gradient: 2

Substitute $(-1,5)$ into $y=2x+c$ to obtain: $5=2(-1)+c$ and solve to obtain $c=7$

Answer: $y=2x+7$

8. Gradient of segment $[RS]=\frac{2+10}{1-7}=-2$

"Perpendicular" so use the negative reciprocal to obtain gradient: $\frac{1}{2}$

"Bisector", so through the mid-point: $\left(\frac{7+1}{2},\frac{-10+2}{2}\right)=(4,-4)$

Substitute $(4,-4)$ into $y=\frac{1}{2}x+c$ to obtain: $-4=\frac{1}{2}(4)+c$ and solve to obtain $c=-6$

Answer: $y=\frac{1}{2}x-6$

9. Gradient of segment $[PQ]=\frac{10-6}{5+3}=\frac{1}{2}$

The line through the apex of an isosceles triangle is a perpendicular bisector to the base line, so use the negative reciprocal to obtain gradient: -2

Mid-point: $\left(\frac{-3+5}{2},\frac{6+10}{2}\right)=(1,8)$

Substitute $(1,8)$ into $y=-2x+c$ to obtain: $8=-2(1)+c$ and solve to obtain $c=10$

Answer: $y=-2x+10$

10. Split the triangle in 2 to get a right-angled triangle with sides 9, 41 and $\sqrt{41^2-9^2}=40$

(a) $\cos B=\frac{9}{41}$

(b) $\sin C=\frac{40}{41}$

(c) $\tan\left(\frac{A}{2}\right)=\frac{9}{40}$

(d) $B=\arccos\left(\frac{9}{41}\right)\approx 77.3°\approx 77°$

(e) $\frac{A}{2}=\arctan\left(\frac{9}{40}\right)\approx 12.6804$ so $A\approx 25.4°\approx 25°$

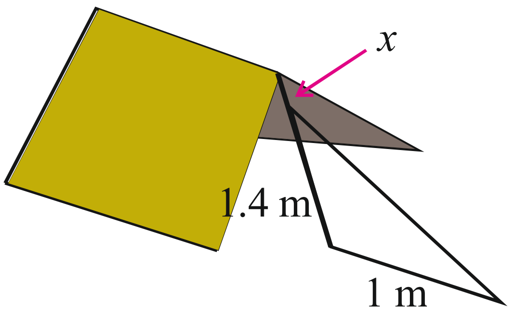
11. Head start = difference in distance = $70+105-\sqrt{70^2+105^2}\approx 49$ to the nearest metres

12. This pentagon is composed of 5 congruent isosceles triangles with base 24 cm and angle at the apex equal to $\frac{360}{5}=72°$. Splitting in 2, we get a right-angled triangle with angle 36° and sides 12 cm and distance vertex-centre equal to $\frac{12}{\sin 36°}\approx 20.4\text{ cm}$

13. Other 2 angles are $\frac{360-2\times 36}{2}=144°$ each. The longer diagonal is opposite the obtuse angles. Use the cosine rule:

$c^2=20^2+20^2-2\cdot 20\cdot 20\cdot\cos(144°)$ so diagonal $\approx\sqrt{1447.21}\approx 38.0\text{ m}$

14. Let x be the height at which the string is tied. Here is a sketch.



$\tan(30°)=\frac{1.4-x}{1}$ so $x\approx 0.823$ metres

15. We have 2 right-angled triangles with angles 60° and 72°, both with opposite side of 40 m.

Let n and f be the distance from the base of the cliff to the near and far bank, respectively.

Then $\tan(72) = \frac{40}{n}$ and $\tan(60) = \frac{40}{f}$

The width of the stream = $f - n = \frac{40}{\tan(60)} - \frac{40}{\tan(72)} \approx 10$ to the nearest metre

16. Let x be the distance between the observation points and h the height of the trees (which we do not need, so will eliminate)

We have 2 right-angled triangles with angles 22° and 40°, both with opposite side h and with adjacent sides 50 and 50-x respectively.

Then $\tan(22) = \frac{h}{50}$ and $\tan(40) = \frac{h}{50 - x}$

Eliminating h, we obtain: $50 \cdot \tan(22) = (50 - x) \cdot \tan(40)$

$$50 \cdot \tan(22) = 50 \cdot \tan(40) - x \cdot \tan(40)$$

$$x \cdot \tan(40) = 50 \cdot \tan(40) - 50 \cdot \tan(22)$$

$$x = \frac{50 \cdot \tan(40) - 50 \cdot \tan(22)}{\tan(40)} \approx 25.9$$

So, 26 metres to the nearest metre.

17. **(a)** $\hat{A} = 180 - (30 + 45) = 105°$

$$\frac{c}{\sin(45°)} = \frac{10}{\sin(105°)} \text{ so } c = \frac{10\sin(45°)}{\sin(105°)} \approx 7.32\,\text{cm}$$

$$\frac{b}{\sin(30°)} = \frac{10}{\sin(105°)} \text{ so } b = \frac{10\sin(30°)}{\sin(105°)} \approx 5.18\,\text{cm}$$

(b) $C = 180 - (110 + 40) = 30°$

$$\frac{c}{\sin(30°)} = \frac{8}{\sin(110°)} \text{ so } c = \frac{8\sin(30°)}{\sin(110°)} \approx 4.26\,\text{cm}$$

$$\frac{b}{\sin(40°)} = \frac{8}{\sin(110°)} \text{ so } b = \frac{8\sin(40°)}{\sin(110°)} \approx 5.47\,\text{cm}$$

(c) $b^2 = 12^2 + 18^2 - 2\cdot 12\cdot 18\cdot \cos(36°)$ so $b = \sqrt{118.5046584} \approx 10.9\,\text{cm}$

Let us substitute directly in the re-arranged form of the cosine rule:

$$\cos(A) = \frac{12^2 + 10.88598449^2 - 18^2}{2\cdot 12\cdot 10.88598449} \approx -0.2353765896$$

$$A \approx \arccos(-0.2353765896) \approx 103.6° \approx 104°$$

Finally, $\hat{C} = 180 - (36 + 104) = 40°$

(d) Let us substitute directly in the re-arranged form of the cosine rule (for either of the 3 missing angles):

$$\cos(A) = \frac{8^2 + 6^2 - 12^2}{2\cdot 8\cdot 6} = \frac{-11}{24}$$

$$A \approx \arccos\left(\frac{-11}{24}\right) \approx 117°$$

Again for either of the 2 missing angles

$$\cos(B) = \frac{8^2 + 12^2 - 6^2}{2\cdot 8\cdot 12} = \frac{43}{48}$$

$$B \approx \arccos\left(\frac{43}{48}\right) \approx 26°$$

Finally, $C = 180 - (117 + 26) = 37°$

18. **(a)** $\text{Area} = \frac{1}{2}\left(\frac{10^2 \sin 30° \sin 45°}{\sin 105°}\right) \approx 18.3\ \text{cm}^2$

alternatively, $\text{Area} = \frac{1}{2}(10)(7.32...)\sin 30° \approx 18.3\ \text{cm}^2$

(b) $\text{Area} = \frac{1}{2}\left(\frac{8^2 \sin 40° \sin 30°}{\sin 110°}\right) \approx 10.9\ \text{cm}^2$

alternatively, $\text{Area} = \frac{1}{2}(8)(4.26...)\sin 40° \approx 10.9\ \text{cm}^2$

(c) $\text{Area} = \frac{1}{2}(12)(18)\sin 36° \approx 63.5\ \text{cm}^2$

(d) $\text{Area} = \frac{1}{2}(6)(8)\sin 117.2796127° \approx 21.3\ \text{cm}^2$

19. With $r = 6$ cm, the triangle is not closed. For this, we need:

$\sin(36) = \frac{r}{14}$ so $r = 14\sin(36) \approx 8.23$ cm

20. **(a)** $\text{Area} = \frac{1}{2}\left(2\sqrt{3}\right)(6)\sin\left(ABC\right)$

so $3\sqrt{3} = \frac{1}{2}\left(2\sqrt{3}\right)(6)\sin\left(ABC\right)$

$\sin\left(ABC\right) = \frac{1}{2}$ so $ABC = \arcsin\left(\frac{1}{2}\right) = 30°$ or $180° - 30° = 150°$

Since ABC is obtuse, we keep the value 150°

(b) $CBD = 180 - 150 = 30° = \frac{\pi}{6}\text{rad}$ so $\text{Area} = r\theta = 6 \cdot \frac{\pi}{6} = \pi\ \text{cm}^2$

21. **(a)** $\text{Area} = \frac{1}{2}(8)(6)\sin\left(A\right)$

$16 = \frac{1}{2}(8)(6)\sin\left(A\right)$

so $\sin\left(A\right) = \frac{2}{3}$ so $A = \arcsin\left(\frac{2}{3}\right) \approx 41.8°$ or $180° - 41.8° \approx 138.2°$

(b) Keep the obtuse angle 138.2° and use the cosine rule:

$BC^2 = 6^2 + 8^2 - 2 \cdot 6 \cdot 8 \cdot \cos(138.2°)$ so $BC = \sqrt{171.565696} \approx 13.1\,\text{cm}$

22. From triangle ABD: $AD^2 = 3^2 + 1^2 - 2 \cdot 3 \cdot 1 \cdot \cos(60°)$ so $AD = \sqrt{7}$

Use this value is triangle ACD: $\cos\left(DAC\right) = \frac{\left(\sqrt{7}\right)^2 + 3^2 - 2^2}{2 \cdot \sqrt{7} \cdot 3} = \frac{12}{6\sqrt{7}} = \frac{2}{\sqrt{7}}$ or $\frac{2\sqrt{7}}{7}$

Exercise 5.1

1. **(a)** arc length $= \frac{120}{360}(2\pi)(6) = 4\pi$ cm

(b) arc length $= \frac{70}{360}(2\pi)(12) = \frac{14}{3}\pi$ cm

2. **(a)** area of sector $= \frac{30}{360}\pi(10)^2 = \frac{25}{3}\pi\ \text{cm}^2$, arc length $= \frac{30}{360}(2\pi)(10) = \frac{5}{3}\pi$ cm

(b) area of sector $= \frac{45}{360}\pi(8)^2 = 8\pi\ \text{m}^2$, arc length $= \frac{45}{360}(2\pi)(8) = 2\pi$ m

(c) area of sector $= \frac{52}{360}\pi(180)^2 = 4680\pi\ \text{mm}^2$

arc length $= \frac{52}{360}(2\pi)(180) = 52\pi$ cm

(d) area of sector $= \frac{n}{360}\pi(15)^2 = \frac{5\pi n}{8}\ \text{cm}^2$, arc length $= \frac{n}{360}(2\pi)(15) = \frac{\pi n}{12}$ cm

3. $12 = \frac{\theta}{360}(2\pi)(8) \Rightarrow 270 = \pi\theta \Rightarrow \theta = \frac{270}{\pi} \approx 85.9°$

4. **(a)** $1.5 \times 360 = 540°\ \text{s}^{-1}$

(b) Bicycle speed is equal to the speed of a point along the circumference of the wheel. Speed $= 1.5 \times 2\pi(35) \approx 330\ \text{cm s}^{-1}$,

$$330\ \frac{\text{cm}}{\text{s}} \times \frac{60\ \text{s}}{1\ \text{min}} \times \frac{60\ \text{min}}{1\ \text{h}} \times \frac{1\ \text{m}}{100\ \text{cm}} \times \frac{1\ \text{km}}{1000\ \text{m}} = 11.9\ \text{km h}^{-1}$$

5. $\frac{16}{5}\pi = \frac{\theta}{360}\pi(4)^2 \Rightarrow \frac{1}{5} = \frac{\theta}{360} \Rightarrow \theta = 72°$

6. $A = l \Rightarrow \frac{\theta}{360}\pi r^2 = \frac{\theta}{360}(2\pi)r \Rightarrow r^2 = r \Rightarrow r = 0$ or $r = 1$

7. Maria $= \frac{112}{360}(2\pi)(230) \approx 450$ m

Norbert = $= \frac{66}{360}(2\pi)(500) \approx 576$ m

$\therefore$ Norbert walks $576 - 450 = 126$ m farther.

8. $\frac{1}{24}\pi(400)^2 \approx 20\ 944 \approx 21\ 000\ \text{m}^2\ \text{h}^{-1}$

9. **(a)** The watered area is $11.8\ \text{m}^2$, so the water requirement is

$2.5\ \text{mm} \times \dfrac{1\ \text{m}}{1000\ \text{mm}} \times 11.8 = 0.0295\ \text{m}^3$. In cm^3,

$0.0295\ \text{m}^3 \times \dfrac{100^3\ \text{cm}^3}{1\ \text{m}^3} = 29\ 500\ \text{cm}^3$

(b) $\dfrac{29\ 500}{800} \approx 36.9\ \text{min}$

(c) New area $= \dfrac{100}{360}\pi(3)^2 \approx 7.85\ \text{m}^2$

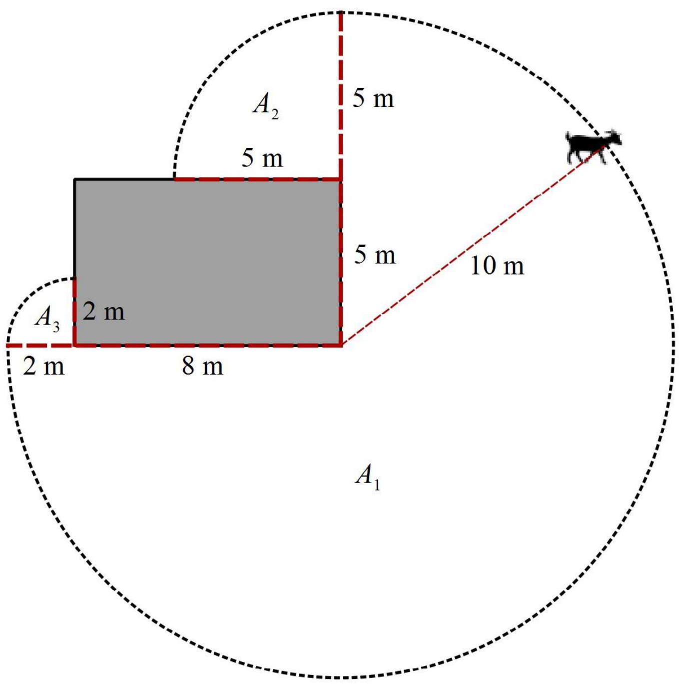
Water requirement $= 7.85 \times \dfrac{2.5}{1000} \approx 0.0196\ \text{m}^3 \approx 19\ 600\ \text{cm}^3$

Required time $= \dfrac{19\ 600}{800} \approx 24.5\ \text{min}$

10. **(a)** $\dfrac{\frac{1}{60}}{360} \times 2\pi(6370) = 1.85\,\text{km}$

(b) $\dfrac{90}{360} \times 2\pi(6370) \approx 10000\,\text{km}$

11. As the leash wraps around the build, three sectors are created.



$A_2 = \dfrac{1}{4}\pi \times 5^2 = 6.25\pi$, $A_3 = \dfrac{1}{4}\pi \times 2^2 = \pi$

$\therefore$ total area $= A_1 + A_2 + A_3 = 82.25\pi \approx 258\ \text{m}^2$

Exercise 5.2

1. $60° \times \frac{\pi}{180°} = \frac{\pi}{3}$

2. $150° \times \frac{\pi}{180°} = \frac{5\pi}{6}$

3. $-270° \times \frac{\pi}{180°} = -\frac{3\pi}{2}$

4. $36° \times \frac{\pi}{180°} = \frac{\pi}{5}$

5. $135° \times \frac{\pi}{180°} = \frac{3\pi}{4}$

6. $50° \times \frac{\pi}{180°} = \frac{5\pi}{18}$

7. $-45° \times \frac{\pi}{180°} = -\frac{\pi}{4}$

8. $400° \times \frac{\pi}{180°} = \frac{20\pi}{9}$

9. $-480° \times \frac{\pi}{180°} = -\frac{8\pi}{3}$

10. $\frac{3\pi}{4} \times \frac{180°}{\pi} = 135°$

11. $-\frac{7\pi}{2} \times \frac{180°}{\pi} = -630°$

12. $2 \times \frac{180°}{\pi} \approx 115°$

13. $\frac{7\pi}{6} \times \frac{180°}{\pi} = 210°$

14. $-2.5 \times \frac{180°}{\pi} \approx -143°$

15. $\frac{5\pi}{3} \times \frac{180°}{\pi} = 300°$

16. $\frac{\pi}{12} \times \frac{180°}{\pi} = 15°$

17. $1.57 \times \frac{180°}{\pi} \approx 90.0°$

18. $\frac{8\pi}{3} \times \frac{180°}{\pi} = 480°$

19. $30 + 360 = 390°$, $30 - 360 = -330°$

20. $\frac{3\pi}{2} + 2\pi = \frac{7\pi}{2}$, $\frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$

21. $175 + 360 = 535°$, $175 - 360 = -185°$

22. $-\frac{\pi}{6} + 2\pi = \frac{11\pi}{6}$, $-\frac{\pi}{6} - 2\pi = -\frac{13\pi}{6}$

23. $\frac{5\pi}{3}+2\pi=\frac{11\pi}{3}$, $\frac{5\pi}{3}-2\pi=-\frac{\pi}{3}$

24. $3.25+2\pi\approx 9.53$, $3.25-2\pi\approx -3.03$

25. $900-2(360)=180°$, $180-360=-180°$

26. $\frac{19\pi}{3}-3(2\pi)=\frac{\pi}{3}$, $\frac{\pi}{3}-2\pi=-\frac{5\pi}{3}$

27. $-51\pi-25(2\pi)=\pi$, $\pi-2\pi=-\pi$

28. $s=6\left(\frac{2\pi}{3}\right)=4\pi\approx 12.6\text{ cm}$

29. $s=1(12)=12\text{ cm}$

30. $12=8\theta\Rightarrow\theta=\frac{3}{2}$, $\theta=\frac{3}{2}\times\frac{180}{\pi}=\frac{270}{\pi}\approx 85.9°$

31. $15=r\left(\frac{2\pi}{3}\right)\Rightarrow r=\frac{45}{2\pi}\approx 7.16$

32. $A=\frac{1}{2}r^2\theta=\frac{1}{2}(4^2)(1.5)=12\text{ cm}^2$

33. $A=\frac{1}{2}r^2\theta=\frac{1}{2}(10^2)\left(\frac{5\pi}{6}\right)=\frac{125\pi}{3}\approx 131\text{ cm}^2$

34. $60=20\alpha\Rightarrow\alpha=\frac{1}{3},\alpha=\frac{1}{3}\times\frac{180}{\pi}\approx 19.1°$

35. $s=r\theta=2(10)=20\text{ cm}$

36. $24=\frac{60}{360}\pi r^2\Rightarrow r=\sqrt{\frac{144}{\pi}}=\frac{12}{\sqrt{\pi}}\text{ cm}\approx 45.8\text{ cm}$

37. **(a)** $A=\frac{1}{2}\times\frac{\pi}{3}\times 20^2=\frac{200\pi}{3}\approx 209\text{ mm}^2$, $l=\frac{\pi}{3}\times 20=\frac{20\pi}{3}\approx 20.9\text{ mm}$

(b) $A=\frac{1}{2}\times\frac{\pi}{6}\times 70^2=\frac{1225\pi}{3}\approx 1280\text{ m}^2$, $l=\frac{\pi}{6}\times 70=\frac{35\pi}{3}\approx 36.7\text{ m}$

(c) $A=\frac{1}{2}\times 2\times 3^2=9\text{ km}^2$, $l=2\times 3=6\text{ km}$

(d) $A=\frac{1}{2}\times 5\times 5^2=62.5\text{ cm}^2$, $l=3\times 5=15\text{ cm}$

38. **(a)** $25\pi = \frac{1}{2}(5)^2\theta \;\Rightarrow\; \theta = 2\pi$ cm

(b) $30\pi = \frac{1}{2}(2)^2\theta \;\Rightarrow\; \theta = 15\pi$ m

(c) $36 = \frac{1}{2}(6)^2\theta \;\Rightarrow\; \theta = 2$ m

(d) $40 = \frac{1}{2}(2)^2\theta \;\Rightarrow\; \theta = 20$ m

(e) $75\pi = (25)\theta \;\Rightarrow\; \theta = 3\pi$ km

(f) $\frac{\pi}{3} = (2)\theta \;\Rightarrow\; \theta = \frac{\pi}{6}$ m

(g) $15 = (5)\theta \;\Rightarrow\; \theta = 3$ cm

(h) $100 = (20)\theta \;\Rightarrow\; \theta = 5$ mm

39. **(a)** $3^2 + 6^2 = l^2 \;\Rightarrow\; l = 3\sqrt{5}$ cm

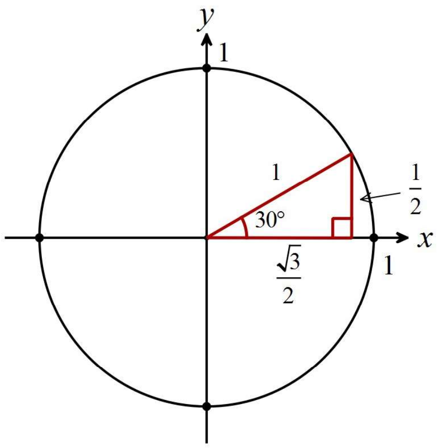
(b) $r = l = 3\sqrt{5}$ cm

(c) $2\pi \times 3 = 6\pi$ cm

(d) $6\pi = (3\sqrt{5})\theta \;\Rightarrow\; \theta = \frac{2\pi\sqrt{5}}{5} \approx 2.81$

Exercise 5.3

1. We can draw a 30-60-90° in the unit circle and then write down the answers for part **(a)**. The solutions to **(b)–(c)** can then be found using symmetry.



(a) $\sin 30° = \frac{1}{2}$, $\cos 30° = \frac{\sqrt{3}}{2}$, $\tan 30° = \frac{\sqrt{3}}{3}$

(b) $\sin 150° = \frac{1}{2}$, $\cos 150° = -\frac{\sqrt{3}}{2}$, $\tan 150° = -\frac{\sqrt{3}}{3}$

(c) $\sin 210° = -\frac{1}{2}$, $\cos 210° = -\frac{\sqrt{3}}{2}$, $\tan 210° = \frac{\sqrt{3}}{3}$

(d) $\sin 330° = -\frac{1}{2}$, $\cos 330° = \frac{\sqrt{3}}{2}$, $\tan 330° = -\frac{\sqrt{3}}{3}$

2. After finding the solutions to **(a)** using a 30-60-90° right triangle in the unit circle, the solutions to **(b)-(c)** can be found using symmetry.

(a) $\sin 60° = \frac{\sqrt{3}}{2}$, $\cos 60° = \frac{1}{2}$, $\tan 60° = \sqrt{3}$

(b) $\sin 120° = \frac{\sqrt{3}}{2}$, $\cos 120° = -\frac{1}{2}$, $\tan 120° = -\sqrt{3}$

(c) $\sin 240° = -\frac{\sqrt{3}}{2}$, $\cos 240° = -\frac{1}{2}$, $\tan 240° = \sqrt{3}$

(d) $\sin 300° = -\frac{\sqrt{3}}{2}$, $\cos 300° = \frac{1}{2}$, $\tan 300° = -\sqrt{3}$

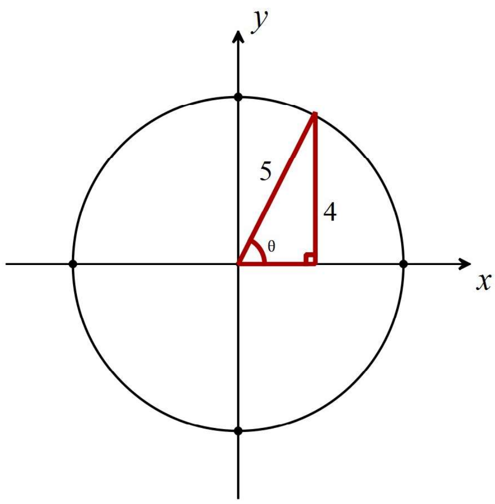
3. **(a)** $\sin 0° = 0$, $\cos 0° = 1$, $\tan 0° = 0$

(b) $\sin 90° = 1$, $\cos 90° = 0$, $\tan 90°$ is undefined

(c) $\sin 180° = 0$, $\cos 180° = -1$, $\tan 180° = 0$

(d) $\sin 270° = -1$, $\cos 270° = 0$, $\tan 270°$ is undefined

4. Using geometry: Start by drawing a right triangle in the coordinate plane with the hypotenuse as the radius of a circle and θ in standard position:



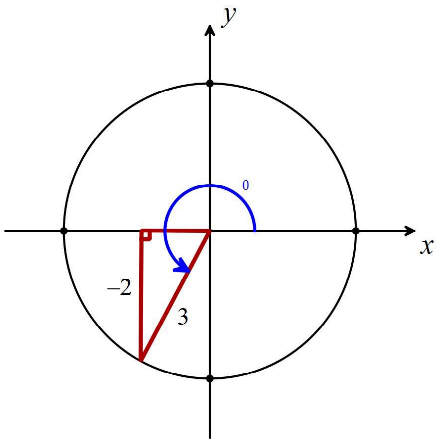
The length of the missing side must be 3 (Pythagorean Theorem or by recognizing the Pythagorean triple 3:4:5).

Therefore, $\cos\theta = \frac{3}{5}$ and $\tan\theta = \frac{4}{3}$.

Using identities: since $\sin^2\theta + \cos^2\theta = 1$, we have $\cos^2\theta = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} \quad \Rightarrow \quad \cos\theta = \pm\frac{3}{5}$;

since θ is in quadrant 1, it must be $\cos\theta = \frac{3}{5}$. Then $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$.

5. **Using geometry:**

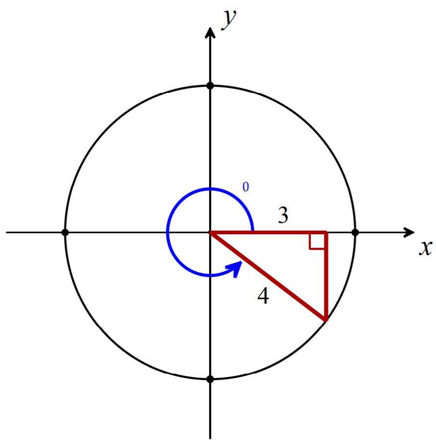


The missing side is $x^2 + 2^2 = 3^2 \Rightarrow x = \sqrt{9-4} = \sqrt{5} \Rightarrow -\sqrt{5}$.

Therefore, $\cos\theta = -\frac{\sqrt{5}}{3}$ and $\tan\theta = \frac{-2}{-\sqrt{5}} = \frac{2\sqrt{5}}{5}$.

6. **Using geometry:**

If $\cos\theta = \frac{3}{4}$, the angle is in quadrants I or IV. Since $\pi < \theta < 2\pi$ implies the angle is in quadrants III or IV, we conclude the angle is in quadrant IV.



The missing side is $y^2 + 3^2 = 4^2 \Rightarrow x = \sqrt{16-9} = \sqrt{7} \Rightarrow -\sqrt{7}$.

Therefore, $\sin\theta = -\frac{\sqrt{7}}{4}$ and $\tan\theta = \frac{-\sqrt{7}}{3} = -\frac{\sqrt{7}}{3}$.

Using identities:

$\sin^2\theta + \cos^2\theta = 1 \quad \Rightarrow \quad \sin^2\theta = 1 - \left(\frac{3}{4}\right)^2 = \frac{7}{16} \quad \Rightarrow \quad \sin\theta = \pm\frac{\sqrt{7}}{4}$; since θ is in quadrant I or IV, it must be $\sin\theta = -\frac{\sqrt{7}}{4}$. Then $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-\frac{\sqrt{7}}{4}}{\frac{3}{4}} = \frac{-\sqrt{7}}{3}$.

7. **(a)** I **(b)** $\left(\cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{6}\right)\right)=\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

8. **(a)** IV **(b)** $\left(\cos\left(\frac{5\pi}{3}\right), \sin\left(\frac{5\pi}{3}\right)\right)=\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

9. **(a)** IV **(b)** $\left(\cos\left(\frac{7\pi}{4}\right), \sin\left(\frac{7\pi}{4}\right)\right)=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$

10. **(a)** on the negative y-axis **(b)** $(0,-1)$

11. **(a)** II **(b)** $(\cos(2), \sin(2))=(-0.416, 0.909)$

12. **(a)** IV **(b)** $\left(\cos\left(-\frac{\pi}{4}\right), \sin\left(-\frac{\pi}{4}\right)\right)=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$

13. **(a)** IV **(b)** $(\cos(-1), \sin(-1))=(0.540,-0.841)$

14. **(a)** III **(b)** $\left(\cos\left(-\frac{3\pi}{4}\right), \sin\left(-\frac{3\pi}{4}\right)\right)=\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$

15. **(a)** III **(b)** $(\cos(3.52), \sin(3.52))=(-0.929,-0.369)$

16. $\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2},\ \cos\frac{\pi}{3}=\frac{1}{2},\ \tan\frac{\pi}{3}=\sqrt{3}$

17. $\sin\frac{5\pi}{6}=\frac{1}{2},\ \cos\frac{5\pi}{6}=-\frac{\sqrt{3}}{2},\ \tan\frac{5\pi}{6}=-\frac{\sqrt{3}}{3}$

18. $\sin\left(-\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2},\ \cos\left(-\frac{3\pi}{4}\right)=-\frac{\sqrt{2}}{2},\ \tan\left(-\frac{3\pi}{4}\right)=1$

19. $\sin\frac{\pi}{2}=1,\ \cos\frac{\pi}{2}=0,\ \tan\frac{\pi}{2}$ is undefined

20. $\sin\left(-\frac{4\pi}{3}\right)=\frac{\sqrt{3}}{2},\ \cos\left(-\frac{4\pi}{3}\right)=-\frac{1}{2},\ \tan\left(-\frac{4\pi}{3}\right)=-\sqrt{3}$

21. $\sin 3\pi=0,\ \cos 3\pi=-1,\ \tan 3\pi=0$

22. $\sin\frac{3\pi}{2}=-1,\ \cos\frac{3\pi}{2}=0,\ \tan\frac{3\pi}{2}$ is undefined

23. $\sin\left(-\frac{7\pi}{6}\right)=\frac{1}{2},\ \cos\left(-\frac{7\pi}{6}\right)=-\frac{\sqrt{3}}{2},\ \tan\left(-\frac{7\pi}{6}\right)=-\frac{\sqrt{3}}{3}$

24. $\sin 1.25\pi=-\frac{\sqrt{2}}{2},\ \cos 1.25\pi=-\frac{\sqrt{2}}{2},\ \tan 1.25\pi=1$

25. $\frac{13\pi}{6}-2\pi=\frac{\pi}{6},\ \sin\frac{\pi}{6}=\frac{1}{2},\ \cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$

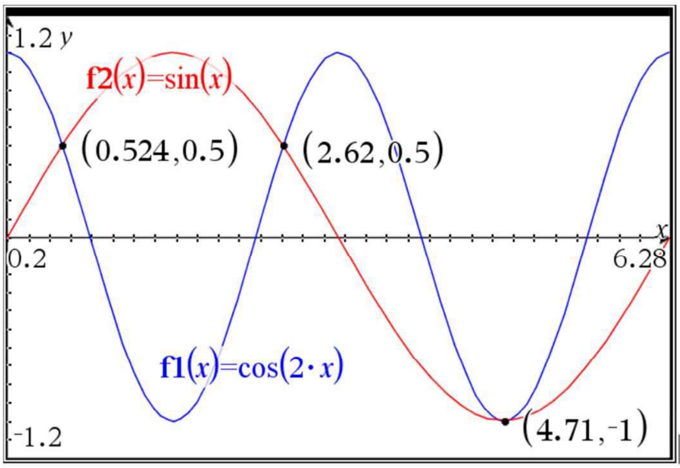
26. $\frac{10\pi}{3}-2\pi=\frac{4\pi}{3},\ \sin\frac{4\pi}{3}=-\frac{\sqrt{3}}{2},\ \cos\frac{4\pi}{3},=-\frac{1}{2}$

27. $\frac{15\pi}{4} - 2\pi = \frac{7\pi}{4}$, $\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$, $\cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$

28. $\frac{17\pi}{6} - 2\pi = \frac{5\pi}{6}$, $\sin\frac{5\pi}{6} = \frac{1}{2}$, $\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$

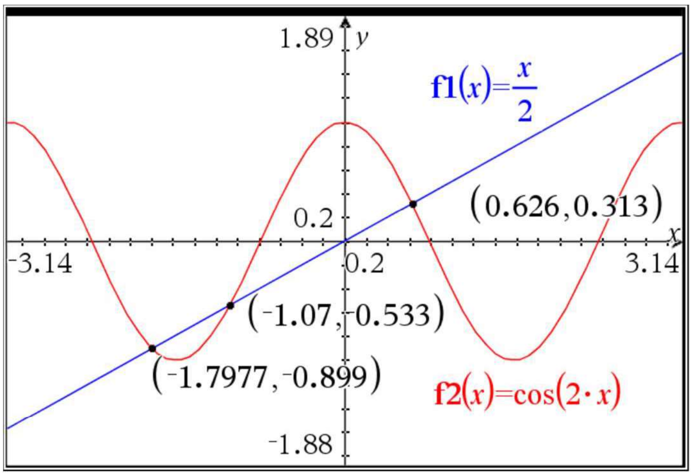
Exercise 5.4

1. GDC screen shown below.



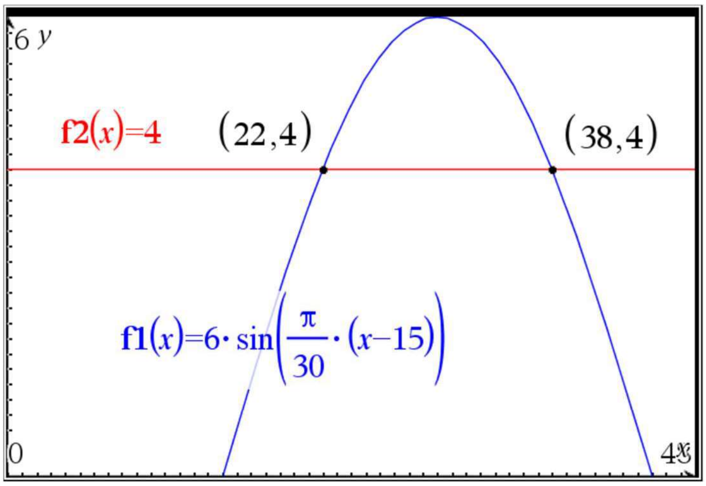
$\therefore x = \{0.524, 2.62, 4.71\}$.

2. GDC screen shown below.



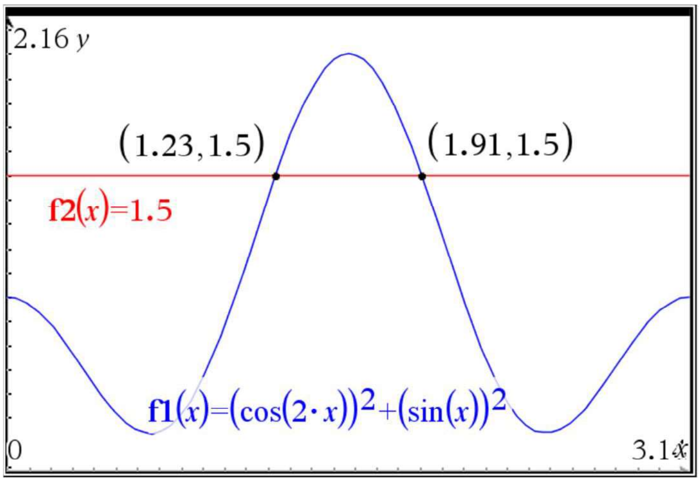
$\therefore x = \{-1.80, -1.07, 0.626\}$.

3. GDC screen shown.



$\therefore x = \{22.0,\ 38.0\}$ (to 3 significant figures)

4. GDC screen shown.

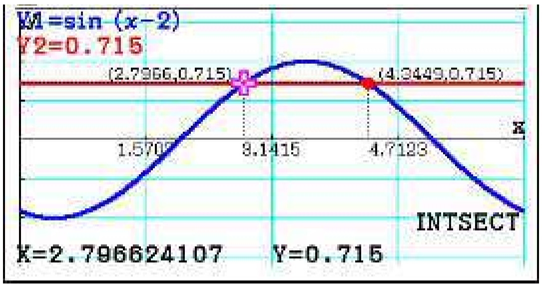


$\therefore x = \{1.23,\ 1.91\}$

5. By the Pythagorean identity, $\cos^2 x + \sin^2 x = 1$, hence $\cos^2 x + \sin^2 x = 2$ has no solution since $1 \neq 2$.

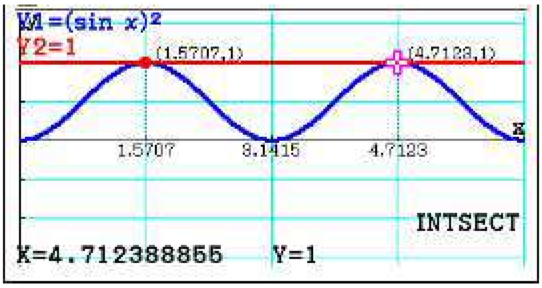
6.

(a) GDC screen shown



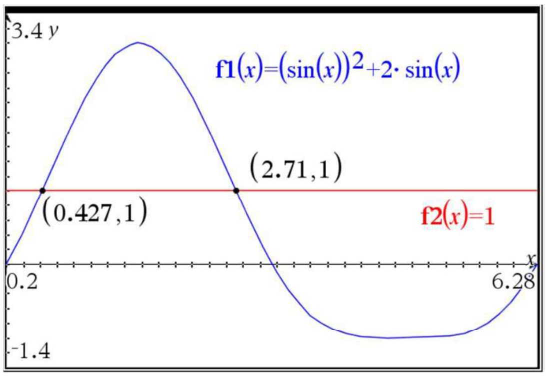
$x = \{2.80,\ 4.34\}$

(b) GDC screen shown



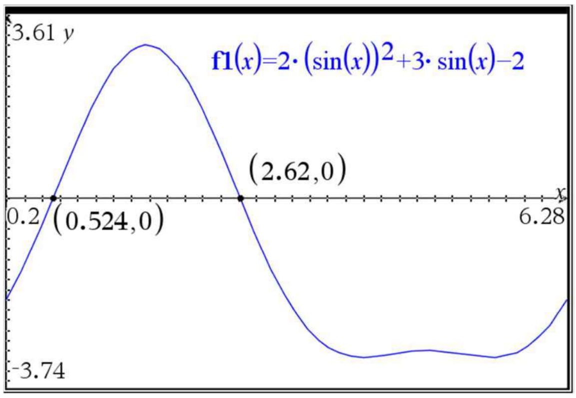
$x = \{1.57,\ 4.71\}$

(c) GDC screen shown



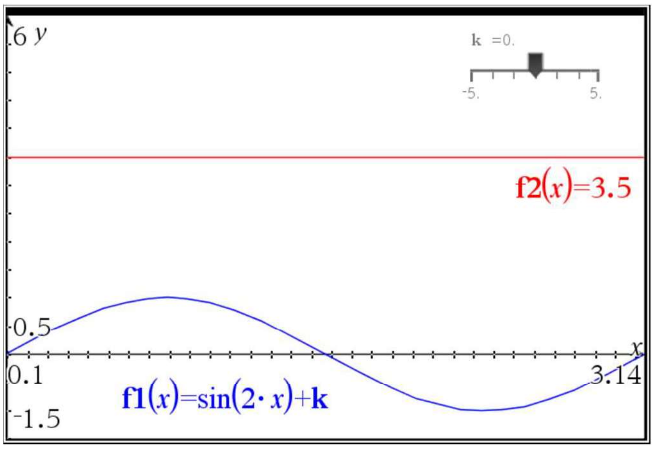
$x = \{0.427,\ 2.71\}$

(d) GDC screen shown

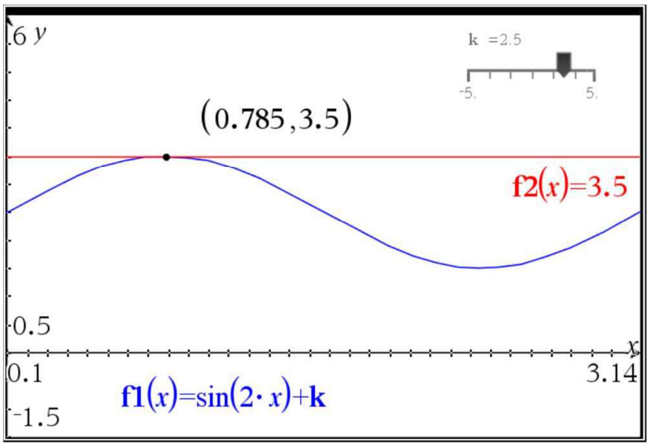


$x = \{0.524,\ 2.62\}$

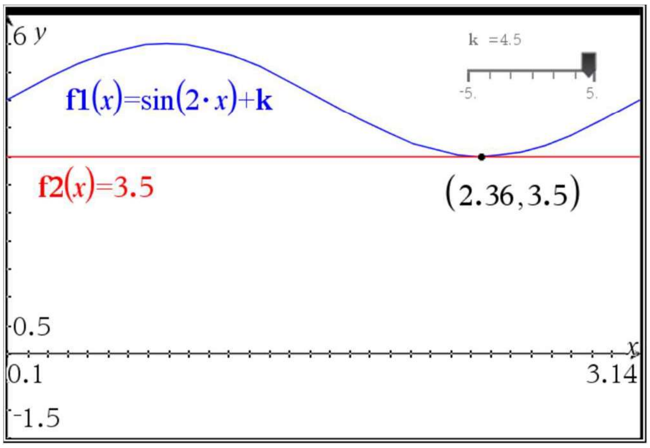
7. We first graph the equation with $k = 0$:



(a) Since k produces a vertical translation, the equation will have one solution when the maximum or minimum of $\sin(2x)+k$ is at 3.5. Since the maximum is 1 when $k = 0$, the equation has one solution when $k = 3.5 - 1 = 2.5$.
Verify graphically:



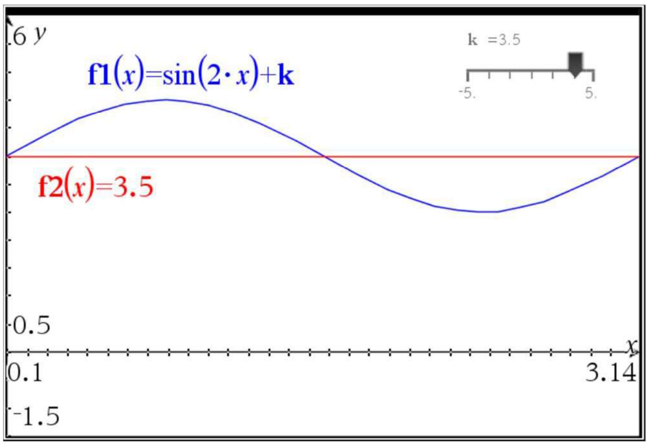
The next solution is when the minimum of -1 is translated to 3.5, so
Verify graphically:



Therefore, the equation has one solution when $k \in \{2.5, 4.5\}$.

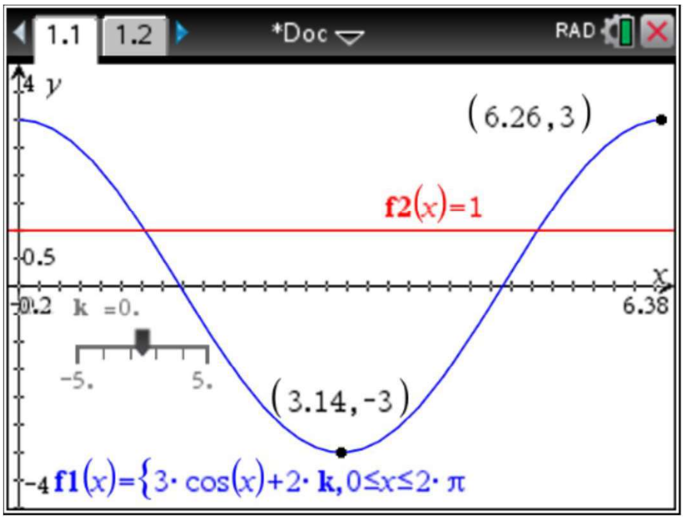
(b) Using the reasoning above, the equation will have two solutions when $2.5 < k < 4.5,\ k \neq 3.5$.

(c) Using the reasoning above, the equation will have three solutions when $k = 3.5$. Verify graphically:



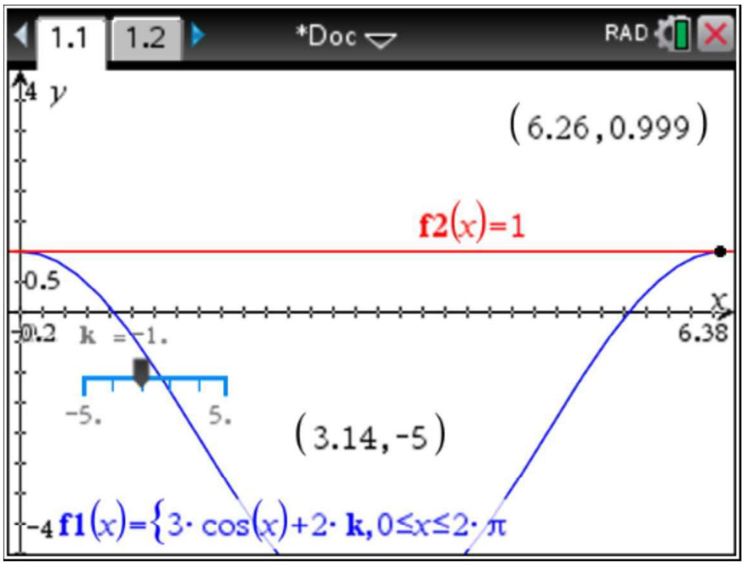
(d) Using the reasoning above, the equation will have no solutions when $k < 2.5$ or $k > 4.5$.

8. We first graph the equation with $k = 0$ and find maxima and minima:



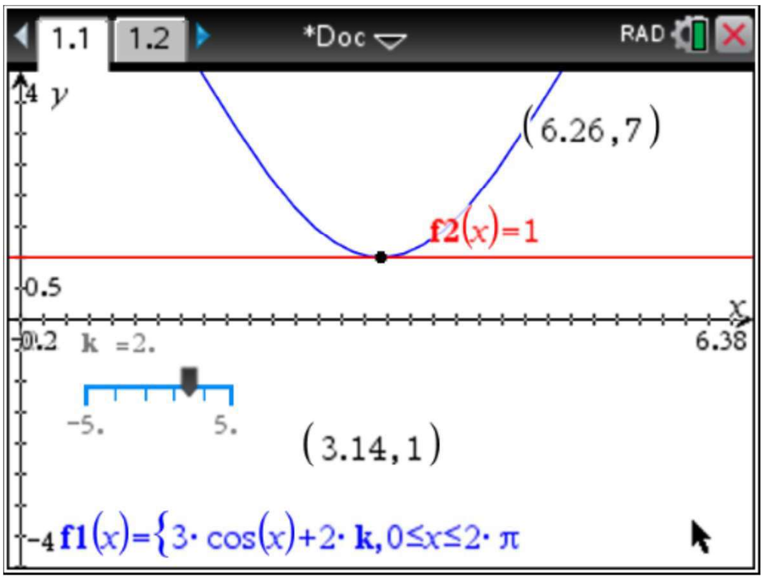
(a) Since $2k$ produces a vertical translation, we can see that the equation will have two solutions when the maximum of $3\cos(x) + 2k$ is at 1. Since the maximum is 3 when $2k = 0$, one solution will be when the maximum is translated $2k = 1 - 3 \Rightarrow k = -1$.

Verify graphically:



As k increases, the function will be translated up, and will continue to have two solutions until the minimum at -3 is reached. Hence, $2k = 1-(-3) \Rightarrow k = 2$.

Verify graphically:



Therefore, the equation has two solutions when $-1 \le k < 2$.

(b) Using the reasoning above, the equation will have one solution when $k = 2$.

(c) Using the reasoning above, the equation will have no solutions when $k < -1$ or $k > 2$.

Exercise 5.5

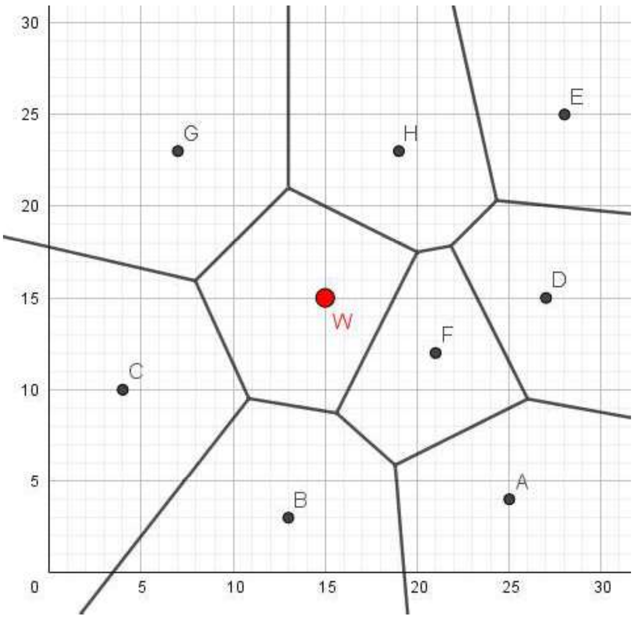
1. **(a)** (15, 15) is in the cell of site F

(b) Midpoint of $\overline{WG} = \left(\dfrac{15+7}{2}, \dfrac{15+23}{2}\right) = (11, 19)$. Gradient of

$\overline{WG} = \dfrac{23-15}{7-15} = \dfrac{8}{-8} = -1 \quad \Rightarrow \quad$ gradient of perpendicular bisector $= -\dfrac{1}{-1} = 1$.

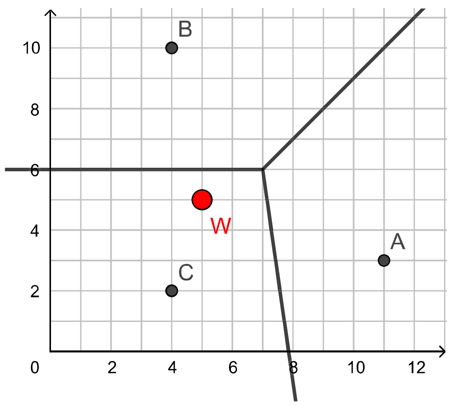
Therefore line is $y - 19 = 1(x - 11) \quad \Rightarrow \quad y = x + 8$.

(c) The cells for sites F, B, C, G, and H will be changed.

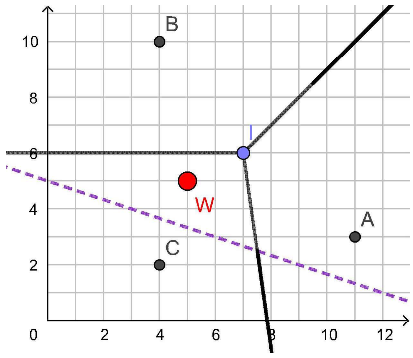


(d) By sketching the perpendicular bisectors of nearby sites, we see that cells of sites B and C will be affected. The new cell will have at least two rays as edges.

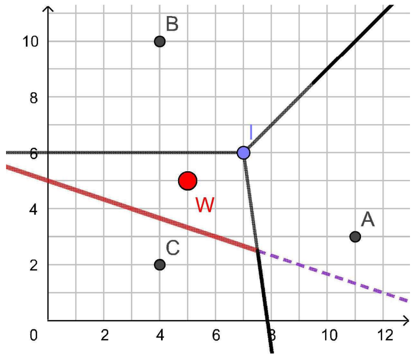
2. The process is shown below.



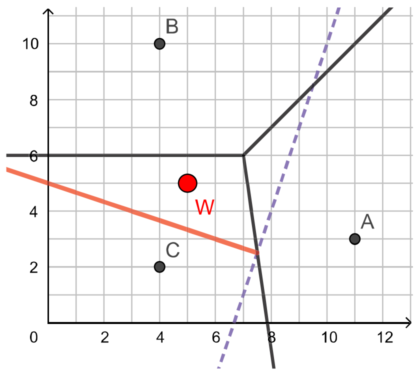
Plot the new site W.



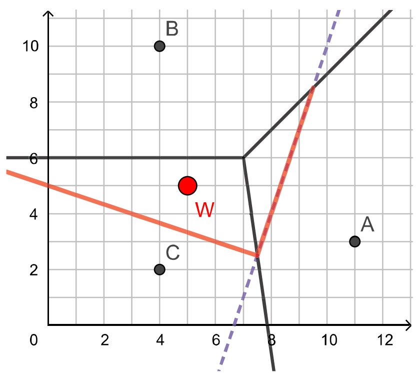
Draw the perpendicular bisector with containing cell's site **(c)**.



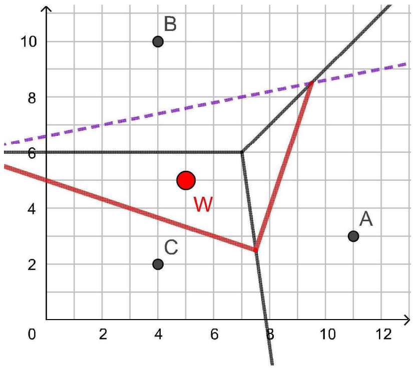
Add ray or line segment between W and C.



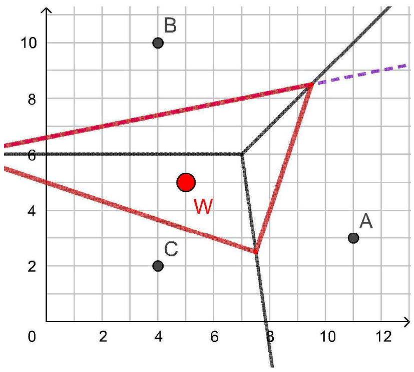
Find perpendicular bisector between W and next adjacent cell site **(a)**.



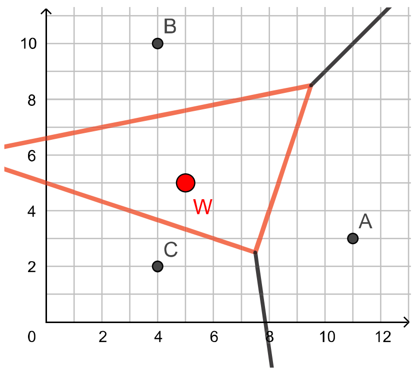
Add ray or line segment between W and A.



Find perpendicular bisector between W and next adjacent cell site **(b)**.



Add ray or line segment between W and B.



Remove edges inside W's new cell.

3. **(a)** We can find Q by reflecting P across the edge shared with the adjacent, empty cell, which gives $Q(4,1)$.

(b) Luckily, the cell for site P is a triangle and we can read the base and height from the graph, hence $A=\frac{1}{2}(4)(3)=6\text{ km}^2$

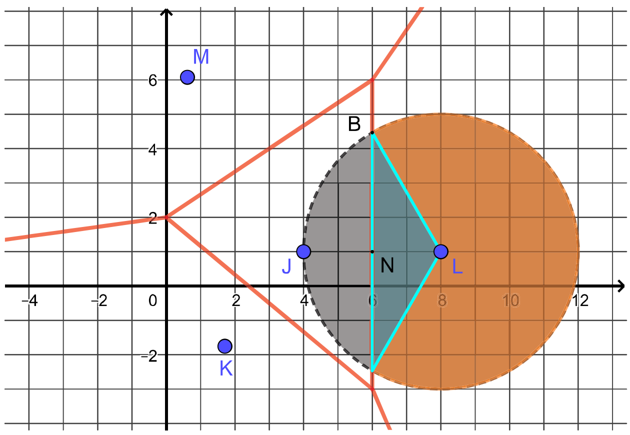
(c) The point $X(9,2)$ is in the cell of site R. Therefore, we find the distance to point $R(5,4)$ using the distance formula:

$d=\sqrt{(9-5)^2+(2-4)^2}=\sqrt{4^2+(-2)^2}=\sqrt{20}\approx 4.47\text{ km}$.

4. **(a)** $A=\frac{1}{2}(9)(6)=27\text{ km}^2$

(b) Since the radius of circle L is 4, we know $BL=4$. From the graph, $NL=2$. Since ΔBLN is a right triangle, it is a special right triangle with sides in the ratio $1:\sqrt{3}:2$, therefore $BLN=60°$.

(c) The region can be seen as a triangle (shown in blue) and the sector of a circle (shown in orange).



Area = Area of triangle + Area of sector

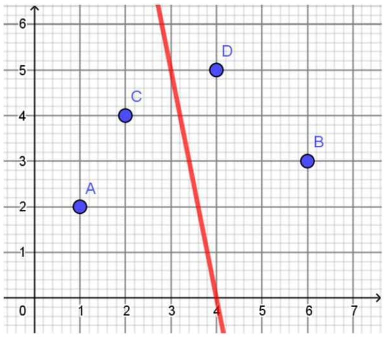
$$=\frac{1}{2}\left(2\times 2\sqrt{2}\right)(2)+\frac{360-120}{360}\pi\left(4^2\right)$$

$$=4\sqrt{2}+\frac{32}{3}\pi$$

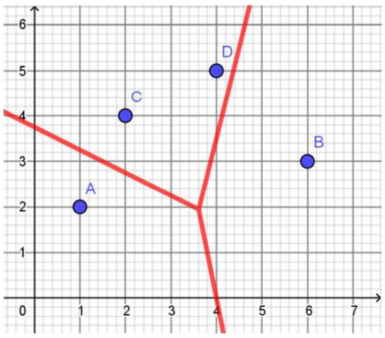
$$\approx 39.2\text{ km}^2$$

5. **(a)** The partial diagrams are shown below.

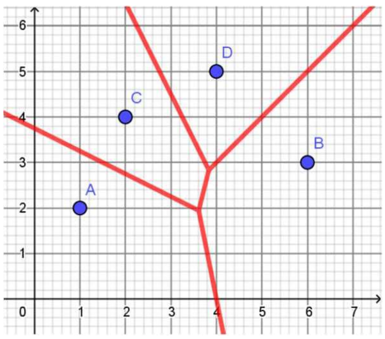
(i)



(ii)



(iii)



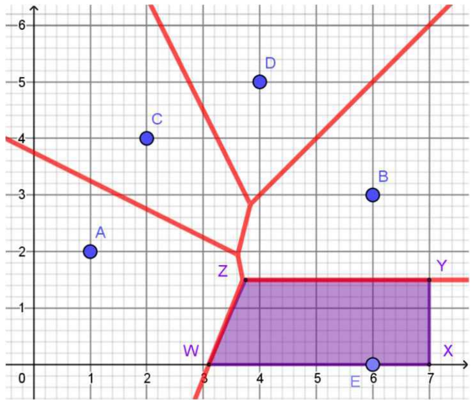
(b) Point $M(3.5, 2.5)$ is in the cell for site *C*. Therefore, it is likely to have soil type **silt**.

(c) $N(5,4)$ is on the boundary between the cells for sites *B* and *C*. Since is it equidistant to sites *B* and *C*, it is not possible to determine the likely soil type.

(d) Cells for sites *A* and *B* would be divided. Cell *B* would be divided because $E(6,0)$ is in the cell *B*. Cell *A* would be divided because the perpendicular bisector of segment *EB* intersects the boundary of cell *A*. There are no more adjacent cells, so no other cells would be divided.

(e) No, the answer to **(b)** does not change because the cell for site *C*, which contains *M*, does not change.

(f) The study area likely to be loam is contained within cell *E*. After we add site *E* to the Voronoi diagram, we have the following:



(The purple shaded area is the cell for site E.)
Cell E forms a trapezoid. To find the area we determine the height and length of the two bases. The height = 1.5 (from the graph). We must find the equation of the boundary between sites A and E to find exact coordinates of the points Z and W.

Midpoint of $AE = \left(\frac{6+1}{2}, \frac{0+2}{2}\right) = (3.5, 1)$.

Slope of $AE = \frac{2-0}{1-6} = -\frac{2}{5} \Rightarrow$ slope of $ZW = \frac{5}{2}$.

$\therefore$ equation of ZW : $y - 1 = \frac{5}{2}(x - 3.5) \Rightarrow y = \frac{5}{2}x - 7.75$.

$\therefore$ when $y = 0 \Rightarrow 0 = \frac{5}{2}x - 7.75 \Rightarrow x = 3.1 \Rightarrow W(3.1, 0)$

$\therefore$ when $y = 1.5 \Rightarrow 1.5 = \frac{5}{2}x - 7.75 \Rightarrow x = 3.7 \Rightarrow Z(3.7, 1.5)$

Therefore the bottom base is $7 - 3.1 = 3.9$, top base is $7 - 3.7 = 3.3$.
Area of cell $E = \frac{1}{2}(b_1 + b_2)(h) = \frac{1}{2}(3.9 + 3.3)(1.5) = 5.4\ \text{m}^2$

6. The centre of the LEC must be found by checking distances to the nearest site of likely candidates. Since all possible centres are vertices adjacent to site E, we can find the distance of each vertex to site E:

$V(10,16)$: $d = \sqrt{(10-11)^2 + (16-8)^2} = \sqrt{1+64} = \sqrt{65}$

$W(15,15)$: $d = \sqrt{(15-11)^2 + (15-8)^2} = \sqrt{16+49} = \sqrt{65}$

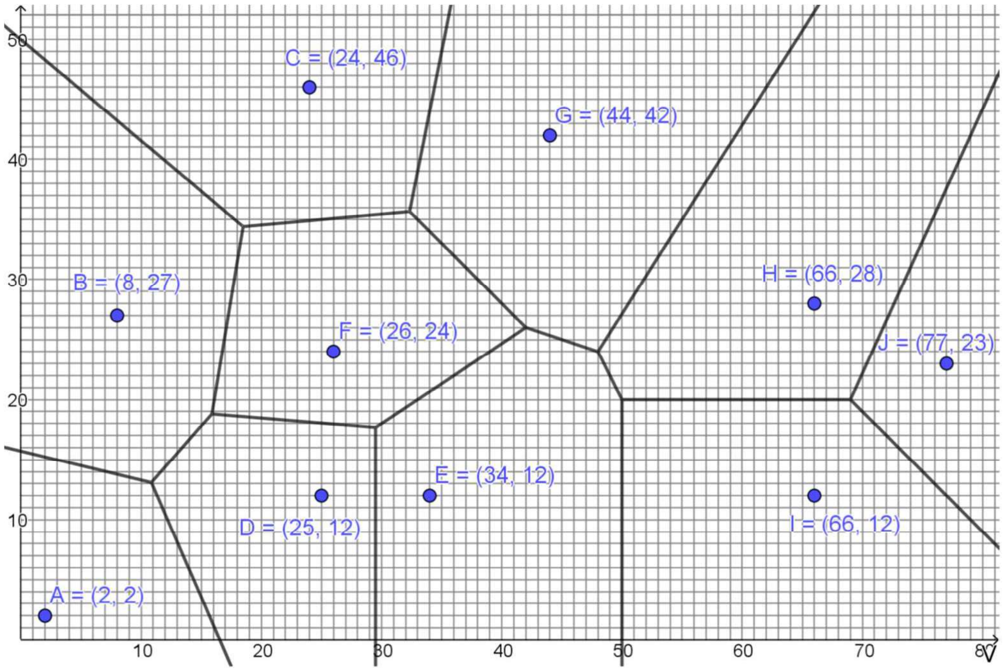
$X(16,14)$: $d = \sqrt{(16-11)^2 + (14-8)^2} = \sqrt{25+36} = \sqrt{61}$

$Y(16,8)$: $d = \sqrt{(16-11)^2 + (8-8)^2} = \sqrt{25+0} = \sqrt{25}$

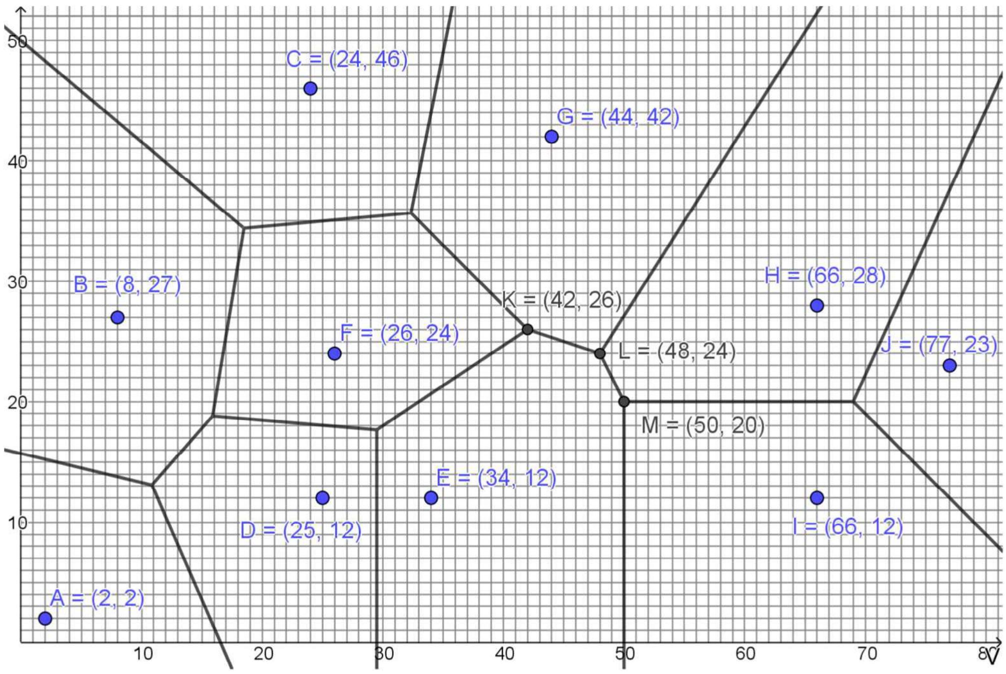
$Z(6,8)$: $d = \sqrt{(6-11)^2 + (8-8)^2} = \sqrt{25+0} = \sqrt{25}$

Since $V(10,16)$ and $W(15,15)$ are both $\sqrt{65}$ units from their nearest site, there are two largest empty circles, centred at each point, both with radius $\sqrt{65} \approx 8.06$.

7. **(a)** After adding site H, the Voronoi diagram should look like this:



(b) The region bounded by points E, G, H, and I contains three vertices, labelled as K, L, and M on the diagram below:

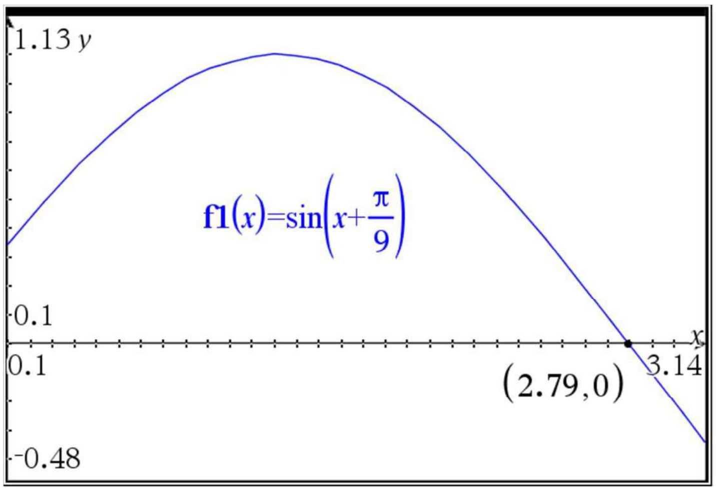


Since all three are adjacent to vertex E, we will find the distance of each to vertex E. For K, $d_K = \sqrt{(42-34)^2 + (26-12)^2} = \sqrt{260}$; for L, $d_L = \sqrt{(48-34)^2 + (24-12)^2} = \sqrt{340}$; for M, $d_M = \sqrt{(50-34)^2 + (20-12)^2} = \sqrt{320}$.

Therefore, the best location for a new bank branch is at $L(48, 24)$.

Practice question answers

1. **(a)** GDC solution:

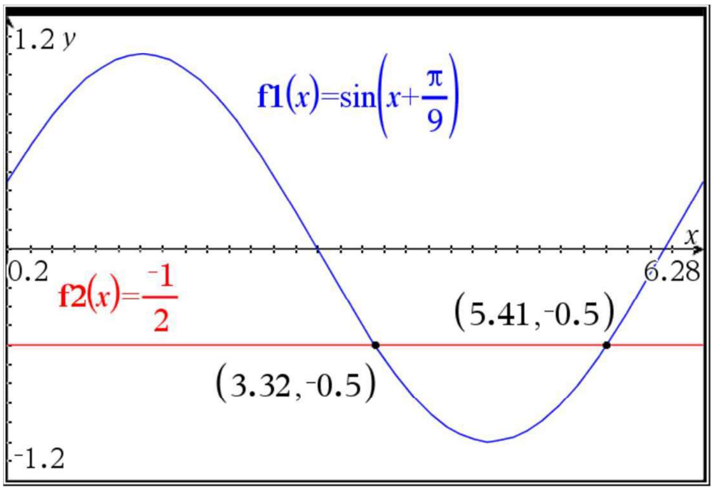


$$\therefore x = 2.79$$

Algebraic solution:

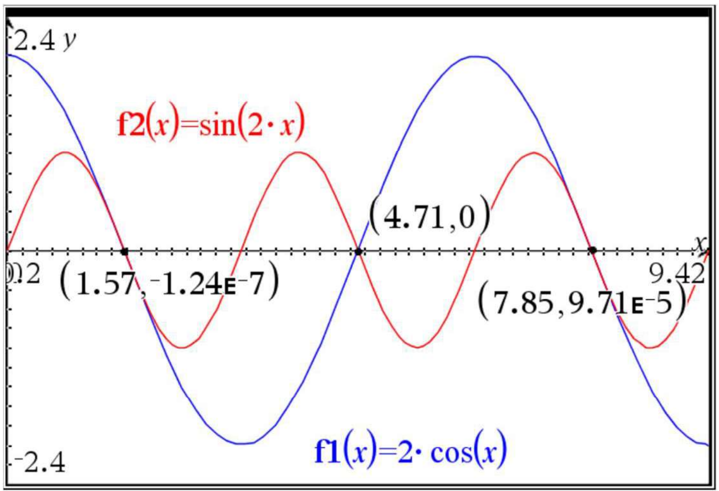
$$\sin\left(x+\frac{\pi}{9}\right)=0 \;\Rightarrow\; x+\frac{\pi}{9}=\pi \;\Rightarrow\; x=\frac{8\pi}{9}\approx 2.79$$

(b) GDC solution:



$$\therefore x = 3.32,\ 5.41$$

2. GDC solution:



$$\therefore x = 1.57,\ 4.71,\ 7.85$$

3. **(a)** $\widehat{ACB} = \theta r = \frac{\pi}{3}(10) = \frac{10\pi}{3} \approx 10.5\ \text{cm}$

(b) Area $= \frac{1}{2}\left(\frac{\pi}{3}\right)(10^2) - \frac{1}{2}\left(\frac{\pi}{3}\right)(8^2) = 6\pi \approx 18.8\ \text{cm}^2$

4. **(a)** $27 = \frac{1}{2}(1.5)r^2 \Rightarrow r = 6\ \text{cm}$

(b) $\widehat{ACB} = 1.5(6) = 9\ \text{cm}$

5. **(a)** $3\pi = \frac{2\pi}{9}r \Rightarrow r = 13.5\ \text{cm}$

(b) $2(13.5) + 3\pi \approx 36.4\ \text{cm}$

(c) $\frac{1}{2}\left(\frac{2\pi}{9}\right)(13.5^2) = \frac{81\pi}{4} \approx 63.6\ \text{cm}$

6. $\theta r = \frac{2\pi}{3} \Rightarrow \theta = \frac{2\pi}{3r},\ \frac{1}{2}\theta r^2 = \frac{4\pi}{3} \Rightarrow \frac{1}{2}\left(\frac{2\pi}{3r}\right)r^2 = \frac{4\pi}{3} \Rightarrow \frac{\pi r}{3} = \frac{4\pi}{3} \Rightarrow r = 4,$

$\theta = \frac{2\pi}{3(4)} = \frac{\pi}{6}$

7. $\theta r = 24 \Rightarrow \theta = \frac{24}{r},\ \frac{1}{2}\theta r^2 = 180 \Rightarrow \frac{1}{2}\left(\frac{24}{r}\right)r^2 = 180 \Rightarrow 12r = 180 \Rightarrow r = 15\ \text{cm},$

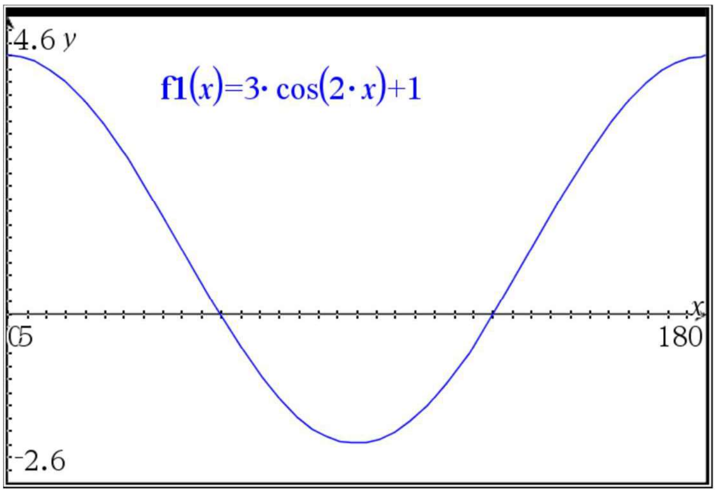
$\theta = \frac{24}{15} = \frac{8}{5} = 1.6\ \text{cm}$

8. **(a)** 15.625 – 3.125 = 12.5 hours

(b) Reading from graph, $\frac{7-1}{2} = 3\ \text{m}$

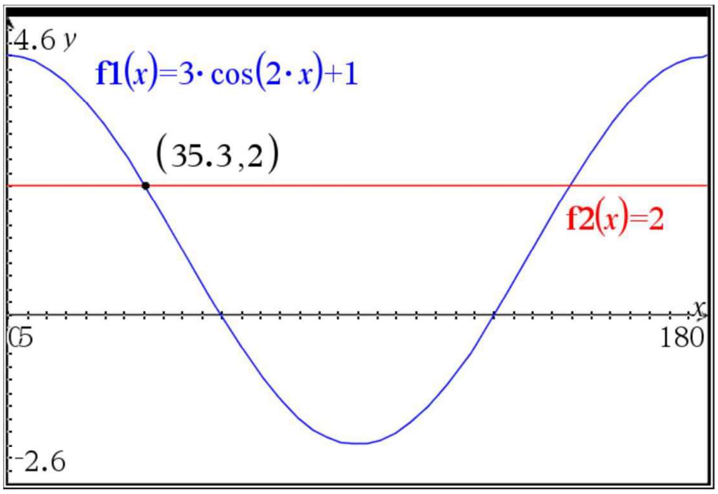
(c) approximately $2 \times 3.25 = 6.5$ hours (between 6.5 and 7 hours)

9. **(a)** GDC screen shown.



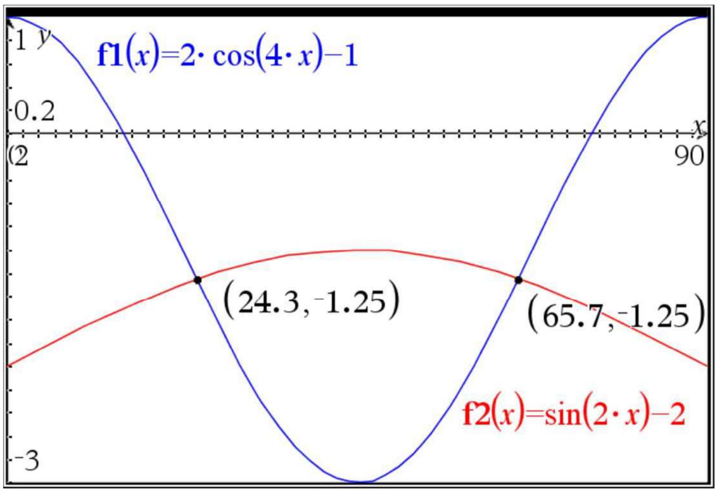
(b) Period = $\frac{360}{2} = 180°$

(c) Use GDC to obtain



$\therefore x = 35.3°$

10. **(a)** GDC screen shown. (Intersection coordinates not required.)



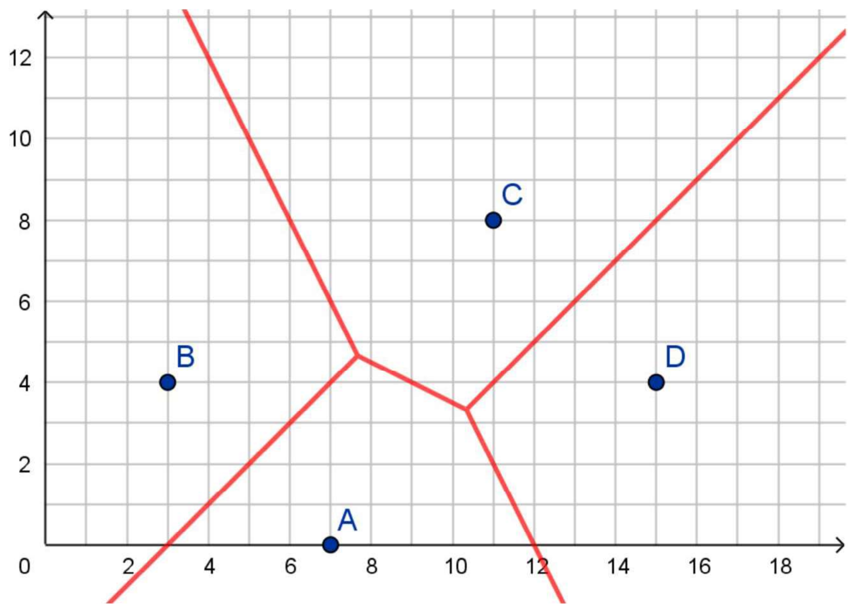
(b) 2

(c) any value $0 \leq x < 24.3°$ or $65.7° < x \leq 90°$

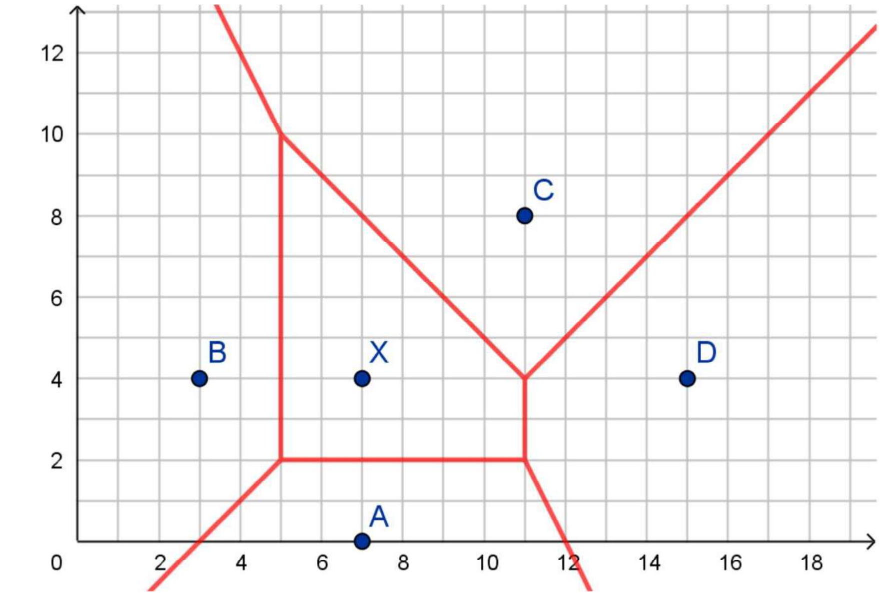
(d) Reading from the graph, **(i)** $a = 24.3°$ **(ii)** $b = 65.7°$

11.

(a)



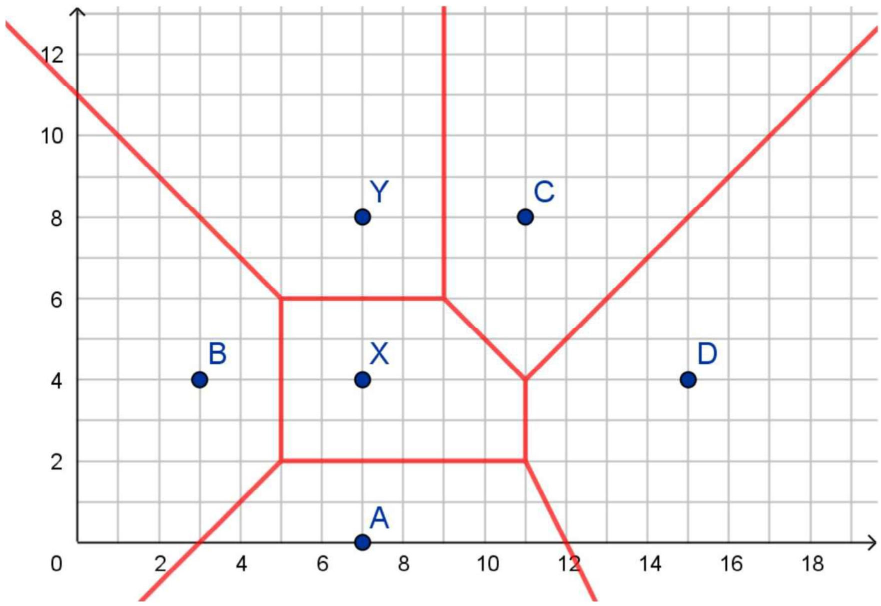
(b)



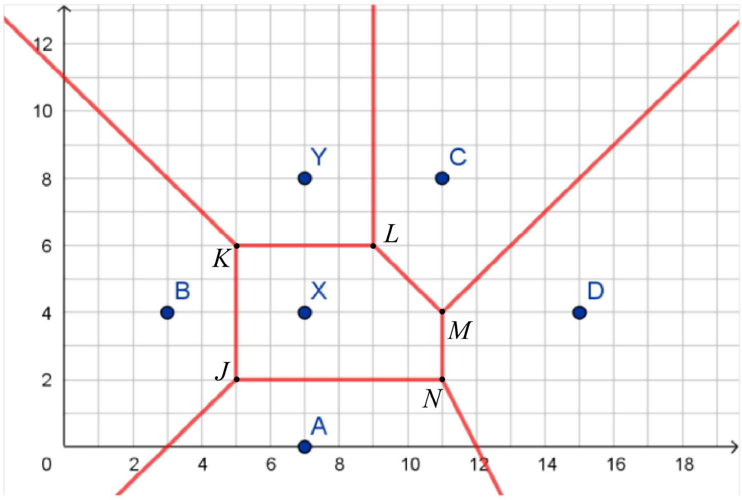
(c) By inspection, the equation is $y = 2$

(d) Midpoint of $\overline{XC}$ is $\left(\frac{7+11}{2}, \frac{4+8}{2}\right) = (9, 6)$. Gradient of $\overline{XC}$ is $m = \frac{8-4}{11-7} = \frac{4}{4} = 1$ therefore gradient of edge is -1. Therefore equation of edge is $y - 6 = -1(x - 9) \Rightarrow \quad y = -x + 15$.

(e)

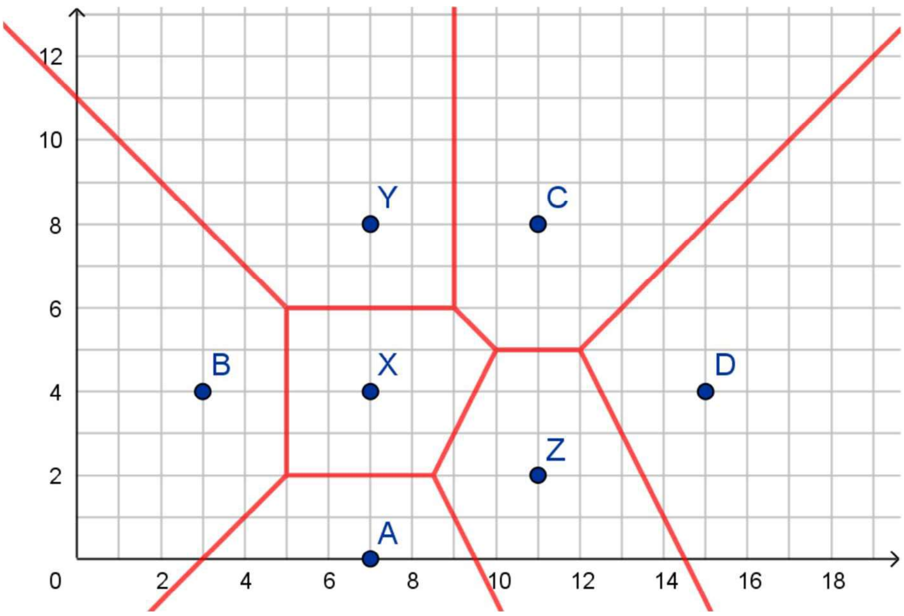


(f) There are 5 vertices to test, labelled as points *J*, *K*, *L*, *M*, and *N*.



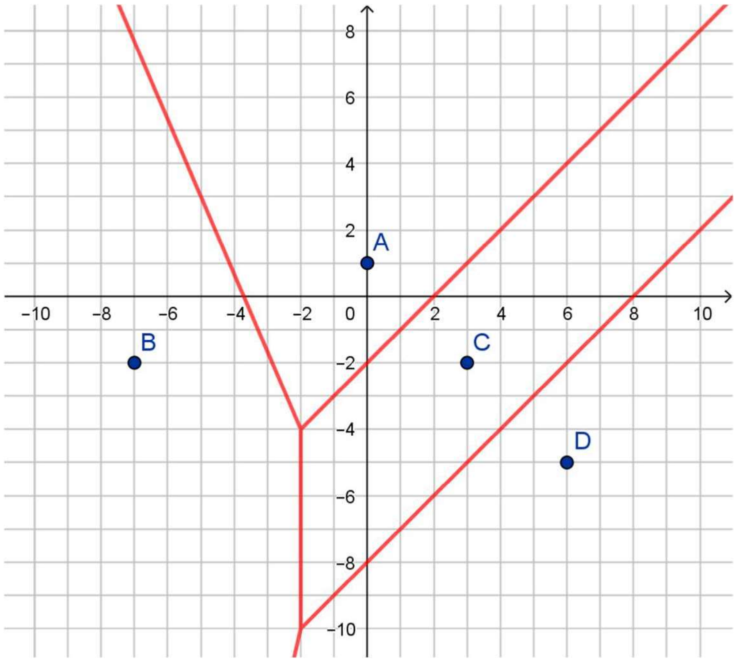
Vertices *J*, *K*, *L* are each $\sqrt{2^2+2^2}=\sqrt{8}$ units from site *X*. Vertex *M* is 4 units from site *X*. Vertex *N* is $\sqrt{2^2+4^2}=\sqrt{20}$. Therefore the centre of the largest empty is at vertex *N* at (11, 2).

(g)

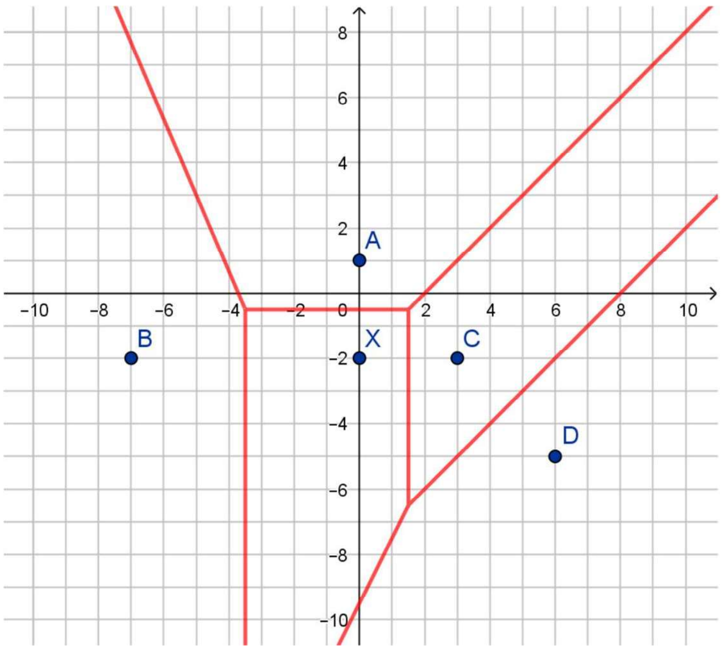


(h) Midpoint of $\overline{ZD}$ is $\left(\frac{11+15}{2},\frac{2+4}{2}\right)=(13,3)$. Gradient of $\overline{ZD}$ is $m=\frac{4-2}{15-11}=\frac{1}{2}$ therefore gradient of edge is –2. Therefore equation of edge is $y-3=-2(x-13) \Rightarrow 2x+y=29$

12. **(a)**

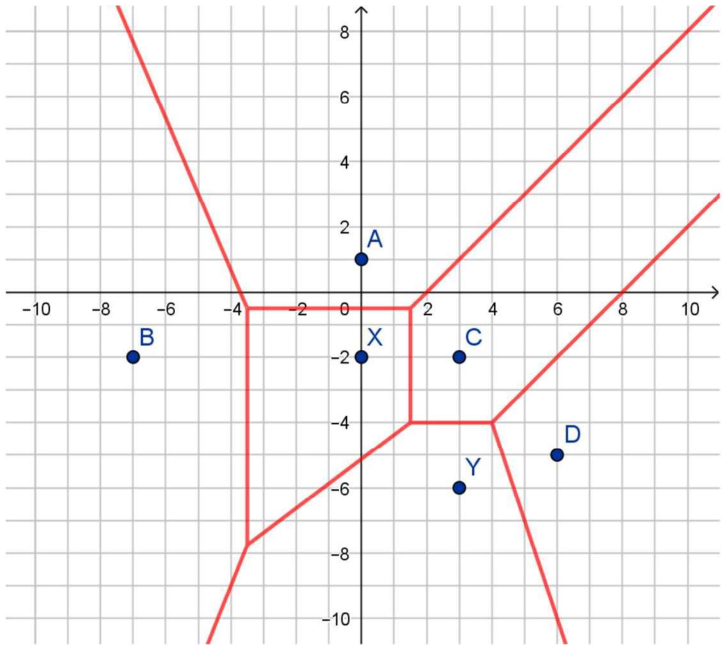


(b)



(c) By inspection, the equation is $x = 1.5$

(d)



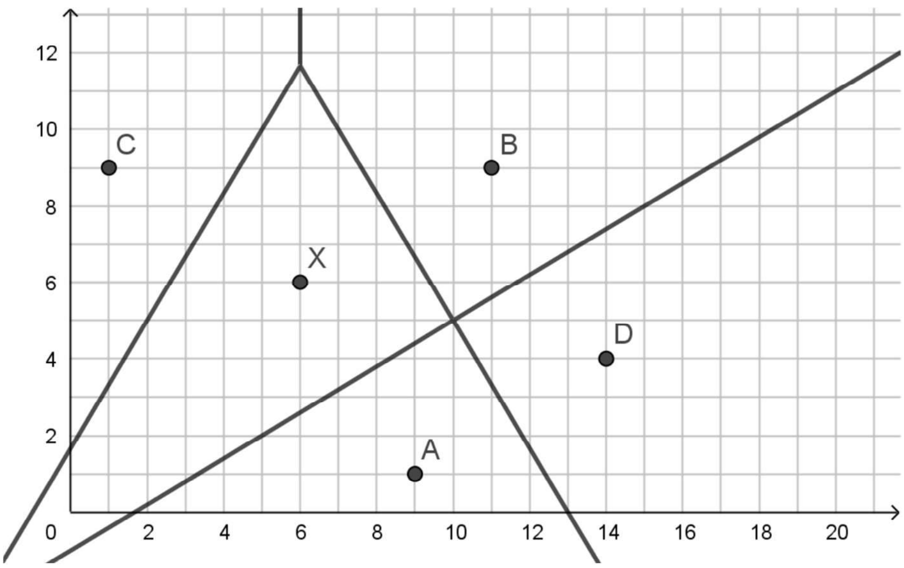
(e) Midpoint of $\overline{XY}$ is $\left(\frac{0+3}{2}, \frac{-2+-6}{2}\right) = (1.5, -4)$. Gradient of $\overline{XY}$ is $m = \frac{-6-(-2)}{3-0} = \frac{-4}{3}$ therefore gradient of edge is $\frac{3}{4}$. Therefore equation of edge is: $y-(-4) = \frac{3}{4}(x-1.5) \Rightarrow -\frac{3}{4}x + y = \frac{-41}{8} \Rightarrow 6x - 8y = 41$

13. **(a)** the largest empty circle (LEC) must be centred on a vertex

(b) All vertices in the diagram are adjacent to cell K and (15,8) is the vertex farthest from site K, therefore it must be the centre of the largest empty circle.

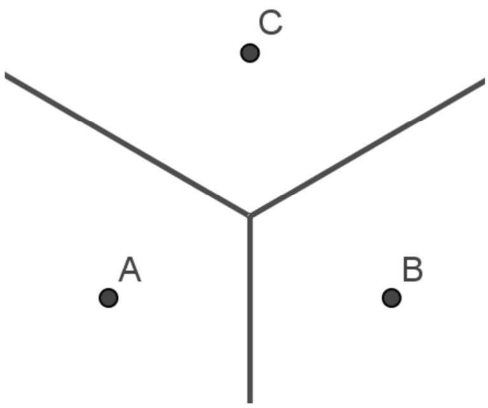
14. **(a)** The LEC must be at (6,6). The LEC must be on a vertex and (6,6) is the vertex farthest from any site.

(b)

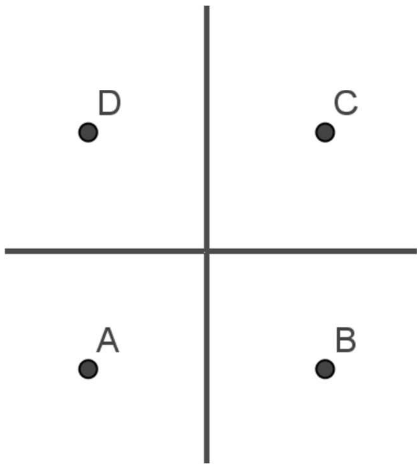


15. Since A, B, C, and D are collinear, the perpendicular bisectors between each pair of points must be parallel. Hence, the edges in the Voronoi diagram are parallel which implies there are no vertices.

16. **(a)** The Voronoi diagram has 3 edges that meet in a single vertex as shown.



(b) The Voronoi diagram has 4 edges that meet in a single vertex as shown.



(c) The Voronoi diagram for a regular n-gon will have n edges that meet at a single vertex in the centre of the n-gon; each edge bisects a side of the polygon. The angle between each pair of adjacent edges will be $\frac{360}{n}$.

Exercise 6.1

1. **(a)** $\sqrt{-36} = \sqrt{36} \times \sqrt{-1} = 6\text{i}$

(b) $\sqrt{-12} = \sqrt{4} \times \sqrt{3} \times \sqrt{-1} = 2\sqrt{3}\text{i}$

(c) $\sqrt{-63} = \sqrt{9 \times 7} \times \sqrt{-1} = 3\sqrt{7}\text{i}$

(d) $\sqrt{-8} \times \sqrt{-18} = \left(\sqrt{4} \times \sqrt{2} \times \sqrt{-1}\right)\left(\sqrt{9} \times \sqrt{2} \times \sqrt{-1}\right) = 2\sqrt{2\text{i}} \times 3\sqrt{2\text{i}} = 12\text{i}^2 = -12$

2. **(a)** $4 + \sqrt{-9} = 4 + 3\text{i}$

(b) $-3 - \sqrt{-4} = -3 - 2\text{i}$

(c) $-18 + \sqrt{-9 \times 2} = -18 + 3\sqrt{2}\text{i}$

(d) $4\sqrt{2} - \sqrt{-4 \times 2} = 4\sqrt{2} - 2\sqrt{2}\text{i}$

(e) $\sqrt{-1 \times 4} = 2\text{i}$

(f) $12 + \sqrt{-4 \times 3} = 12 + 2\sqrt{3}\text{i}$

(g) -7

(h) $2\text{i} + \left(\sqrt{-4}\right)\text{i} = 2\text{i} + 2\text{i} \times \text{i} = 2\text{i} - 2$

3. **(a)** $\text{i}^7 = \left(\text{i}^2\right)^3 \times \text{i} = \left(-1\right)^3 \times \text{i} = -\text{i}$

(b) $\text{i}^{66} = (\text{i}^2)^{33} = \left(-1\right)^{33} = \left(-1\right)^{32} \times \left(-1\right) = -1$

(c) $\text{i}^{721} = \left(\text{i}^2\right)^{360} \times \text{i} = \text{i}$

(d) $\text{i}^{-24} = \left(\text{i}^{2 \times 12}\right)^{-1} = ((-1)^{12})^{-1} = 1$

4. **(a)** $x = \dfrac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 10}}{2} = -1 \pm \dfrac{1}{2}\sqrt{-36} = -1 \pm 3\text{i}$

(b) $x = \dfrac{-4 \pm \sqrt{4^2 - 4 \times 1 \times 7}}{2} = 2 \pm \dfrac{1}{2}\sqrt{-12} = 2 \pm 2\sqrt{3}\text{i}$

(c) $x = \dfrac{-4 \pm \sqrt{4^2 - 4 \times 2 \times 5}}{2} = -1 \pm \dfrac{1}{4}\sqrt{-24} = -1 \pm \dfrac{1}{2}\sqrt{6}\text{i}$

(d) $x = \dfrac{2 \pm \sqrt{2^2 - 4 \times 1 \times 10}}{2} = 1 \pm \dfrac{1}{2}\sqrt{-36} = 1 \pm 3\text{i}$

(e) $x = \dfrac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 10}}{2} = -3 \pm \dfrac{1}{2}\sqrt{-4} = -3 \pm \text{i}$

(f) $x = \dfrac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 3}}{2} = -\dfrac{3}{2} \pm \dfrac{1}{2}\sqrt{-3} = -\dfrac{3}{2} \pm \dfrac{\sqrt{3}}{2}\mathrm{i}$

5. **(a)** $(x^4 - 1) = (x^2 + 1)(x^2 - 1) = (x^2 + 1)(x - 1)(x + 1)$

So, the four roots are $x = \pm 1, \quad x = \pm\mathrm{i}$

(b) $(x^3 - 1) = (x - 1)(x^2 + x + 1)$

so $x - 1 = 0$ or $x^2 + x + 1 = 0$

so $x = 1$ or $x = \dfrac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 1}}{2} = -1 \pm \dfrac{1}{2}\sqrt{-3} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}\mathrm{i}$

So, the three roots are 1, $-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}\mathrm{i}, \quad -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\mathrm{i}$

Exercise 6.2

1. **(a)** $2 - 3 - 4\mathrm{i} + 2\mathrm{i} = -1 - 2\mathrm{i}$

(b) $-1 + 3 + \mathrm{i} - 2i = 2 - \mathrm{i}$

(c) $2\sqrt{2} - \sqrt{2} + \mathrm{i} - 2\mathrm{i} = \sqrt{2} - \mathrm{i}$

2. **(a)** $1 + 4\mathrm{i} - 4\mathrm{i} - 16\mathrm{i}^2 = 1 + 16 = 17$

(b) $-12 + 2\sqrt{3}\mathrm{i} - 2\sqrt{3}\mathrm{i} + \mathrm{i}^2 = -12 - 1 = -13$

(c) $9 + 12\mathrm{i} - 12\mathrm{i} - 16\mathrm{i}^2 = 9 + 16 = 25$

3. **(a)** $(4 - 3\mathrm{i})(4 + 3\mathrm{i}) = 16 - 9\mathrm{i}^2 = 16 + 9 = 25$

(b) $(-5 + 12\mathrm{i})(-5 - 12\mathrm{i}) = 25 - 144\mathrm{i}^2 = 25 + 144 = 169$

(c) $(-4 - 2\sqrt{5}\mathrm{i})(-4 + 2\sqrt{5}\mathrm{i}) = 16 - 20\mathrm{i}^2 = 16 + 20 = 36$

4. **(a)** $\dfrac{5}{2 - \mathrm{i}} \times \dfrac{2 + \mathrm{i}}{2 + \mathrm{i}} = \dfrac{10 + 5\mathrm{i}}{4 + 1} = 2 + \mathrm{i}$

(b) $\dfrac{1 - 2\mathrm{i}}{1 + 2\mathrm{i}} \times \dfrac{1 - 2\mathrm{i}}{1 - 2\mathrm{i}} = \dfrac{1 - 4\mathrm{i} + 4\mathrm{i}^2}{1 + 4} = \dfrac{1 - 4\mathrm{i} - 4}{5} = -\dfrac{3}{5} - \dfrac{4}{5}\mathrm{i}$

(c) $\dfrac{2 - 4\mathrm{i}}{-3 + 2\mathrm{i}} \times \dfrac{-3 - 2\mathrm{i}}{-3 - 2\mathrm{i}} = \dfrac{-6 + 8\mathrm{i} + 8\mathrm{i}^2}{9 + 4} = \dfrac{-14 + 8\mathrm{i}}{13} = -\dfrac{14}{13} + \dfrac{8}{13}\mathrm{i}$

5. **(a)** Since a, b, and c are real, then the other root is $1+\mathrm{i}$.

Assume $a=1$. Then

$-b=$ sum of roots $=1-\mathrm{i}+1+\mathrm{i}=2$

$c=$ product of roots $=(1+\mathrm{i})(1-\mathrm{i})=1+1=2$

so function is $y=x^2-2x+2$

(b) other root is $-7-\mathrm{i}$

$-b=$ sum of roots $=-7+\mathrm{i}-7-\mathrm{i}=-14$

$c=$ product of roots $=(-7+\mathrm{i})(-7-\mathrm{i})=49+1=50$

so function is $y=x^2+14x+50$

(c) other root is $-2\sqrt{3}-3\mathrm{i}$

$-b=$ sum of roots $=-2\sqrt{3}+\sqrt{3}\mathrm{i}-2\sqrt{3}-\sqrt{3}\mathrm{i}=-4\sqrt{3}$

$c=$ product of roots $=\left(-2\sqrt{3}+\sqrt{3}\mathrm{i}\right)\left(-2\sqrt{3}-\sqrt{3}\mathrm{i}\right)=12+3=15$

so function is $y=x^2+4\sqrt{3}x+15$

6. **(a)** assuming $a=1$, $-b=2+\sqrt{3}+2-\sqrt{3}=4$, $c=2^2-3=1$

So function is $y=x^2-4x+1$

(b) $-b=\frac{1}{2}+\frac{1}{2}=1$, $c=\left(\frac{1-5}{4}\right)=-1$

So function is $y=x^2-x-1$

(c) $-b=-2$, $c=1^2+2^2=5$

So function is $y=x^2+2x+5$

(d) $-b=3$, $c=\frac{9}{4}+\frac{25}{4}=\frac{34}{4}$

So function is $y=x^2-3x+\frac{17}{2}$

(e) $-b=1$, $c=\frac{1+5}{4}=\frac{3}{2}$

So function is $y=x^2-x+\frac{3}{2}$

(f) $-b=-4\sqrt{3}$, $c=4\times3+3=15$

So function is $y=x^2+4\sqrt{3}x+15$

7. The conditions in this exercise do not include real coefficients, thus, the roots do not have to be conjugates.

(a) $-b = 8 + \text{i}$, $c = (5+2\text{i})(3-\text{i}) = 15 + \text{i} + 2$

So, function is $y = x^2 - (8+\text{i})x + 17 + \text{i}$

(b) $b = 0$, $c = (3+2\text{i})(-3-2\text{i}) = -9 - 12\text{i} + 4 = -5 - 12\text{i}$

So, function is $y = x^2 - 5 - 12\text{i}$

(c) $b = 0$, $c = \left(3+\sqrt{2}\text{i}\right)\left(-3-\sqrt{2}\text{i}\right) = -9 - 6\sqrt{2}\text{i} + 2 = -7 - 6\sqrt{2}\text{i}$

So, function is $y = x^2 - 7 - 6\sqrt{2}\text{i}$

8. Let $z = a + b\text{i}$.

Then $(2+3\text{i})(a+b\text{i}) = 7 + \text{i}$

$(2a-3b) + (2b+3a)\text{i} = 7 + \text{i}$

Equate coefficients of real and imaginary coefficients:

$2a - 3b = 7$, $3a + 2b = 1$

Solve simultaneously: $a = \dfrac{17}{13}$, $b = \dfrac{-19}{13}$

So, $z = \dfrac{17}{13} - \dfrac{19}{13}\text{i}$

9. $2x + 2\text{i} + xy\text{i} - y = 1 + 3\text{i}$

Equate coefficients of real and imaginary coefficients:

$2x - y = 1$, $xy + 2 = 3$

Solve simultaneously: $y = 2x - 1$, $y = \dfrac{1}{x}$

$\Rightarrow 2x^2 - x - 1 = 0$

$\Rightarrow \quad x = \dfrac{1 \pm 3}{4} = 1 \text{ or } -\dfrac{1}{2}$

$\Rightarrow \quad x = 1,\ y = 1 \text{ or } x = -\dfrac{1}{2},\ y = -2$

10. **(a)** $\left(1+\sqrt{3}\mathrm{i}\right)^3 = 1+3\times1^2\times\sqrt{3}\mathrm{i}+3\times1\times\left(\sqrt{3}\mathrm{i}\right)^2+\left(\sqrt{3}\mathrm{i}\right)^3$

$= 1+3\sqrt{3}\mathrm{i}-9-3\sqrt{3}\mathrm{i} = -8$

(b) $z^{6n} = \left(z^3\right)^{2n} = \left(-8\right)^{2n} = \left(-1\right)^2\times\left(8\right)^{2n} = 8^{2n}$

(c) $z^{48} = z^{6\times8} = 8^{2\times8} = 8^{16} = (2^3)^{16} = 2^{48}$

11. **(a)** $\left(-\sqrt{2}+\sqrt{2}\mathrm{i}\right)^2 = 2-4\mathrm{i}-2 = -4\mathrm{i}$

(b) $z^{4k} = \left(z^2\right)^{2k} = \left(-4\mathrm{i}\right)^{2k} = (16\mathrm{i}^2)^k = (-16)^k$

(c) $z^{46} = z^{4\times11}\times z^2 = \left(-16\right)^{11}\times -4\mathrm{i} = (16)^{11}\times4\mathrm{i} = (4^2)^{11}\times4\mathrm{i} = 4^{23}\mathrm{i} = 2^{46}\mathrm{i}$

12. Let $z = a+b\mathrm{i}$

Then $|z+4\mathrm{i}|^2 = a^2+(b+4)^2$

and $|z+\mathrm{i}|^2 = a^2+(b+1)^2$

$\sqrt{a^2+(b+4)^2} = 2\sqrt{a^2+(b+1)^2}$

$\Rightarrow a^2+(b+4)^2 = 4\times(a^2+\left(b+1\right)^2)$

$\Rightarrow a^2+b^2+8b+16 = 4a^2+4b^2+8b+4$

$\Rightarrow 3a^2+3b^2 = 12$

$\Rightarrow a^2+b^2 = 4$

$\Rightarrow |z|^2 = 4 \Rightarrow |z| = 2$

13. $\dfrac{2\mathrm{i}}{2-\sqrt{2}\mathrm{i}} = \dfrac{2\mathrm{i}}{2-\sqrt{2}\mathrm{i}}\times\dfrac{2+\sqrt{2}\mathrm{i}}{2+\sqrt{2}\mathrm{i}} = \dfrac{4\mathrm{i}-2\sqrt{2}}{4+2} = -\dfrac{2\sqrt{2}}{6}+\dfrac{2}{6}\mathrm{i} = -\dfrac{\sqrt{2}}{3}+\dfrac{1}{3}\mathrm{i}$

$\Rightarrow \dfrac{2\mathrm{i}}{2-\sqrt{2}\mathrm{i}}+3 = -\dfrac{\sqrt{2}}{3}+\dfrac{2}{3}\mathrm{i}+3 = \dfrac{9-\sqrt{2}}{3}+\dfrac{2}{3}\mathrm{i}$

14. expand and equate Re and Im coefficients:

Re: $4x+7y=3$ Im: $-7x+4y=2$

Solve simultaneously: $x = -\dfrac{2}{65}$, $y = \dfrac{29}{65}$

15. **(a)** Let $z = a + bi$

Then $(a + bi + 1) \times i = 3(a + bi) - 2$

Equate Re and Im coefficients:

Re: $-b = 3a - 2$ Im: $a + 1 = 3b$

Solve simultaneously: $a = \frac{1}{2}$, $b = \frac{1}{2}$

So $z = \frac{1}{2} + \frac{1}{2}i$

(b) $(a + bi) \times (a + bi) = 3 - 4i$

Equate Re and Im coefficients:

Re: $a^2 - b^2 = 3$ Im: $2ab = -4$

Solve simultaneously: $b = -\frac{2}{a}$, substitute: $a^2 - \left(\frac{-2}{a}\right)^2 = 3$

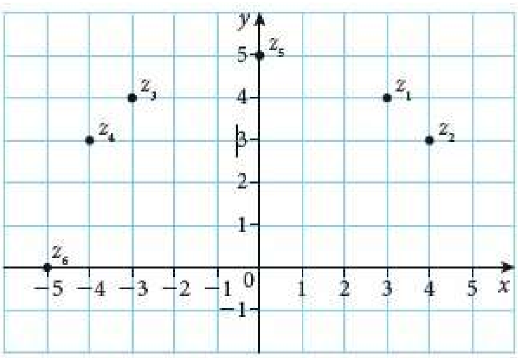
$a^2 - \frac{4}{a^2} = 3 \Rightarrow a^4 - 4 = 3a^2 \Rightarrow a^4 - 3a^2 - 4 = 0 \Rightarrow$

$(a^2 - 4)(a^2 + 1) = 0$, real roots $a = \pm 2$

So $z = 2 - i$ *or* $z = -2 + i$

Exercise 6.3

1. **(a)** modulus of all these complex numbers = 5



(b) e.g. $-4 - 3i$

(c) Yes. $|z|^2 = (2\sqrt{3})^2 + (\sqrt{13})^2 = 12 + 13 = 25$, $|z| = 5$

(d) $a^2 + (3\sqrt{2})^2 = 25$, $a^2 = 25 - 18$, $a = \pm\sqrt{7}$

So complex numbers are $\pm\sqrt{7} + 3\sqrt{2}i$

2. **(a)** $|z|^2 = (1)^2 + (1)^2, \quad |z| = \sqrt{2}$

(b) $|z|^2 = (\sqrt{3})^2 + (1)^2 \quad , |z| = 2$

(c) $|z|^2 = (2)^2, \quad |z| = 2$

(d) $|z|^2 = (-1)^2, \quad |z| = 2$

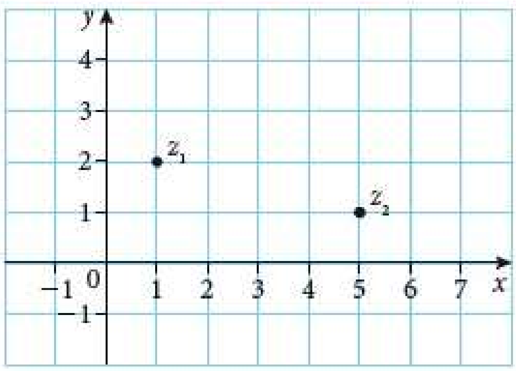
(e) $|z|^2 = (-5)^2 + (-12)^2, \quad |z| = 13$

(f) $|z|^2 = (-5)^2 + (12)^2, \quad |z| = 13$

(g) $|z|^2 = (-21)^2 + (20)^2, \quad |z| = \sqrt{841} = 29$

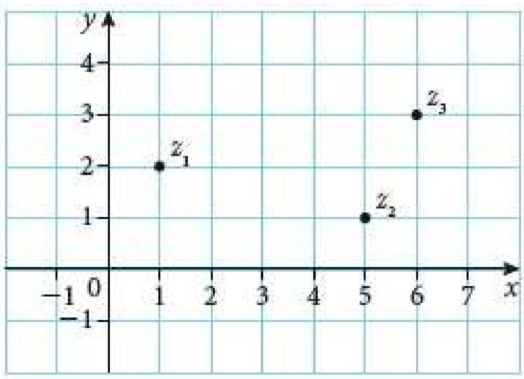
(h) $|z|^2 = \left(-2\sqrt{3}\right)^2 + (4\sqrt{6})^2 = 12 + 96, \quad |z| = \sqrt{108} = 6\sqrt{3}$

3. **(a)**

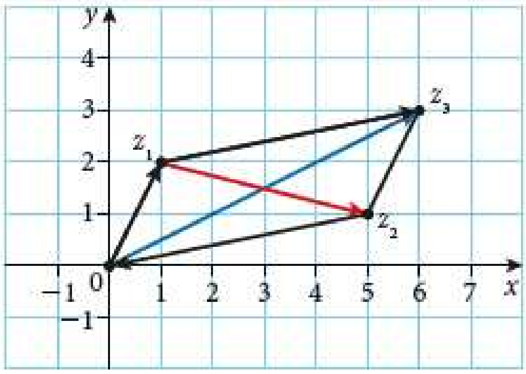


(b) $|z_1| = \sqrt{5} \quad ,|z_2| = \sqrt{26}$

(c) $z_3 = 6 + 3\mathrm{i}$



(d)



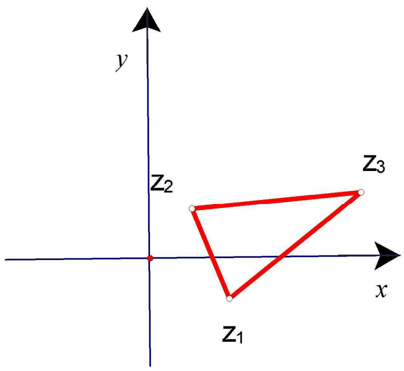
(e) diagonal $Oz_3 = z_1 + z_2 = 6 + 3\text{i}$, (sum)

diagonal $z_1z_2 = z_2 - z_1 = 4 - \text{i}$ (difference)

4. In **3(d),** the triangle with vertices O, z_1, z_3 has sides $|z_1|, |z_2|, |z_1 + z_2|$.

In any triangle, the sum of 2 sides is greater than the 3rd side.

Hence $|z_1| + |z_2| \geq |z_1 + z_2|$ $\quad$ ($|z_1| + |z_2| = |z_1 + z_2| \Rightarrow Oz_1, Oz_2$ parallel)



5. For z_1: $|z_1| = \sqrt{12 + 4} = 4$, $\tan\theta = -\frac{1}{\sqrt{3}}$; hence, $\arg z_1 = -\frac{\pi}{6}$

For z_2: $|z_2| = \sqrt{4 + 4} = 2\sqrt{2}$, $\tan\theta = 1$; hence, $\arg z_2 = \frac{\pi}{4}$

Since $z_3 = z_1 z_2$, $|z_3| = 8\sqrt{2}$, and $\arg z_3 = -\frac{\pi}{6} + \frac{\pi}{4} = \frac{\pi}{12}$

For the area: There are several methods available. Some will give you numerical approximate answers using a GDC and others may require knowledge about matrix applications.

Below are 2 such methods.

Method I: (A GDC can be used for calculations)

We can find the side lengths and the angles of the triangle:

$$|z_1z_2| = \sqrt{\left(2-2\sqrt{3}\right)^2 + 4^2} = \sqrt{32-8\sqrt{3}}$$

$$|z_1z_3| = \sqrt{\left(2\sqrt{3}+4\right)^2 + \left(4\sqrt{3}-2\right)^2} = \sqrt{80}$$

$$|z_2z_3| = \sqrt{\left(4\sqrt{3}+2\right)^2 + \left(4\sqrt{3}-6\right)^2} = \sqrt{136-32\sqrt{3}}$$

Angle θ between sides $|z_1z_2|$ and $|z_1z_3|$ is:

$$\cos\theta = \frac{|z_1z_2|^2 + |z_1z_3|^2 - |z_2z_3|^2}{2|z_1z_2||z_1z_3|} \Rightarrow \theta = 76.6689^\circ$$

Hence, the area is: $A = \frac{1}{2}|z_1z_2||z_1z_3|\sin\theta \approx 18.5$.

Method II:

There is a formula for the area of a triangle using the coordinates of the vertices. If the vertices are: $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$, then the area is:

$$A = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} 2 & 2 & 1 \\ 2\sqrt{3} & -2 & 1 \\ 4\left(1+\sqrt{3}\right) & 4\left(\sqrt{3}-1\right) & 1 \end{vmatrix} = 22 - 2\sqrt{3} \approx 18.5$$

6. **(a)** Let $z = x + y\mathrm{i}$

Then $|z|^2 = x^2 + y^2$

So equation of locus is $x^2 + y^2 = 9$: circle radius 3 centre origin

(b) $x + y\mathrm{i} = x - y\mathrm{i}$

$\Rightarrow y = 0$ and x may take any value: locus is the Im axis

(c) $x + y\mathrm{i} + x - y\mathrm{i} = 8$

$\Rightarrow x = 4$ and y may take any value: line $x = 4$

(d) $|z-3|^2 = (x-3)^2 + y^2 = 4$: circle radius 2, centre (3, 0)

(e) If z has an imaginary component, then $(z-1)$ and $(z-3)$ form 2 sides of a triangle with base of length 2, (1, 0) to (3, 0).

$\Rightarrow |z-1|+|z-3|>2$

So to satisfy the equality $|z-1|+|z-3|=2$, zust have no imaginary component and lie between (1, 0) and (3, 0)

So the locus is a line segment from (1, 0) to (3, 0)

Exercise 6.4

1. **(a)** $r=\sqrt{3^2+3^2}=3\sqrt{2}, \quad \theta=\tan^{-1}\left(\frac{3}{3}\right)=\frac{\pi}{4}, \quad z=3\sqrt{2}\text{cis}\frac{\pi}{4}$

(b) $r=\sqrt{(-3)^2+3^2}=3\sqrt{2}, \quad \theta=\tan^{-1}\left(\frac{-3}{3}\right)=\frac{3\pi}{4}, \quad z=3\sqrt{2}\text{cis}\frac{3\pi}{4}$

(c) $r=\sqrt{3^2+(-3)^2}=3\sqrt{2}, \quad \theta=\tan^{-1}\left(\frac{3}{-3}\right)=\frac{7\pi}{4}, \quad z=3\sqrt{2}\text{cis}\frac{7\pi}{4}$

(d) $r=\sqrt{(-3)^2+(-3)^2}=3\sqrt{2}, \quad \theta=\tan^{-1}\left(\frac{-3}{-3}\right)=\frac{5\pi}{4}, \quad z=3\sqrt{2}\text{cis}\frac{5\pi}{4}$

(e) $r=\sqrt{5^2+(5\sqrt{3})^2}=10, \quad \theta=\tan^{-1}\left(\frac{5\sqrt{3}}{5}\right)=\frac{\pi}{3}, \quad z=10\text{cis}\frac{\pi}{3}$

(f) $r=\sqrt{(5\sqrt{3})^2+5^2}=10, \quad \theta=\tan^{-1}\left(\frac{5}{5\sqrt{3}}\right)=\frac{\pi}{6}, \quad z=10\text{cis}\frac{\pi}{6}$

(g) $r=\sqrt{(-5)^2+(5\sqrt{3})^2}=10, \quad \theta=\tan^{-1}\left(\frac{5\sqrt{3}}{-5}\right)=\frac{2\pi}{3}, \quad z=10\text{cis}\frac{2\pi}{3}$

(h) $r=\sqrt{(-5\sqrt{3})^2+(-5)^2}=10, \quad \theta=\tan^{-1}\left(\frac{-5}{-5\sqrt{3}}\right)=\frac{7\pi}{6}, \quad z=10\text{cis}\frac{7\pi}{6}$

2. **(a)** $|z_1z_2|=5\times3=15$, $\arg(z_1z_2)=\frac{\pi}{3}+\frac{\pi}{6}=\frac{\pi}{2}$

So, $z_1z_2=15\text{cis}\frac{\pi}{2}=15\cos\frac{\pi}{2}+15\sin\frac{\pi}{2}\text{i}=15\text{i}$

(b) $|z_1z_2|=4\times2=8$, $\arg(z_1z_2)=\frac{2\pi}{3}+\frac{2\pi}{3}=\frac{4\pi}{3}$

So, $z_1z_2=8\text{cis}\frac{4\pi}{3}=8\cos\frac{4\pi}{3}+8\sin\frac{4\pi}{3}\text{i}=8\times-\frac{1}{2}+8\times-\frac{\sqrt{3}}{2}\text{i}=-4-4\sqrt{3}\text{i}$

3. **(a)** $|z_1/z_2| = \frac{6}{2} = 3$, $\arg(z_1/z_2) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

So, $z_1/z_2 = 3\text{cis}\frac{\pi}{3} = 3\cos\frac{\pi}{3} + 3\sin\frac{\pi}{3}\text{i} = \frac{3}{2} + \frac{3\sqrt{3}}{2}\text{i}$

(b) $|z_1/z_2| = \frac{16}{4} = 4$, $\arg(z_1/z_2) = \frac{3\pi}{2} - \frac{\pi}{6} = \frac{4\pi}{3}$

So, $z_1/z_2 = 4\text{cis}\frac{4\pi}{3} = 4\cos\frac{4\pi}{3} + 4\sin\frac{4\pi}{3}\text{i} = -2 - 2\sqrt{3}\text{i}$

(c) $|z_1/z_2| = \frac{8}{2} = 4$, $\arg(z_1/z_2) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

So $z_1/z_2 = 4\text{cis} - \frac{\pi}{4} = 4\cos\frac{-\pi}{4} + 4\sin\frac{-\pi}{4}\text{i} = 2\sqrt{2} - 2\sqrt{2}\text{i}$

(d) $|z_1/z_2| = \frac{16}{2} = 8$, $\arg(z_1/z_2) = \frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6}$

So $z_1/z_2 = 8\text{cis} - \frac{\pi}{6} = 8\cos\frac{-\pi}{6} + 8\sin\frac{-\pi}{6}\text{i} = 4\sqrt{3} - 4\text{i}$

4. **(a)** $|-1+\text{i}| = \sqrt{2}$, $\arg(-1+\text{i}) = \frac{3\pi}{4}$

$$\left(\sqrt{2}\text{cis}\frac{3\pi}{4}\right)^5 = (\sqrt{2})^5\text{cis}\left(5\times\frac{3\pi}{4}\right)$$
$$= 4\sqrt{2}\left(\cos\frac{15\pi}{4} + \text{i}\sin\frac{15\pi}{4}\right)$$
$$= 4\sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\text{i}\right)$$
$$= 4 - 4\text{i}$$

(b) $|-1-\text{i}| = \sqrt{2}$, $\arg(-1-\text{i}) = \frac{5\pi}{4}$

$$\left(\sqrt{2}\text{cis}\frac{5\pi}{4}\right)^4 = (\sqrt{2})^4\text{cis}\left(4\times\frac{5\pi}{4}\right)$$
$$= 4(\cos 5\pi + \text{i}\sin 5\pi)$$
$$= -4$$

(c) $\left|-\sqrt{3}+\mathrm{i}\right|=2, \quad \arg\left(-\sqrt{3}+\mathrm{i}\right)=\frac{5\pi}{6}$

$$\left(2\text{cis}\frac{5\pi}{6}\right)^4=(2)^4\text{cis}\left(4\times\frac{5\pi}{6}\right)$$
$$=16\left(\cos\frac{10\pi}{3}+\mathrm{i}\sin\frac{10\pi}{3}\right)$$
$$=16\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)$$
$$=-8-8\sqrt{3}\mathrm{i}$$

(d) $\left|-\sqrt{3}-\mathrm{i}\right|=2, \quad \arg\left(-\sqrt{3}-\mathrm{i}\right)=\frac{7\pi}{6}$

$$\left(2\text{cis}\frac{7\pi}{6}\right)^6=(2)^6\text{cis}\left(6\times\frac{7\pi}{6}\right)$$
$$=64\left(\cos 7\pi+\mathrm{i}\sin 7\pi\right)$$
$$=64(-1)$$
$$=-64$$

5. **(a)** $\sqrt[3]{2^3}=2\text{cis}\frac{2k\pi}{3}$

$k=0,1,2$

$k=0\Rightarrow \quad \sqrt[3]{2^3}=2\text{cis}0=2$

$k=1\Rightarrow \quad \sqrt[3]{2^3}=2\text{cis}\frac{2\pi}{3}=2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}\mathrm{i}\right)=-1+\sqrt{3}\mathrm{i}$

$k=2\Rightarrow \quad \sqrt[3]{2^3}=2\text{cis}\frac{4\pi}{3}=2\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\mathrm{i}\right)=-1-\sqrt{3}\mathrm{i}$

(b) $\sqrt{2\text{cis}\left(\frac{\pi}{3}+2k\pi\right)}=\sqrt{2}\left(\text{cis}\frac{\frac{\pi}{3}+2k\pi}{2}\right)$

$k=0,\ 1$

$k=0\Rightarrow \quad \sqrt{2}\left(\text{cis}\frac{\pi}{6}\right)=\sqrt{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\mathrm{i}\right)=\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}$

$k=1\Rightarrow \quad \sqrt{2}\left(\text{cis}\frac{7\pi}{6}\right)=\sqrt{2}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}\mathrm{i}\right)=-\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}\mathrm{i}$

(c) $\sqrt[4]{2^4}\text{cis}(\pi+2k\pi)=2\text{cis}\left(\frac{\pi+2k\pi}{4}\right)$

$k=0 \Rightarrow\ 2\text{cis}\left(\frac{\pi}{4}\right)=\sqrt{2}+\sqrt{2}\text{i}$

$k=1 \Rightarrow 2\text{cis}\left(\frac{\pi+2\pi}{4}\right)=2\text{cis}\left(\frac{3\pi}{4}\right)=-\sqrt{2}+\sqrt{2}\text{i}$

$k=2 \Rightarrow 2\text{cis}\left(\frac{\pi+4\pi}{4}\right)=2\text{cis}\left(\frac{5\pi}{4}\right)=-\sqrt{2}-\sqrt{2}\text{i}$

$k=3 \Rightarrow 2\text{cis}\left(\frac{\pi+6\pi}{4}\right)=2\text{cis}\left(\frac{7\pi}{4}\right)=\sqrt{2}-\sqrt{2}\text{i}$

6. **(a)** $\text{mod}\, z=6\sqrt{2},\quad \arg z=\frac{3\pi}{4}\ ,\ \textit{so}\ z=6\sqrt{2}\text{e}^{\frac{3\pi\text{i}}{4}}$

(b) $\text{mod}\, z=4\sqrt{3},\quad \arg z=\frac{\pi}{3}\ ,\ \textit{so}\ z=4\sqrt{3}\text{e}^{\frac{\pi\text{i}}{3}}$

(c) $\text{mod}\, z=2,\quad \arg z=\frac{5\pi}{3}\ ,\ \text{so}\ z=2\text{e}^{\frac{5\pi\text{i}}{3}}$

7. **(a)** $|z_1z_2|=3\times2=6\ ,\ \arg(z_1z_2)=\frac{\pi}{8}+\frac{3\pi}{8}=\frac{\pi}{2}$

So $z_1z_2=6e^{\frac{\pi\text{i}}{2}}$

(b) $|z_1z_2|=4\times3=12\ ,\ \arg(z_1z_2)=\frac{3\pi}{4}+\frac{\pi}{2}=\frac{5\pi}{4}$

So $z_1z_2=12\text{e}^{\frac{5\pi\text{i}}{4}}$

8. **(a)** $|z_1/z_2|=\frac{12}{3}=4\ ,\ \arg\left(\frac{z_1}{z_2}\right)=\frac{3\pi}{4}-\frac{\pi}{8}=\frac{5\pi}{8}$

So $z_1/z_2=4\text{e}^{\frac{5\pi\text{i}}{8}}=4\text{cis}\frac{5\pi}{8}=4\left(\cos\frac{5\pi}{8}+\text{i}\sin\frac{5\pi}{8}\right)$

(b) $|z_1/z_2|=\frac{16}{2}=8\ ,\ \arg\left(\frac{z_1}{z_2}\right)=\frac{3\pi}{2}-\frac{\pi}{4}=\frac{5\pi}{4}$

So $z_1/z_2=8\text{e}^{\frac{5\pi\text{i}}{4}}=8\text{cis}\frac{5\pi}{4}=8\left(\cos\frac{5\pi}{4}+\text{i}\sin\frac{5\pi}{4}\right)=-4\sqrt{2}-4\sqrt{2}\text{i}$

9. **(a)** 2 roots

(b) $\sqrt{\mathrm{i}} = \sqrt{\mathrm{e}^{\left(\frac{\pi}{2}+2k\pi\right)\mathrm{i}}} = \mathrm{e}^{\left(\frac{\pi}{4}+k\pi\right)\mathrm{i}}$

$k = 0,1$

$k = 0 \Rightarrow \sqrt{\mathrm{i}} = \mathrm{e}^{\left(\frac{\pi}{4}\right)\mathrm{i}} = \cos\frac{\pi}{4} + \mathrm{i}\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\mathrm{i}$

$k = 1 \Rightarrow \sqrt{\mathrm{i}} = \mathrm{e}^{\left(\frac{5\pi}{4}\right)\mathrm{i}} = \cos\frac{5\pi}{4} + \mathrm{i}\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{i}$

(c) $\sqrt{-\mathrm{i}} = \sqrt{\mathrm{e}^{\left(\frac{3\pi}{2}+2k\pi\right)\mathrm{i}}} = \mathrm{e}^{\left(\frac{3\pi}{4}+k\pi\right)\mathrm{i}}$

$k = 0 \Rightarrow \sqrt{-\mathrm{i}} = \mathrm{e}^{\left(\frac{3\pi}{4}\right)\mathrm{i}}$ and $k = 1 \Rightarrow \sqrt{-\mathrm{i}} = \mathrm{e}^{\left(\frac{7\pi}{4}\right)\mathrm{i}}$, so not the negative.

10. $z = -1 = \mathrm{e}^{(\pi)\mathrm{i}}$, hence $\mathrm{e}^{\pi\mathrm{i}} + 1 = 0$

Exercise 6.5

1. **(a)** $Z = R + \mathrm{j}(X_L - X_C) = 12 + \mathrm{j}(12-3) = 12 + 9\mathrm{j}$

(b) $|12+9\mathrm{j}| = 15$, $\arg(12+9\mathrm{j}) = \tan^{-1}\left(\frac{3}{4}\right) = 36.9^\circ$

So $Z = 15\mathrm{e}^{36.9^\circ \mathrm{i}}$

2. $Z = R + \mathrm{j}(X_L - X_C) = 4 + \mathrm{j}(2-5) = 4 - 3\mathrm{j}$

3. $V = V_R + \mathrm{j}(V_L - V_C) = 9 + \mathrm{j}(15-3) = 9 + 12\mathrm{j}$

$|V| = \sqrt{9^2 + 12^2} = 15$ volts

4. $V = V_R + \mathrm{j}(V_L - V_C) = 6 + \mathrm{j}(11.5-3.5) = 6 + 8\mathrm{j}$

$|V| = \sqrt{6^2 + 8^2} = 10$ volts

5. $|V| = |I| \times |Z| = |6-3\mathrm{j}| \times |8+4\mathrm{j}| = \sqrt{45} \times \sqrt{80} = 60$ volts

6. $|V| = |I| \times |Z| \Rightarrow 100 = |I| \times |4-3\mathrm{j}| \Rightarrow |I| = \frac{100}{5} = 20$ amps

7. $Z = \frac{Z_1 \times Z_2}{Z_1 + Z_2} = \frac{(12+5\mathrm{j})\times(8-6\mathrm{j})}{(12+5\mathrm{j})+(8-6\mathrm{j})} = \frac{13\mathrm{e}^{22.6^\circ \mathrm{i}} \times 10\mathrm{e}^{-36.9^\circ \mathrm{i}}}{20-\mathrm{j}} = \frac{130\mathrm{e}^{(22.6^\circ - 36.9^\circ)\mathrm{i}}}{\sqrt{401}\mathrm{e}^{-2.9^\circ \mathrm{i}}} = 6.49\mathrm{e}^{-11.4^\circ \mathrm{i}}$

$= 6.49(\cos(-11.4^\circ) + \mathrm{j}\sin(-11.4^\circ)) = 6.36 - 1.28\mathrm{j}$

8. $Z = \frac{Z_1 \times Z_2}{Z_1 + Z_2} = \frac{(8+6\mathrm{j})\times(8-6\mathrm{j})}{(8+6\mathrm{j})+(8-6\mathrm{j})} = \frac{100}{16} = 6.25$ ohms

Chapter 6 practice questions

1. **(a)** $\sqrt{-49} = 7\sqrt{-1} = 7\text{i}$

(b) $\sqrt{-18} = \sqrt{9\times 2\times -1} = 3\sqrt{2}\text{i}$

(c) $\sqrt{-9}\times\sqrt{-1} = 3\text{i}\times\text{i} = 3\times -1 = -3$

(d) $\sqrt{-12}\times\sqrt{-27} = 2\sqrt{3}\text{i}\times 3\sqrt{3}\text{i} = 6\times 3\text{i}^2 = 18\times -1 = -18$

2. **(a)** $\sqrt{-16} = 4\text{i}$

(b) $25 + 5\sqrt{-1} = 25 + 5\text{i}$

(c) $5 + 5\sqrt{-1} = 5 + 5\text{i}$

(d) $-3\sqrt{2} + \sqrt{9\times 2\times -1} = -3\sqrt{2} + 3\sqrt{2}\text{i}$

(e) $2\sqrt{3} - \sqrt{4\times 3\times -1} = 2\sqrt{3} - 2\sqrt{3}\text{i}$

(f) $-\sqrt{\text{i}^2} = -\text{i}$

(g) $-\sqrt{\text{i}^4} = -\sqrt{1} = -1$

(h) $-3\text{i} + \sqrt{-9}\text{i} = -3\text{i} + 3\text{i}\times\text{i} = -3 - 3\text{i}$

3. **(a)** $\text{i}^{22} = (\text{i}^2)^{11} = (-1)^{11} = -1$

(b) $\text{i}^{21} = (\text{i}^2)^{10}\text{i} = (-1)^{10}\text{i} = \text{i}$

(c) $\text{i}^{20} = (\text{i}^2)^{10} = (-1)^{10} = 1$

(d) $\text{i}^{19} = (\text{i}^2)^{9}\text{i} = (-1)^{9}\text{i} = -1\times\text{i} = -\text{i}$

4. **(a)** $x = \dfrac{-4\pm\sqrt{4^2 - 4\times 1\times 8}}{2} = -2 \pm \dfrac{1}{2}\sqrt{-16} = -2 \pm 2\text{i}$

(b) $x = \dfrac{6\pm\sqrt{6^2 - 4\times 1\times 10}}{2} = 3 \pm \dfrac{1}{2}\sqrt{-4} = 3 \pm \text{i}$

(c) $x = \dfrac{8\pm\sqrt{8^2 - 4\times 1\times 25}}{2} = 4 \pm \dfrac{1}{2}\sqrt{-36} = 4 \pm 3\text{i}$

(d) $x = \dfrac{4\pm\sqrt{4^2 - 4\times 1\times 8}}{2} = 2 \pm \dfrac{1}{2}\sqrt{-16} = 2 \pm 2\text{i}$

(e) $x = \dfrac{10\pm\sqrt{10^2 - 4\times 1\times 29}}{2} = 5 \pm \dfrac{1}{2}\sqrt{-16} = 5 \pm 2\text{i}$

(f) $x = \dfrac{-8\pm\sqrt{8^2 - 4\times 1\times 32}}{2} = -4 \pm \dfrac{1}{2}\sqrt{-64} = -4 \pm 4\text{i}$

5. **(a)** $5-4-\text{i}+3\text{i}=1+2\text{i}$

(b) $4-2+2\text{i}+\text{i}=2+3\text{i}$

(c) $\sqrt{5}+2\sqrt{5}-3\text{i}+\text{i}=3\sqrt{5}-2\text{i}$

6. **(a)** $36+16-24\text{i}+24\text{i}=52$

(b) $-49-4+14\text{i}-14\text{i}=-53$

(c) $27+1-3\sqrt{3}\,\text{i}+3\sqrt{3}\,\text{i}=28$

7. **(a)** $(9^2+12^2)=225$

(b) $(6^2+8^2)=100$

(c) $(3^2+\left(3\sqrt{2})^2\right)=27$

8. **(a)** $\frac{10}{3+\text{i}}\times\frac{3-\text{i}}{3-\text{i}}=\frac{30-10\text{i}}{9+1}=3-\text{i}$

(b) $\frac{14}{\sqrt{3}-2\text{i}}\times\frac{\sqrt{3}+2\text{i}}{\sqrt{3}+2\text{i}}=\frac{14\sqrt{3}+28\text{i}}{3+4}=2\sqrt{3}+4\text{i}$

(c) $\frac{1-3\text{i}}{-3+9\text{i}}\times\frac{-3-9\text{i}}{-3-9\text{i}}=\frac{-3-27}{9+81}=-\frac{1}{3}$

9. **(a)** other zero is $(1-2\text{i})$

$-b=$ sum of roots $=1+2\text{i}+1-2\text{i}=2$

$c=$ product of roots $=(1+2\text{i})(1-2\text{i})=1+4=5$

so function is $y=x^2-2x+5$

(b) other zero is $(4-3\text{i})$

$-b=$ sum of roots $=4+3\text{i}+4-3\text{i}=8$

$c=$ product of roots $=(4+3\text{i})(4-3\text{i})=16+9=25$

so function is $y=x^2-8x+25$

(c) so other zero is $(-3\sqrt{}2-\sqrt{}2\text{i})$

$-b=$ b = sum of roots $=-3\sqrt{2}+\sqrt{2\text{i}}-3\sqrt{2}-\sqrt{2\text{i}}=-6\sqrt{}2$

$c=$ c = product of roots $=\left(-3\sqrt{2}+\sqrt{2\text{i}}\right)\left(-3\sqrt{2}-\sqrt{2\text{i}}\right)=18+2=20$

so function is $y=x^2+6\sqrt{}2x+20$

10. **(a)** $-b =$ sum of roots $= -2$

$c =$ product of roots $= (-1-\mathrm{i})(-1+\mathrm{i}) = 1+1 = 2$

so function is $y = x^2 + 2x + 2$

(b) $-b =$ sum of roots $= 4$

$c =$ product of roots $= (2+\sqrt{2}\mathrm{i})(2-\sqrt{2}\mathrm{i}) = 4+2 = 6$

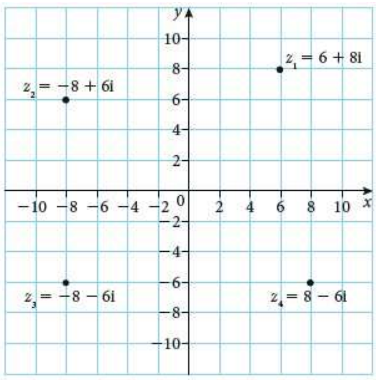
so function is $y = x^2 - 4x + 6$

(c) $-b =$ sum of roots $= 7+\mathrm{i}$

c product of roots $= (3+2\mathrm{i})(4-\mathrm{i}) = 12+2+8\mathrm{i}-3\mathrm{i} = 14+5\mathrm{i}$

so function is $y = x^2 - (7+\mathrm{i})x + 14 + 5\mathrm{i}$

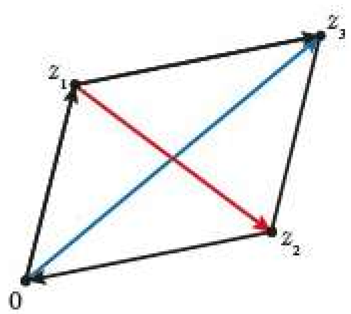
11.



12. $\left|a + 2\sqrt{2}\mathrm{i}\right| = 10$

$\Rightarrow a^2 + 8 = 100 \Rightarrow a = \pm\sqrt{92} = \pm 2\sqrt{23}$

13.



diagonal $Oz_3 = z_1 + z_2$, diagonal $z_1z_2 = z_2 - z_1$

14. **(a)** $r=\sqrt{2^2+2^2}=2\sqrt{2}$, $\quad\theta=\tan^{-1}\left(\dfrac{-2}{2}\right)=\dfrac{7\pi}{4}$, $\quad 2\sqrt{2}\text{cis}\dfrac{7\pi}{4}$

(b) $r=\sqrt{2^2+2^2}=2\sqrt{2}$, $\quad\theta=\tan^{-1}\left(\dfrac{-2}{-2}\right)=\dfrac{5\pi}{4}$, $\quad 2\sqrt{2}\text{cis}\dfrac{5\pi}{4}$

(c) $r=\sqrt{2^2+2^2}=2\sqrt{2}$, $\quad\theta=\tan^{-1}\left(\dfrac{2}{2}\right)=\dfrac{\pi}{4}$, $\quad 2\sqrt{2}\text{cis}\dfrac{\pi}{4}$

(d) $r=\sqrt{2^2+2^2}=2\sqrt{2}$, $\quad\theta=\tan^{-1}\left(\dfrac{2}{-2}\right)=\dfrac{3\pi}{4}$, $\quad 2\sqrt{2}\text{cis}\dfrac{3\pi}{4}$

(e) $r=\sqrt{2^2+\left(2\sqrt{3}\right)^2}=\sqrt{16}=4$, $\quad\theta=\tan^{-1}\left(\dfrac{2\sqrt{3}}{2}\right)=\dfrac{\pi}{3}$, $\quad 4\text{cis}\dfrac{\pi}{3}$

(f) $r=\sqrt{2^2+\left(2\sqrt{3}\right)^2}=\sqrt{16}=4$, $\quad\theta=\tan^{-1}\left(\dfrac{-2\sqrt{3}}{2}\right)=\dfrac{5\pi}{3}$, $\quad 4\text{cis}\dfrac{5\pi}{3}$

(g) $r=\sqrt{2^2+\left(2\sqrt{3}\right)^2}=\sqrt{16}=4$, $\quad\theta=\tan^{-1}\left(\dfrac{2\sqrt{3}}{-2}\right)=\dfrac{2\pi}{3}$, $\quad 4\text{cis}\dfrac{2\pi}{3}$

(h) $r=\sqrt{2^2+\left(2\sqrt{3}\right)^2}=\sqrt{16}=4$, $\quad\theta=\tan^{-1}\left(\dfrac{-2\sqrt{3}}{-2}\right)=\dfrac{4\pi}{3}$, $\quad 4\text{cis}\dfrac{4\pi}{3}$

15. **(a)** $|z_1z_2|=6\times 2=12$, $\arg(z_1z_2)=\dfrac{\pi}{2}+\dfrac{3\pi}{4}=\dfrac{5\pi}{4}$

S $z_1z_2=12\text{cis}\dfrac{5\pi}{4}=12\left(-\dfrac{\sqrt{2}}{2}-\dfrac{\sqrt{2}}{2}\text{i}\right)=-6\sqrt{2}-6\sqrt{2}\text{i}$

(b) $|z_1z_2|=8\times 2=16$, $\arg(z_1z_2)=\dfrac{\pi}{3}+\dfrac{\pi}{6}=\dfrac{\pi}{2}$

So $z_1z_2=16\text{cis}\dfrac{\pi}{2}=16\text{i}$

16. **(a)** $|z_1/z_2|=\dfrac{9}{3}=3$, $\arg\left(\dfrac{z_1}{z_2}\right)=\dfrac{3\pi}{4}-\dfrac{\pi}{4}=\dfrac{\pi}{2}$

So $z_1/z_2=3\text{cis}\dfrac{\pi}{2}=3\left(\cos\dfrac{\pi}{2}+\text{i}\sin\dfrac{\pi}{2}\right)=3\text{i}$

(b) $|z_1/z_2|=\dfrac{10}{2}=5$, $\arg\left(\dfrac{z_1}{z_2}\right)=\dfrac{5\pi}{4}-\dfrac{\pi}{2}=\dfrac{3\pi}{4}$

So $z_1/z_2=5\text{cis}\dfrac{3\pi}{4}=5\left(\cos\dfrac{3\pi}{4}+\text{i}\sin\dfrac{3\pi}{4}\right)=-\dfrac{5}{2}\sqrt{2}+\dfrac{5}{2}\sqrt{2}\text{i}$

(c) $|z_1 / z_2| = \frac{2}{4} = \frac{1}{2}$, $\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$

So $z_1 / z_2 = \frac{1}{2}\text{cis}\frac{\pi}{6} = \frac{1}{2}\left(\cos\frac{\pi}{6} + \text{i}\sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{4} + \frac{1}{4}\text{i}$

(d) $|z_1 / z_2| = \frac{12}{3} = 4$, $\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{4} - \frac{\pi}{2} = -\frac{\pi}{4}$

So $z_1 / z_2 = 4\text{cis} - \frac{\pi}{4} = 4\left(\cos - \frac{\pi}{4} + \text{i}\sin - \frac{\pi}{4}\right) = 4\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\text{i}\right) = 2\sqrt{2} - 2\sqrt{2}\text{i}$

17. (a) $\left(\sqrt{2}\text{cis}\frac{\pi}{4}\right)^4 = \left(\sqrt{2}\right)^4 \text{cis}\left(4 \times \frac{\pi}{4}\right) = 4\text{cis}\pi = -4$

(b) $(\sqrt{2}\text{cis}\frac{7\pi}{4})^4 = \left(\sqrt{2}\right)^4 \text{cis}\left(4 \times \frac{7\pi}{4}\right) = 4\text{cis}7\pi = -4$

(c) $\left(2\text{cis}\frac{2\pi}{3}\right)^3 = (2)^3\text{cis}\left(3 \times \frac{2\pi}{3}\right) = 8\text{cis}2\pi = 8$

(d) $\left(4\text{cis}\frac{\pi}{6}\right)^4 = (4)^4\text{cis}\left(4 \times \frac{\pi}{6}\right) = 256\text{cis}\frac{2\pi}{3} = -128 + 128\sqrt{3}\text{i}$

18. (a) $\sqrt[3]{-8} = (2^3\text{cis}\pi)^{\frac{1}{3}} = 2\text{cis}\left(\frac{\pi + 2k\pi}{3}\right)$

$k = 0, 1, 2$

$k = 0 : 2\text{cis}\frac{\pi}{3} = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}\right) = 1 + \sqrt{3}\text{i}$

$k = 1 : 2\text{cis}(\pi) = 2(-1) = -2$

$k = 2 : 2\text{cis}\frac{5\pi}{3} = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}\right) = 1 - \sqrt{3}\text{i}$

(b) $\sqrt[3]{64} = (4^3\text{cis}\,0)^{\frac{1}{3}} = 4\text{cis}\left(\frac{2k\pi}{3}\right)$

$k = 0, 1, 2$

$k = 0 : 4\text{cis}\,0 = 4$

$k = 1 : 4\,\text{cis}\left(\frac{2\pi}{3}\right) = 4\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\text{i}\right) = -2 + 2\sqrt{3}\text{i}$

$k = 2 : 4\,\text{cis}\frac{4\pi}{3} = 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\text{i}\right) = -2 - 2\sqrt{3}\text{i}$

(c) $\sqrt[2]{\left(-3+3\sqrt{3}\text{i}\right)} = 6\text{cis}\left(\frac{2\pi}{3}+2k\pi\right)^{\frac{1}{2}} = \sqrt{6}\text{cis}\left(\frac{2\pi}{6}+k\pi\right)$

$k = 0,1$

$k = 0:\ \sqrt{6}\ \text{cis}\left(\frac{\pi}{3}\right) = \sqrt{6}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\text{i}\right) = \frac{\sqrt{6}}{2}+\frac{3\sqrt{2}}{2}\text{i}$

$k = 1:\ \sqrt{6}\ \text{cis}\left(\frac{4\pi}{3}\right) = \sqrt{6}\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}\right) = -\frac{\sqrt{6}}{2}-\frac{3\sqrt{2}}{2}\text{i}$

19. **(a)** $\text{mod}\, z = \sqrt{2}$, $\arg z = \frac{7\pi}{4}$, so $z = \sqrt{2}\text{e}^{\frac{7\pi\text{i}}{4}}$

(b) $\text{mod}\, z = \sqrt{8}$, $\arg z = \frac{5\pi}{4}$, so $z = 2\sqrt{2}\text{e}^{\frac{5\pi\text{i}}{4}}$

(c) $\text{mod}\, z = 2\sqrt{3}$, $\arg z = \frac{2\pi}{3}$, so $z = 2\sqrt{3}\text{e}^{\frac{2\pi\text{i}}{3}}$

20. **(a)** $|z_1z_2| = 2\times 5 = 10$, $\arg(z_1z_2) = \frac{\pi}{6}+\frac{3\pi}{2} = \frac{10\pi}{6}$

So $z_1z_2 = 10e^{\frac{5\pi\text{i}}{3}} = 10\text{cis}\frac{5\pi}{3} = 10\left(\frac{1}{2}-\frac{\sqrt{3}}{2}\text{i}\right) = 5-5\sqrt{3}\text{i}$

(b) $|z_1z_2| = 2\times 4 = 8$, $\arg(z_1z_2) = \frac{2\pi}{3}+\frac{\pi}{6} = \frac{5\pi}{6}$

So $z_1z_2 = 8\text{e}^{\frac{5\pi\text{i}}{6}} = 8\text{cis}\frac{5\pi}{6} = 8\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}\text{i}\right) = -4\sqrt{3}+4\text{i}$

21. **(a)** $|z_1/z_2| = \frac{10}{2} = 5$, $\arg\left(\frac{z_1}{z_2}\right) = \frac{3\pi}{2}-\frac{\pi}{4} = \frac{5\pi}{4}$

So $z_1/z_2 = 5\text{e}^{\frac{5\pi\text{i}}{4}} = 5\text{cis}\frac{5\pi}{4} = 5\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\text{i}\right) = -\frac{5}{2}\sqrt{2}-\frac{5}{2}\sqrt{2}\text{i}$

(b) $|z_1/z_2| = \frac{12}{3} = 4$, $\arg\left(\frac{z_1}{z_2}\right) = \frac{\pi}{2}-\frac{3\pi}{4} = -\frac{\pi}{4}$

So $z_1/z_2 = 4\text{e}^{-\frac{\pi\text{i}}{4}} = 4\text{cis}\left(-\frac{\pi}{4}\right) = 4\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}\text{i}\right) = 2\sqrt{2}-2\sqrt{2}\text{i}$

22 **(a)** $Z = R + \mathrm{j}(X_L - X_C) = 12 + \mathrm{j}(3-8) = 12 - 5\mathrm{j}$

(b) $|12-5\mathrm{j}| = 13$, $\arg(12-5\mathrm{j}) = \tan^{-1}\left(\frac{-5}{12}\right) = -0.395$ rad

So $Z = 13\mathrm{e}^{-0.395\mathrm{i}}$

23. $Z = R + \mathrm{j}(X_L - X_C) = 4 + \mathrm{j}(7-3) = 4 + 4\mathrm{j}$

24. $V = V_R + \mathrm{j}(V_L - V_C) = 12 + \mathrm{j}(4.5 - 9.5) = 12 - 5\mathrm{j}$

$|V| = \sqrt{12^2 + 5^2} = 13$ volts

25. $V = V_R + \mathrm{j}(V_L - V_C) = 8 + \mathrm{j}(4.5 - 10.5) = 8 - 6\mathrm{j}$

$|V| = \sqrt{8^2 + 6^2} = 10$ volts

26. $|V| = |I| \times |Z| = |2+\mathrm{j}| \times |4 + 2\mathrm{j}| = \sqrt{5} \times \sqrt{20} = 10$ volts

27. $|V| = |I| \times |Z|$

So, $65 = |I| \times |12 - 5\mathrm{j}| = |I| \times 13$

$\Rightarrow \quad |I| = \frac{65}{13} = 5$ amps

28. $Z = \frac{Z_1 \times Z_2}{Z_1 + Z_2} = \frac{(6+8\mathrm{j}) \times (6-3\mathrm{j})}{(6+8\mathrm{j}) + (6-3\mathrm{j})} = \frac{36 + 24 + 48\mathrm{j} - 18\mathrm{j}}{12 + 5\mathrm{j}} = \frac{60 + 30\mathrm{j}}{12+5\mathrm{j}} = \frac{60+30\mathrm{j}}{12+5\mathrm{j}} \times \frac{12-5\mathrm{j}}{12-5\mathrm{j}}$

$= \frac{10(72 + 15 - 30\mathrm{j} + 36\mathrm{j})}{169} = 10 \times \frac{87 + 6\mathrm{j}}{169} = \frac{10}{169} \times (87 + 6\mathrm{j})$

$= 5.15 + 0.355\mathrm{j}$ ohms

29. **(a)** $\frac{1}{2+3\mathrm{i}} + \frac{1}{3+2\mathrm{i}} = \frac{1}{2+3\mathrm{i}} \times \frac{2-3\mathrm{i}}{2-3\mathrm{i}} + \frac{1}{3+2\mathrm{i}} \times \frac{3-2\mathrm{i}}{3-2\mathrm{i}} = \frac{2-3\mathrm{i}}{13} + \frac{3-2\mathrm{i}}{13} = \frac{5-5\mathrm{i}}{13}$

So $\frac{10}{w} = \frac{5-5\mathrm{i}}{13}$

$\Rightarrow \frac{w}{10} = \frac{13}{5-5\mathrm{i}}$

$\Rightarrow \frac{w}{10} = \frac{13}{5-5\mathrm{i}} \times \frac{5+5\mathrm{i}}{5+5\mathrm{i}}$

$\Rightarrow \frac{w}{10} = \frac{13(5+5\mathrm{i})}{50}$

$\Rightarrow w = 13 + 13\mathrm{i}$

(b) $w^* = 13 - 13\text{i}$

$|w^*| = 13$, $\arg w^* = \tan^{-1}\dfrac{-13}{13}$

$w^* = 13e^{-\frac{\pi}{4}\text{i}}$

30. **(a)** $z^3 = -\dfrac{27}{8} = -\dfrac{3^3}{2^3} = -\left(\dfrac{3}{2}\right)^3$, so $|z| = \dfrac{3}{2}$

Let $z = \dfrac{3}{2}\text{cis}\,\theta$

Then $\left(\dfrac{3}{2}\text{cis}\,\theta\right)^3 = -\left(\dfrac{3}{2}\right)^3$

$\Rightarrow \text{cis}(3\theta) = -1$

$\Rightarrow 3\theta = \pi, 3\pi, 5\pi \quad (\pi + 2k\pi \text{ for } k = 0,1,2) +$

$\Rightarrow \theta = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$

So 3 roots are $\dfrac{3}{2}\text{cis}\dfrac{\pi}{3}$, $\dfrac{3}{2}\text{cis}\,\pi$, $\dfrac{3}{2}\text{cis}\dfrac{5\pi}{3}$

(b) equilateral triangle with sides of length $2\times\dfrac{3}{2}\cos 30^\circ = \dfrac{3\sqrt{3}}{2}$

area = $\dfrac{1}{2}ab\sin C = \dfrac{1}{2}\times\dfrac{3\sqrt{3}}{2}\times\dfrac{3\sqrt{3}}{2}\times\sin 60^\circ = \dfrac{27\sqrt{3}}{16}$

31. **(a)** $\left(\text{cis}\dfrac{2\pi}{7}\right)^7 = \text{cis}\left(7\times\dfrac{2\pi}{7}\right) = \text{cis}\,2\pi = 1$

Thus $z^7 = 1$

(b) **(i)** $(w-1)(1+w+w^2+w^3+w^4+w^5+w^6)$

$= (w+w^2+w^3+w^4+w^5+w^6+w^7) - (1+w+w^2+w^3+w^4+w^5+w^6)$

$= w^7 - 1$

(ii) From part i, $w^7 - 1 = 0$

$\Rightarrow (w-1)(1+w+w^2+w^3+w^4+w^5+w^6) = 0$

$\Rightarrow w = 1$ or $(1+w+w^2+w^3+w^4+w^5+w^6) = 0$

But $w \neq 0$ and hence $(1+w+w^2+w^3+w^4+w^5+w^6) = 0$

(c) roots are $cis\left(k\times\frac{2\pi}{7}\right)$, $k=0,1,2,3,4,5,6$ and correspond to

$1,w\ ,w^2,w^3,w^4,w^5,w^6$ (all roots lie on unit circle, angle $\frac{2\pi}{7}$ between each.

(d) **(i)** From symmetry of Argand diagram of roots (reflection in Re axis):

$w^*=w^6, w^{2*}=w^5, w^{3*}=w^4$, and visa versa.

Hence $\alpha^*=(w+w^2+w^4)^*=w^*+w^{2*}+w^{4*}=w^6+w^5+w^3$

(ii) sum of roots $=-b=\alpha+\alpha^*=\left(w+w^2+w^4\right)+\left(w^6+w^5+w^3\right)=-1$

(from **b(ii)** above)

Hence $b=1$

Product of roots $=c=\alpha\times\alpha^*$

$$=\left(w+w^2+w^4\right)\left(w^6+w^5+w^3\right)$$

$$=w^7+w^6+w^4+w^8+w^7+w^5+w^{10}+w^9+w^7$$

$$=w^4+w^5+w^6+3w^7+w^8+w^9+w^{10}$$

But $w^8=\text{cis}\left(8\times\frac{2\pi}{7}\right)=\text{cis}\left(2\pi+\frac{2\pi}{7}\right)=\text{cis}\frac{2\pi}{7}=w$

Similarly $w^9=w^2, w^{10}=w^3$ and $w^7=1$

Hence

$c=3+w+w^2+w^3+w^4+w^5+w^6=2+\left(1+w+w^2+w^3+w^4+w^5+w^6\right)=2+0$

$\Rightarrow c=2$

(e) $z^2+z+2=0$

roots are $z=\frac{-1\pm\sqrt{1-4\times1\times2}}{2}=-\frac{1}{2}\pm\frac{1}{2}\sqrt{-7}=-\frac{1}{2}\pm\frac{1}{2}\sqrt{7}\text{i}$,

Im part $\alpha=\frac{1}{2}\sqrt{7}\,\text{i}$

32. Real coefficients so sum roots $=-a$, product = b

Roots are $2+3\text{i},\quad 2-3\text{i}$ (conjugates)

Hence $2+3\text{i}+2-3\text{i}=-a$ and $(2+3\text{i})(2-3\text{i})=b$

$\Rightarrow a=-4,\ b=13$

Exercise 7.1

1. For matrices $\boldsymbol{A}=\begin{pmatrix}-2 & x\\ y-1 & 3\end{pmatrix}$ and $\boldsymbol{B}=\begin{pmatrix}x+1 & -3\\ 4 & y-2\end{pmatrix}$, we have:

(a) **(i)** $\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix}x-1 & x-3\\ y+3 & y+1\end{pmatrix}$

(ii) $3\boldsymbol{A}-\boldsymbol{B}=\begin{pmatrix}-6 & 3x\\ 3y-3 & 9\end{pmatrix}-\begin{pmatrix}x+1 & -3\\ 4 & y-2\end{pmatrix}=\begin{pmatrix}-x-7 & 3x+3\\ 3y-7 & 11-y\end{pmatrix}$

(b) If $\boldsymbol{A}=\boldsymbol{B}$, their corresponding elements are equal; this leads to a system of equations:

$x+1=-2$

$x=-3$

$y-1=4$

$y-2=3$

From each of the first two equations we get $x=-3$, and from the last two $y=5$.

(c) If $\boldsymbol{A}+\boldsymbol{B}$ is a diagonal matrix, the elements on the opposite diagonal are zeroes; hence,

$x-3=0$

$y+3=0$

Therefore, $x=3, y=-3$.

(d)

$$\boldsymbol{AB}=\begin{pmatrix}-2(x+1)+4x & -2(-3)+x(y-2)\\ (y-1)(x+1)+3\cdot4 & (y-1)(-3)+3(y-2)\end{pmatrix}=\begin{pmatrix}2x-2 & xy-2x+6\\ xy-x+y+11 & -3\end{pmatrix}$$

$$\boldsymbol{BA}=\begin{pmatrix}(x+1)(-2)-3(y-1) & (x+1)x-3\cdot3\\ 4(-2)+(y-2)(y-1) & 4x+3(y-2)\end{pmatrix}=\begin{pmatrix}-2x-3y+1 & x^2+x-9\\ y^2-3y-6 & 4x+3y-6\end{pmatrix}$$

2. **(a)** $\begin{pmatrix} 3 & 0 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix} \Rightarrow \begin{pmatrix} 3x \\ 4x+2y \end{pmatrix} = \begin{pmatrix} 6 \\ -12 \end{pmatrix}$

By comparing the corresponding elements, we obtain the following equations:

$$\left.\begin{array}{l} 3x = 6 \\ 4x+2y=-12 \end{array}\right\} \Rightarrow \begin{array}{c} x=2 \\ 8+2y=-12 \end{array}$$

Therefore, $x = 2, y = -10$.

(b) $\begin{pmatrix} 2 & p \\ 3 & q \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \end{pmatrix} \Rightarrow \begin{pmatrix} 8+5p \\ 12+5q \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \end{pmatrix}$

By comparing the corresponding elements, we obtain the following equations:

$8+5p=18$

$12+5q=-8$

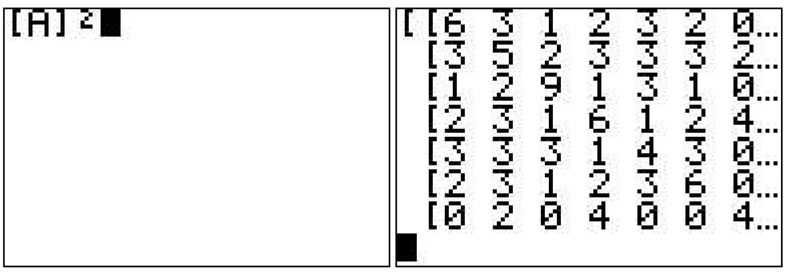
Therefore, $p = 2, q = -4$.

3. **(a)** Here is the completed matrix

$$\begin{array}{c} \\ \boldsymbol{V} \\ \boldsymbol{M} \\ \boldsymbol{F} \\ \boldsymbol{S} \\ \boldsymbol{Z} \\ \boldsymbol{L} \\ \boldsymbol{P} \end{array} \begin{array}{c} \begin{array}{ccccccc} \boldsymbol{V} & \boldsymbol{M} & \boldsymbol{F} & \boldsymbol{S} & \boldsymbol{Z} & \boldsymbol{L} & \boldsymbol{P} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(b) Matrix A showing the number of direct routes between each pair of cities is input into a GDC.

We square matrix A and obtain the answer shown in the GDC screenshot below right.



Matrix A^2 represents the number of routes between each pair of cities that go via another city. For example, the entry $a_{1,2} = 3$ means that there are three different routes from Vienna to Munich: two of them through Milano and one through Zurich. Also, $a_{7,7} = 4$ means that there are four different routes from Paris to another city (all to Frankfurt) and back to Paris.

4. **(a)**

$$\boldsymbol{A}+\boldsymbol{C}=\begin{pmatrix}2&5&1\\0&-3&2\\7&0&-1\end{pmatrix}+\begin{pmatrix}x-1&5&y\\0&-x&y+1\\2x+y&x-3y&2y-x\end{pmatrix}=\begin{pmatrix}x+1&10&y+1\\0&-x-3&y+3\\2x+y+7&x-3y&-x+2y-1\end{pmatrix}$$

(b) $$\boldsymbol{AB}=\begin{pmatrix}2&5&1\\0&-3&2\\7&0&-1\end{pmatrix}\cdot\begin{pmatrix}m&-2\\3m&-1\\2&3\end{pmatrix}=\begin{pmatrix}17m+2&-6\\-9m+4&9\\7m-2&-17\end{pmatrix}$$

(c) ***BA*** cannot be found because ***B*** is a 3×2 matrix and *A* is a 3×3 matrix; as such, the number of columns of matrix *B* does not match the number of rows of matrix *A*, so the product is not defined.

(d) If ***A*** = ***C***, their corresponding elements are equal; this leads to a system of equations:

$x-1=2;\quad y=1;\quad x=3;\quad y+1=2$

$2x+y=7;\quad x-3y=0;\quad 2y-x=-1$

These equations all give the same solution: $x=3, y=1$.

(e) ***B*** + ***C*** cannot be found because ***B*** is a 3×2 matrix and ***C*** is a 3×3 matrix; the sum of two matrices of different order is not defined.

(f) $$3\boldsymbol{B}+2\begin{pmatrix}-1&m^2\\-5&2\\1&-1\end{pmatrix}=\begin{pmatrix}7&12\\17&1\\2m+2&7\end{pmatrix}$$

By comparing the corresponding elements from

$$3\begin{pmatrix}m&-2\\3m&-1\\2&3\end{pmatrix}+2\begin{pmatrix}-1&m^2\\-5&2\\1&-1\end{pmatrix}=\begin{pmatrix}7&12\\17&1\\2m+2&7\end{pmatrix},$$

we obtain the following system of equations:

$3m-2=7$

$-6+2m^2=12$

$9m-10=17$

$8=2m+2$

All of the equations give us the unique solution: m = 3.

5. By comparing the corresponding elements from

$$2\begin{pmatrix} a-1 & b \\ c+2 & 3 \end{pmatrix}+\begin{pmatrix} 3 & -1 \\ 0 & 5 \end{pmatrix}=\begin{pmatrix} -5 & 5 \\ 8 & c+9 \end{pmatrix},$$

we obtain the following system of equations:

$$2(a-1)+3=-5$$
$$2b-1=5$$
$$2(c+2)=8$$
$$2\cdot 3+5=c+9$$

Therefore, $a=-3, b=3, c=2$.

6. $$\begin{pmatrix} 2 & -3 \\ -5 & 7 \end{pmatrix}\begin{pmatrix} x-11 & 1-x \\ -5 & x+2y \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

After performing matrix multiplication, we get:

$$\begin{pmatrix} 2x-7 & -5x-6y+2 \\ -5x+20 & 12x+14y-5 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

By comparing the corresponding elements, we obtain the following system of equations:

$$2x-7=1$$
$$-5x-6y+2=0$$
$$-5x+20=0$$
$$12x+14y-5=1$$

The first and third equation give us $x = 4$.

Substituting $x = 4$ into the second and fourth equation gives us the same solution: $y = -3$.

7. By comparing the corresponding elements, we obtain the following system of equations:

$$m^2-1=3$$
$$m+2=n+1$$
$$n-5=-2$$

Therefore, $m=2, n=3$.

8. Let matrix $\boldsymbol{L}=\begin{pmatrix}2 & 5 & 3\end{pmatrix}$ represent the quantities from your shopping list, and

$\boldsymbol{P}=\begin{pmatrix}1.66 & 1.58\\ 2.55 & 2.6\\ 0.90 & 0.95\end{pmatrix}$ represent the prices in shops A and B.

Then $\boldsymbol{LP}=\begin{pmatrix}2 & 5 & 3\end{pmatrix}\begin{pmatrix}1.66 & 1.58\\ 2.55 & 2.6\\ 0.90 & 0.95\end{pmatrix}=\begin{pmatrix}18.77 & 19.01\end{pmatrix}$ gives us the total cost of the shopping in shop A (€18.77) and B (€19.01) respectively.

Therefore, you should go to shop A because the total cost is cheaper.

9. **(a)**

$$\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=\begin{pmatrix}2 & 0\\ -5 & 1\end{pmatrix}+\left(\begin{pmatrix}3 & -1\\ 1 & 4\end{pmatrix}+\begin{pmatrix}-3 & 5\\ 2 & 7\end{pmatrix}\right)=\begin{pmatrix}2 & 0\\ -5 & 1\end{pmatrix}+\begin{pmatrix}0 & 4\\ 3 & 11\end{pmatrix}=\begin{pmatrix}2 & 4\\ -2 & 12\end{pmatrix}$$

$$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\left(\begin{pmatrix}2 & 0\\ -5 & 1\end{pmatrix}+\begin{pmatrix}3 & -1\\ 1 & 4\end{pmatrix}\right)+\begin{pmatrix}-3 & 5\\ 2 & 7\end{pmatrix}=\begin{pmatrix}5 & -1\\ -4 & 5\end{pmatrix}+\begin{pmatrix}-3 & 5\\ 2 & 7\end{pmatrix}=\begin{pmatrix}2 & 4\\ -2 & 12\end{pmatrix}$$

(b) We conclude that the addition of 2×2 matrices is associative, which can be proved as follows:

$$\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=\begin{pmatrix}a & b\\ c & d\end{pmatrix}+\left(\begin{pmatrix}m & n\\ p & q\end{pmatrix}+\begin{pmatrix}r & s\\ t & u\end{pmatrix}\right)$$

$$=\begin{pmatrix}a & b\\ c & d\end{pmatrix}+\begin{pmatrix}m+r & n+s\\ p+t & q+u\end{pmatrix}=\begin{pmatrix}a+m+r & b+n+s\\ c+p+t & d+q+u\end{pmatrix}$$

$$(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}+\begin{pmatrix}m & n\\ p & q\end{pmatrix}\right)+\begin{pmatrix}r & s\\ t & u\end{pmatrix}$$

$$=\begin{pmatrix}a+m & b+n\\ c+p & d+q\end{pmatrix}+\begin{pmatrix}r & s\\ t & u\end{pmatrix}=\begin{pmatrix}a+m+r & b+n+s\\ c+p+t & d+q+u\end{pmatrix}$$

So, $\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})=(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}$ for all real numbers *a, b, c, d, m, n, p, q, r, s, t, u.*

(c) $$\boldsymbol{A}(\boldsymbol{BC})=\begin{pmatrix}2 & 0\\ -5 & 1\end{pmatrix}\left(\begin{pmatrix}3 & -1\\ 1 & 4\end{pmatrix}\begin{pmatrix}-3 & 5\\ 2 & 7\end{pmatrix}\right)=\begin{pmatrix}2 & 0\\ -5 & 1\end{pmatrix}\begin{pmatrix}-11 & 8\\ 5 & 33\end{pmatrix}=\begin{pmatrix}-22 & 16\\ 60 & -7\end{pmatrix}$$

$$(\boldsymbol{AB})\boldsymbol{C}=\left(\begin{pmatrix}2 & 0\\ -5 & 1\end{pmatrix}\begin{pmatrix}3 & -1\\ 1 & 4\end{pmatrix}\right)\begin{pmatrix}-3 & 5\\ 2 & 7\end{pmatrix}=\begin{pmatrix}6 & -2\\ -14 & 9\end{pmatrix}\begin{pmatrix}-3 & 5\\ 2 & 7\end{pmatrix}=\begin{pmatrix}-22 & 16\\ 60 & -7\end{pmatrix}$$

(d) We conclude that the multiplication of 2×2 matrices is associative, which can be proved as follows:

$$\boldsymbol{A}(\boldsymbol{BC})=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\begin{pmatrix} m & n \\ p & q \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix}\right)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} mr+nt & ms+nu \\ pr+qt & ps+qu \end{pmatrix}$$

$$=\begin{pmatrix} amr+ant+bpr+bqt & ams+anu+bps+bqu \\ cmr+cnt+dpr+dqt & cms+cnu+dps+dqu \end{pmatrix}$$

$$(\boldsymbol{AB})\boldsymbol{C}=\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} m & n \\ p & q \end{pmatrix}\right)\begin{pmatrix} r & s \\ t & u \end{pmatrix}=\begin{pmatrix} am+bp & an+bq \\ cm+dp & cn+dq \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

$$=\begin{pmatrix} amr+ant+bpr+bqt & ams+anu+bps+bqu \\ cmr+cnt+dpr+dqt & cms+cnu+dps+dqu \end{pmatrix}$$

So, $\boldsymbol{A}(\boldsymbol{BC})=(\boldsymbol{AB})\boldsymbol{C}$ for all real numbers *a, b, c, d, m, n, p, q, r, s, t, u.*

10. $$\boldsymbol{AB}=\begin{pmatrix} 235 & 562 & 117 \end{pmatrix}\begin{pmatrix} 120 \\ 95 \\ 56 \end{pmatrix}=(235\cdot 120+562\cdot 95+117\cdot 56)=(88<t>142)$$

AB represents the total profit ($€88< t >142$).

11. $$r\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{A}\Rightarrow r\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}+\begin{pmatrix} -12 & -18 \\ s-8 & -42 \end{pmatrix}=\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}\Rightarrow\begin{pmatrix} 2r-12 & 3r-18 \\ 5r+s-8 & 7r-42 \end{pmatrix}=\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$$

By comparing the corresponding elements, we obtain the following system of equations:

$2r-12=2;\quad 3r-18=3$

$5r+s-8=5;\quad 7r-42=7$

From the first two equations we get $r = 7$ and substituting this value in the third equation gives us $s = –22$. $r = 7$ satisfies the fourth equation too.

12. For $\boldsymbol{A}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

(a) **(i)** $\boldsymbol{A}^2=\boldsymbol{A}\times\boldsymbol{A}=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

(ii) $\boldsymbol{A}^3=\boldsymbol{A}^2\times\boldsymbol{A}=\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

(iii) $\boldsymbol{A}^4=\boldsymbol{A}^3\times\boldsymbol{A}=\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

(iv) By observing the pattern of entries in the resulting matrices, we would conclude that

$$\boldsymbol{A}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

For $\boldsymbol{B} = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}$:

(b) **(i)** $\boldsymbol{B}^2 = \boldsymbol{B} \times \boldsymbol{B} = \begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 9 & 18 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 9 & 2\cdot 9 \\ 0 & 9 \end{pmatrix}$

(ii) $\boldsymbol{B}^3 = \boldsymbol{B}^2 \times \boldsymbol{B} = \begin{pmatrix} 9 & 18 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 27 & 81 \\ 0 & 27 \end{pmatrix} = \begin{pmatrix} 27 & 3\cdot 27 \\ 0 & 27 \end{pmatrix}$

(iii) $\boldsymbol{B}^4 = \boldsymbol{B}^3 \times \boldsymbol{B} = \begin{pmatrix} 27 & 81 \\ 0 & 27 \end{pmatrix}\begin{pmatrix} 3 & 3 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 81 & 324 \\ 0 & 81 \end{pmatrix} = \begin{pmatrix} 81 & 4\cdot 81 \\ 0 & 81 \end{pmatrix}$

(iv) By rewriting the result matrices and observing the pattern of entries, we can conclude that $\boldsymbol{B}^n = \begin{pmatrix} 3^n & n\cdot 3^n \\ 0 & 3^n \end{pmatrix}$.

13. For matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ we have:

$$\boldsymbol{AB} = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x & 2 \\ y & 3 \end{pmatrix} = \begin{pmatrix} 2x+3y & 13 \\ 4x+y & 11 \end{pmatrix}$$

$$\boldsymbol{BA} = \begin{pmatrix} x & 2 \\ y & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 2x+8 & 3x+2 \\ 2y+12 & 3y+3 \end{pmatrix}$$

From $\boldsymbol{AB} = \boldsymbol{BA}$ we obtain the following system of equations:

$2x+3y = 2x+8;\quad 13 = 3x+2$

$4x+y = 2y+12;\quad 11 = 3y+3$

Solutions: $x = \frac{11}{3}, y = \frac{8}{3}$.

14. For matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ we have:

$$\boldsymbol{AB} = \begin{pmatrix} 3 & x \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 15+xy & 6+x \\ -10+y & -3 \end{pmatrix}$$

$$\boldsymbol{BA} = \begin{pmatrix} 5 & 2 \\ y & 1 \end{pmatrix}\begin{pmatrix} 3 & x \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 5x+2 \\ 3y-2 & xy+1 \end{pmatrix}$$

From $\boldsymbol{AB} = \boldsymbol{BA}$ we get the following system of equations:

$15 + xy = 11; \quad 6 + x = 5x + 2$

$-10 + y = 3y - 2; \quad -3 = xy + 1$

From the second equation we get $x = 1$, and from the third $y = -4$.

This solution satisfies the remaining two equations as well, so we have the unique solution: $x = 1$, $y = -4$.

15. For matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ we have:

$$\boldsymbol{AB} = \begin{pmatrix} 1 & 2 & 3 \\ x & 2 & -3 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix} = \begin{pmatrix} 44 & 3x-15 & 0 \\ -8x+40 & x^2+5x-6 & 12x-60 \\ 0 & x-5 & 44 \end{pmatrix}$$

$$\boldsymbol{BA} = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ x & 2 & -3 \\ 1 & 0 & 4 \end{pmatrix} = \begin{pmatrix} x^2+3x+4 & 2x-10 & -3x+15 \\ x^2-6x+5 & 2x+34 & -3x+15 \\ -2x+10 & 0 & 44 \end{pmatrix}$$

From $\boldsymbol{AB} = \boldsymbol{BA}$, and by comparing the corresponding elements, we get a system of eight equations. The only solution that satisfies all of the equations is $x = 5$. Some of the equations have two solutions, but our final solution must satisfy all the equations and therefore we cannot accept them.

16. For matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ we have:

$$\boldsymbol{AB} = \begin{pmatrix} y & 2 & y+2 \\ x & 2 & -3 \\ 1 & y-1 & 4 \end{pmatrix}\begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix} = \begin{pmatrix} -6y+50 & xy+y+2x-16 & 20y-20 \\ -8x+40 & x^2+5x-6 & 12x-60 \\ 23y-23 & xy-6y+1 & -18y+62 \end{pmatrix}$$

$$\boldsymbol{BA} = \begin{pmatrix} -8 & x+3 & 12 \\ 23 & x-6 & -18 \\ 2 & -2 & 8 \end{pmatrix}\begin{pmatrix} y & 2 & y+2 \\ x & 2 & -3 \\ 1 & y-1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} -8y+x^2+3x+12 & 2x+12y-22 & -8y-3x+23 \\ 23y+x^2-6x-18 & 2x-18y+52 & 23y-3x-8 \\ 2y-2x+8 & 8y-8 & 2y+42 \end{pmatrix}$$

From $\boldsymbol{AB} = \boldsymbol{BA}$, and by comparing the corresponding elements, we get a system of nine equations. The last equation, $-18y + 62 = 2y + 42$, gives us $y = 1$. Then, for example, by substituting $y = 1$ into the penultimate equation ($xy - 6y + 1 = 8y - 8$) we get $x = 5$. By **checking all of the remaining** equations, we verify that (5, 1) is the unique solution of this system.

Exercise 7.2

1. **(a)** Your GDC will give you the inverse. However, for demonstration of the concept, we will describe the method in this exercise.

To find the inverse of matrix $\boldsymbol{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we use the formula

$\boldsymbol{A}^{-1}=\dfrac{1}{\det(\boldsymbol{A})}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, where $\det(\boldsymbol{A})=\begin{vmatrix} a & b \\ c & d \end{vmatrix}=ad-bc$.

$$\begin{vmatrix} 3 & 7 \\ -4 & -9 \end{vmatrix}=3(-9)-7(-4)=1$$

$$\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix}^{-1}=\frac{1}{1}\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}=\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}$$

(b) As the product of two matrices, $\boldsymbol{M}=\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix}^{-1}\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}=\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$.

(c) $\boldsymbol{M}=\begin{pmatrix} (-9)\cdot 2+(-7)\cdot 3 & (-9)\cdot 1+(-7)\cdot 5 \\ 4\cdot 2+3\cdot 3 & 4\cdot 1+3\cdot 5 \end{pmatrix}=\begin{pmatrix} -39 & -44 \\ 17 & 19 \end{pmatrix}$

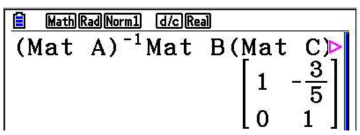
(d) **(i)** As the product of two matrices,

$\boldsymbol{N}=\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 3 & 7 \\ -4 & -9 \end{pmatrix}^{-1}=\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}$.

(ii) $\boldsymbol{N}=\begin{pmatrix} -14 & -11 \\ -7 & -6 \end{pmatrix}$

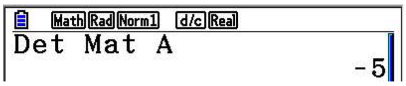
(e) In parts **(b)–(c)** we multiplied $\begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$ by the inverse from the left to determine matrix $\boldsymbol{M}$, whilst in **(d)** we multiplied it from the right to obtain matrix $\boldsymbol{N}$. The results are two different matrices as multiplication of matrices is not commutative.

2. $\boldsymbol{E}=\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}^{-1}=\frac{1}{1}\begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}\left(-\frac{1}{5}\right)\begin{pmatrix} -5 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & -\frac{3}{5} \\ 0 & 1 \end{pmatrix}$



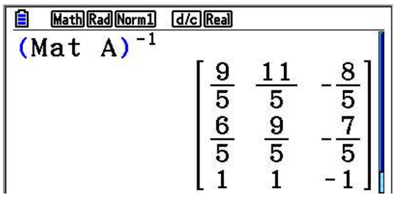
3. **(a)** Matrix $\boldsymbol{A}$ should have an inverse if $\det(\boldsymbol{A}) \neq 0$.

$$\det(\boldsymbol{A}) = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 1 & -3 \\ 3 & -2 & -3 \end{vmatrix} = -5 \neq 0$$



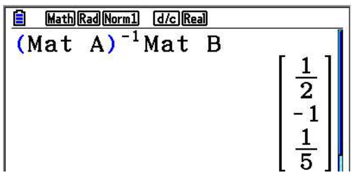
Therefore, $\boldsymbol{A}$ is non-singular, i.e. it has an inverse.

(b) For matrix $\boldsymbol{A} = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -3 \\ 3 & -2 & -3 \end{pmatrix}$, we have:



(c) For $\boldsymbol{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 4.2 \\ -1.1 \\ 2.9 \end{pmatrix}$, the system can be written as $\boldsymbol{AX} = \boldsymbol{B}$;

therefore, the solution will be $\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B}$.



So, the solution is $x = \frac{1}{2}, y = -1, z = \frac{1}{5}$.

4. **(a)** $\det(\boldsymbol{A}) = \begin{vmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix} = \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1 \Rightarrow \boldsymbol{A}^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$

(b) $\det(\boldsymbol{B}) = \begin{vmatrix} a & 1 \\ a+2 & \frac{3}{a}+1 \end{vmatrix} = a\left(\frac{3}{a}+1\right) - (a+2) = 1 \Rightarrow \boldsymbol{B}^{-1} = \begin{pmatrix} \frac{3}{a}+1 & -1 \\ -a-2 & a \end{pmatrix}$

5. Matrix $\boldsymbol{A}$ is singular if det($\boldsymbol{A}$) = 0. We have:

$$\det(\boldsymbol{A}) = \begin{vmatrix} x+1 & 3 \\ 3x-1 & x+3 \end{vmatrix} = (x+1)(x+3) - 3(3x-1) = x^2 - 5x + 6$$

$$x^2 - 5x + 6 = 0 \Rightarrow x = \frac{5 \pm \sqrt{(-5)^2 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2}$$

Therefore, $\boldsymbol{A}$ is singular for $x = 2$ or $x = 3$.

6. If one matrix is the inverse of the other, then their product is the identity matrix:

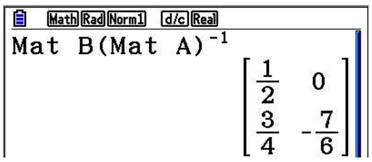
$$\begin{pmatrix} 2 & -1 & 4 \\ 2n & 2 & 0 \\ 2 & 1 & 4n \end{pmatrix}\begin{pmatrix} -2 & -3 & 4 \\ 1 & 2 & -2 \\ 3n & 2 & -5n \end{pmatrix} = \begin{pmatrix} -2 & -3 & 4 \\ 1 & 2 & -2 \\ 3n & 2 & -5n \end{pmatrix}\begin{pmatrix} 2 & -1 & 4 \\ 2n & 2 & 0 \\ 2 & 1 & 4n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 12n-5 & 0 & 10-20n \\ 2-4n & 4-6n & 8n-4 \\ -3+12n^2 & 8n-4 & 6-20n^2 \end{pmatrix} = \begin{pmatrix} 4-6n & 0 & 16n-8 \\ 4n-2 & 1 & 4-8n \\ 0 & 4-8n & 12n-20n^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

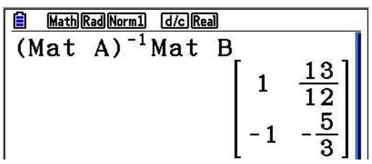
By comparing the corresponding elements, we can see that all equations are satisfied for $n = \frac{1}{2}$.

7. For $\boldsymbol{A} = \begin{pmatrix} 4 & 2 \\ 0 & -3 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$, we have:

(a) $\boldsymbol{XA} = \boldsymbol{B} \Rightarrow \boldsymbol{X} = \boldsymbol{BA}^{-1}$



(b) $\boldsymbol{AY} = \boldsymbol{B} \Rightarrow \boldsymbol{Y} = \boldsymbol{A}^{-1}\boldsymbol{B}$



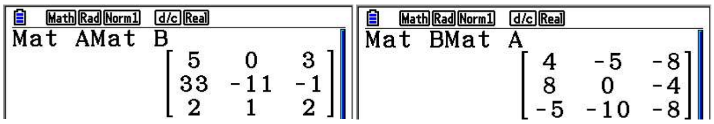
(c) We can see that $\boldsymbol{X} \neq \boldsymbol{Y}$ because, in general, multiplication of matrices is not commutative.

8. For this question we will use a GDC and, as such, we need to rename our matrices:

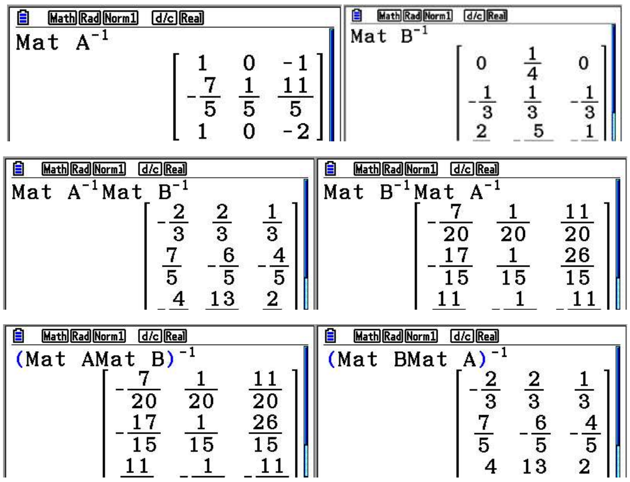
$$\boldsymbol{P}=\boldsymbol{A}=\begin{pmatrix}2 & 0 & -1\\ 3 & 5 & 4\\ 1 & 0 & -1\end{pmatrix} \text{ and } \boldsymbol{Q}=\boldsymbol{B}=\begin{pmatrix}3 & -1 & 1\\ 4 & 0 & 0\\ 1 & -2 & -1\end{pmatrix}$$

Then:

(a)



(b)



(c) We can see that the multiplication of matrices in general is not commutative: $\boldsymbol{PQ}\neq\boldsymbol{QP}$.

For the inverse matrices, the following is valid:
$(\boldsymbol{PQ})^{-1}=\boldsymbol{Q}^{-1}\boldsymbol{P}^{-1},(\boldsymbol{QP})^{-1}=\boldsymbol{P}^{-1}\boldsymbol{Q}^{-1}$.

9. For matrices $\boldsymbol{A}=\begin{pmatrix}3 & -2 & 1\\ -4 & 1 & -3\\ 1 & -5 & 1\end{pmatrix}$ and $\boldsymbol{B}=\begin{pmatrix}-29\\ 37\\ -24\end{pmatrix}$:

(a) If $\boldsymbol{AC}=\boldsymbol{B}$ then $\boldsymbol{C}=\boldsymbol{A}^{-1}\boldsymbol{B}$.

(b) By multiplying the second and third equation by (–1) the system is equivalent to the matrix notation $\boldsymbol{AX}=\boldsymbol{B}$, where $\boldsymbol{X}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$. So, $\boldsymbol{X}=\boldsymbol{A}^{-1}\boldsymbol{B}$, which means that the solution is $x=-7, y=3, z=-2$.

10. $\begin{pmatrix} 2 & 2+x \\ 5 & 4+x \end{pmatrix}\begin{pmatrix} 3 & x \\ x-4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & x \\ x-4 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2+x \\ 5 & 4+x \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x^2-2x-2 & 4x+4 \\ x^2-1 & 7x+8 \end{pmatrix} = \begin{pmatrix} 5x+6 & x^2+7x+6 \\ 2x+2 & x^2 \end{pmatrix}$

By comparing the corresponding elements and rearranging the equations, we get:

$x^2-7x-8=0 \Rightarrow x \in \{-1,8\}$; $x^2+3x+2=0 \Rightarrow x \in \{-1,-2\}$

$x^2-2x-3=0 \Rightarrow x \in \{-1,3\}$; $x^2-7x-8=0 \Rightarrow x \in \{-1,8\}$

$x=-1$ is the only common solution for all the equations, so it is the solution.

11. **(a)** $\boldsymbol{AB} = \boldsymbol{BA}$

$\Rightarrow \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} 2-x & 1 \\ 5x & y \end{pmatrix} = \begin{pmatrix} 2-x & 1 \\ 5x & y \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 3x+4 & y+2 \\ 10x+10 & 3y+5 \end{pmatrix} = \begin{pmatrix} 9-2x & 5-x \\ 10x+5y & 5x+3y \end{pmatrix}$

By comparing the corresponding elements and rearranging the equations, we get:

$5x=5$; $x+y=3$

$5y=10$; $5x=5$

The solution that satisfies all the equations is: $x=1, y=2$.

(b) $\boldsymbol{AB} = \boldsymbol{BA}$

$\Rightarrow \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} 1-x & x \\ 5x & y \end{pmatrix} = \begin{pmatrix} 1-x & x \\ 5x & y \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2x+3 & 3x+y \\ 15x-5 & -5x+2y \end{pmatrix} = \begin{pmatrix} 3-8x & x+1 \\ 15x-5y & 5x+2y \end{pmatrix}$

By comparing the corresponding elements and rearranging the equations, we get:

$10x=0$; $2x+y=1$

$5y=5$; $10x=0$

The solution that satisfies all the equations is: $x=0, y=1$.

(c) $\boldsymbol{AB = BA}$

$$\Rightarrow \begin{pmatrix} 3+x & 1 \\ -5 & 2 \end{pmatrix}\begin{pmatrix} y-x & x \\ 5x-y+1 & y+x \end{pmatrix} = \begin{pmatrix} y-x & x \\ 5x-y+1 & y+x \end{pmatrix}\begin{pmatrix} 3+x & 1 \\ -5 & 2 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} xy-x^2+2x+2y+1 & x^2+4x+y \\ 15x-7y+2 & 2y-3x \end{pmatrix} = \begin{pmatrix} xy-x^2-8x+3y & x+y \\ 5x^2-xy+11x-8y+3 & 7x+y+1 \end{pmatrix}$$

By comparing the corresponding elements and rearranging the equations, we get:

$10x-y=-1;\quad x^2+3x=0$

$5x^2-xy-4x-y=-1;\quad 10x-y=-1$

From the second equation, we get the solutions $x = 0$ or $x = -3$.

Substituting our solutions for x into the first equation, we get: $x = 0$, $y = 1$, and $x = -3$, $y = -29$.

Both solutions satisfy the third equation, so both pairs are valid.

12. **(a)** $\begin{vmatrix} x & y & 1 \\ -5 & -6 & 1 \\ 3 & 11 & 1 \end{vmatrix} = 0 \Rightarrow x(-6-11)-y(-5-3)+(-55+18)=0 \Rightarrow 17x-8y+37=0$

(b) $\begin{vmatrix} x & y & 1 \\ 5 & -2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 0 \Rightarrow x(-2+2)-y(5-3)+(-10+6)=0 \Rightarrow 2y+4=0 \Rightarrow y+2=0$

(c) $\begin{vmatrix} x & y & 1 \\ -5 & 3 & 1 \\ -5 & 8 & 1 \end{vmatrix} = 0 \Rightarrow x(3-8)-y(-5+5)+(-40+15)=0 \Rightarrow -5x-25=0 \Rightarrow x+5=0$

13. **(a)** Area $= \left\| \begin{matrix} -5 & -6 & 1 \\ 3 & 11 & 1 \\ 8 & 1 & 1 \end{matrix} \right\| = \left|[-5(11-1)+6(3-8)+(3-88)]\right| = |-165| = 165$

(b) Area $= \left\| \begin{matrix} 3 & -5 & 1 \\ 3 & 11 & 1 \\ 8 & 11 & 1 \end{matrix} \right\| = \left|[3(11-11)+5(3-8)+(33-88)]\right| = |-80| = 80$

(c) Area $= \left\| \begin{matrix} 4 & -6 & 1 \\ -3 & 9 & 1 \\ 7 & 7 & 1 \end{matrix} \right\| = \left|[4(9-7)+6(-3-7)+(-21-63)]\right| = |-136| = 136$

14. **(a)** $\text{Area} = \left|\frac{1}{2}\begin{vmatrix} x & -6 & 1 \\ 3 & 11 & 1 \\ 8 & 3 & 1 \end{vmatrix}\right| = \left|\frac{1}{2}\left[x(11-3)+6(3-8)+(9-88)\right]\right| = \frac{1}{2}|8x-109|$

$\Rightarrow \frac{1}{2}|8x-109| = 10 \Rightarrow |8x-109| = 20$

$\Rightarrow 8x-109 = 20 \text{ or } 8x-109 = -20 \Rightarrow x = \frac{129}{8} \text{ or } x = \frac{89}{8}$

(b) $\text{Area} = \left|\frac{1}{2}\begin{vmatrix} -5 & x & 1 \\ 3 & x+2 & 1 \\ x^2+2x-3 & 1 & 1 \end{vmatrix}\right|$ which implies that

$\text{Area} = \left|\frac{1}{2}\left[-5(x+2-1) - x(3-x^2-2x+3) + (3-(x+2)(x^2+2x-3))\right]\right|$

$= \left|-x^2-6x+2\right|$

We have two possibilities:

$-x^2-6x+2 = 10 \quad \text{or} \quad -x^2-6x+2 = -10 \Rightarrow$

$x_1 = -2, x_2 = -4, x_3 = -3+\sqrt{21}, x_4 = -3-\sqrt{21}$

15. **(a)** $\begin{vmatrix} 2 & -5 & 1 \\ 4 & k & 1 \\ 5 & -2 & 1 \end{vmatrix} = 0 \Rightarrow 2(k+2)+5(4-5)+1(-8-5k) = 0 \Rightarrow -3k-9 = 0 \Rightarrow k = -3$

(b) $\begin{vmatrix} -6 & 2 & 1 \\ -5 & k & 1 \\ -3 & 5 & 1 \end{vmatrix} = 0 \Rightarrow -6(k-5)-2(-5+3)+1(-25+3k) = 0 \Rightarrow -3k+9 = 0 \Rightarrow k = 3$

16. **(a)** $\det(\boldsymbol{A}) = \begin{vmatrix} 2 & 7 \\ 5 & 5 \end{vmatrix} = 2\cdot5 - 7\cdot5 = -25$

(b) $x\boldsymbol{I} - \boldsymbol{A} = x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 7 \\ 5 & 5 \end{pmatrix} = \begin{pmatrix} x-2 & -7 \\ -5 & x-5 \end{pmatrix}$

$f(x) = \det(x\boldsymbol{I} - \boldsymbol{A}) = \begin{vmatrix} x-2 & -7 \\ -5 & x-5 \end{vmatrix} = (x-2)(x-5)-(-7)(-5) = x^2-7x-25$

We can see that the constant term is equal to det(***A***).

(c) Coefficient of x appears to be the opposite of the sum of entries on the main diagonal.

(d) Replace x with $\boldsymbol{A}$

$$f(\boldsymbol{A})=\begin{pmatrix}2 & 7\\ 5 & 5\end{pmatrix}^2-7\begin{pmatrix}2 & 7\\ 5 & 5\end{pmatrix}-25\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}39 & 49\\ 35 & 60\end{pmatrix}-\begin{pmatrix}14 & 49\\ 35 & 35\end{pmatrix}-\begin{pmatrix}25 & 0\\ 0 & 25\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

(e) Generally, we have:

$$\det(\boldsymbol{A})=\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$$

$$f(x)=\det\left(x\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}-\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)=\begin{vmatrix}x-a & -b\\ -c & x-d\end{vmatrix}$$
$$=(x-a)(x-d)-bc=x^2-(a+d)x+ad-bc$$

The constant term is equal to det(**A**).

Coefficient of $x=-(a+d)$.

$$f(\boldsymbol{A})=\begin{pmatrix}a & b\\ c & d\end{pmatrix}^2-(a+d)\begin{pmatrix}a & b\\ c & d\end{pmatrix}+(ad-bc)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$
$$=\begin{pmatrix}a^2+bc & ab+bd\\ ac+cd & bc+d^2\end{pmatrix}-\begin{pmatrix}a^2+ad & ab+bd\\ ac+cd & ad+d^2\end{pmatrix}+\begin{pmatrix}ad-bc & 0\\ 0 & ad-bc\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

17. For $\boldsymbol{A}=\begin{pmatrix}2 & 7 & 1\\ -1 & 3 & 2\\ 5 & 5 & -4\end{pmatrix}$:

(a)

```
det([A])
                -22
```

(b) $x\boldsymbol{I}-\boldsymbol{A}=\begin{pmatrix}x-2 & -7 & -1\\ 1 & x-3 & -2\\ -5 & -5 & x+4\end{pmatrix}$

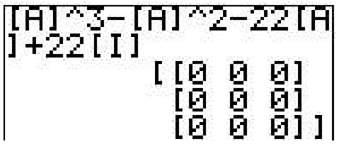
$$f(x)=\begin{vmatrix}x-2 & -7 & -1\\ 1 & x-3 & -2\\ -5 & -5 & x+4\end{vmatrix}=x^3-x^2-22x+22$$

We can see that the constant term in the expansion of $f(x)$ is equal to –det(**A**).

(c) The coefficient of x^2 is –1, which is the opposite of the sum of the main diagonal.

(d) $f(\boldsymbol{A}) = \boldsymbol{A}^3 - \boldsymbol{A}^2 - 22\boldsymbol{A} + 22\boldsymbol{I}$

We can calculate this with a GDC, knowing that $\boldsymbol{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.



f(***A***) is again the zero matrix.

(e) For $\boldsymbol{B} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ we have :

$$\det \boldsymbol{B} = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh - bdi + bfg + cdh - ceg$$

$$f(x) = \det(x\boldsymbol{I} - \boldsymbol{B}) = \begin{vmatrix} x-a & -b & -c \\ -d & x-e & -f \\ -g & -h & x-i \end{vmatrix}$$

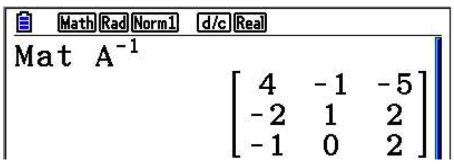
$$= x^3 - (a+e+i)x^2 + x\left(ae + ai + ei - bd - cg - fh\right) - (aei - afh - bdi + bfg + cdh - ceg)$$

We can see that the constant term in the expansion of $f(x)$ is again equal to det(***B***).

The coefficient of x^2 is $-(a+e+i)$, which is the opposite of the sum of the main diagonal.

$f(\boldsymbol{B})$ is the zero matrix.

18. **(a)** First, we need the inverse of the coding matrix. Store it into matrix ***B***.



Next step is to set up the message in matrix form using table 7.4. The matrix will be 10×3 since we have 30. Let us store it in matrix ***C***. Then multiply ***B*** and ***C***, and "clean" the result from numbers beyond [0, 29] by adding or subtracting multiples of 30. Thus, the result is

$$\begin{pmatrix} 7 & 19 & 9 & 1 & 18 & 20 & 1 & 5 & 20 & 9 \\ 1 & 19 & 19 & 0 & 5 & 0 & 20 & 13 & 9 & 1 \\ 21 & 0 & 0 & 7 & 1 & 13 & 8 & 1 & 3 & 14 \end{pmatrix}$$

Then reading the message column wise:

GAUSS IS A GREAT MATHEMATICIAN.

Exercise 7.3

1. Given matrix $\boldsymbol{A}=\begin{pmatrix} 5 & 6 \\ -1 & 0 \end{pmatrix}$:

$$\det\left(\boldsymbol{A}-m\boldsymbol{I}\right)=\det\left(\begin{pmatrix} 5 & 6 \\ -1 & 0 \end{pmatrix}-m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)=\begin{vmatrix} 5-m & 6 \\ -1 & -m \end{vmatrix}=m^2-5m+6$$

$$\Rightarrow m^2-5m+6=0 \Rightarrow m_1=2,\ m_2=3$$

2. **(a)** If ***A*** is the inverse of matrix ***B***, then $\boldsymbol{AB}=\boldsymbol{BA}=\boldsymbol{I}$ must be satisfied. So, we have:

$$\boldsymbol{AB}=\begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}$$

$$=\begin{pmatrix} a-6 & 2a-4b-6 & -2a+14 \\ 0 & 5b-9 & 0 \\ 0 & 3b-6 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{BA}=\begin{pmatrix} 1 & 2 & -2 \\ 3 & b & 1 \\ -1 & 1 & -3 \end{pmatrix}\begin{pmatrix} a & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}$$

$$=\begin{pmatrix} a-6 & 0 & 0 \\ 3a-8b-5 & 5b-9 & 7b-14 \\ -a+7 & 0 & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By comparing the corresponding elements, we find the solution $a=7$, $b=2$ satisfies all the conditions.

(b) The system of linear equations for $a = 7$, $b = 2$ can be written as: $\boldsymbol{B}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix}$.

So, we have: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \boldsymbol{B}^{-1}\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} = \boldsymbol{A}\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix}\begin{pmatrix} 5 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

3. If the matrix is singular, its determinant is zero.

$$\begin{vmatrix} 1 & m & 1 \\ 3 & 1-m & 2 \\ m & -3 & m-1 \end{vmatrix} = -m^2 + 4m - 4 = 0 \Rightarrow (m-2)^2 = 0 \Rightarrow m = 2$$

4. For each of the given systems, we will reduce the augmented matrix of the system to reduced row echelon form. From this we can conclude whether the solution is unique (and which one it is), or if there are an infinite number of solutions, or no solution. **(a)** will be done algebraically, while the rest will be done using a GDC.

(a)

$$\left(\begin{array}{ccc|c} 4 & -1 & 1 & -5 \\ 2 & 2 & 3 & 10 \\ 5 & -2 & 6 & 1 \end{array}\right) \begin{cases} 2R_2 - R_1 \\ 5R_2 - 2R_3 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 4 & -1 & 1 & -5 \\ 0 & 5 & 5 & 25 \\ 0 & 14 & 3 & 48 \end{array}\right) \left\{\frac{1}{5}R_2 \Rightarrow\right.$$

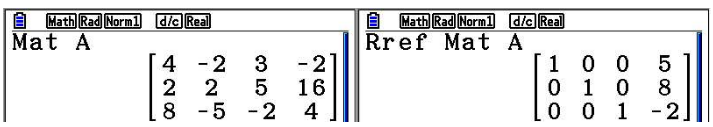
$$\left(\begin{array}{ccc|c} 4 & -1 & 1 & -5 \\ 0 & 1 & 1 & 5 \\ 0 & 14 & 3 & 48 \end{array}\right) \{14R_2 - R_3 \Rightarrow \left(\begin{array}{ccc|c} 4 & -1 & 1 & -5 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 11 & 22 \end{array}\right) \left\{\frac{1}{11}R_3 \Rightarrow\right.$$

$$\left(\begin{array}{ccc|c} 4 & -1 & 1 & -5 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right) \begin{cases} R_1 - R_3 \\ R_2 - R_3 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 4 & -1 & 0 & -7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right) \{R_1 + R_3 \Rightarrow$$

$$\left(\begin{array}{ccc|c} 4 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right) \left\{\frac{1}{4}R_1 \Rightarrow\right. \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

We can read the unique solution as: $x = -1, y = 3, z = 2$.

(b) The unique solution is: $x = 5, y = 8, z = -2$. (Rref stands for row reduced echelon form)



(c) This system has an infinite number of solutions.

$$z = 16t \Rightarrow y - \frac{19}{16}z = \frac{11}{16} \Rightarrow y = \frac{11}{16} + 19t$$

$$x - \frac{5}{16}z = \frac{13}{16} \Rightarrow x = \frac{13}{16} + 5t$$

```
Rref Mat A
 [1  0  -5/16   13/16]
 [0  1  -19/16  11/16]
 [0  0   0      0    ]
```

Notice how the last row is all zeros.

(d) The system has a unique solution is: $x = -7, y = 3, z = -2$.

```
rref([A])
  [[1 0 0 -7]
   [0 1 0 3 ]
   [0 0 1 -2]]
```

(e) Since the last row is all zeros, the system has an infinite number of solutions:
$z = t \Rightarrow y = 2 - 3t$, $x = -1 + 2t$

```
rref([A])
  [[1 0 -2 -1]
   [0 1 3   2 ]
   [0 0 0   0 ]]
```

(f) The last row shows inconsistency as the coefficients are zeros, and the answer entry is different from zero. So, inconsistent system with no solutions.

```
rref([A])
  [[1 0 -2 0]
   [0 1 3  0]
   [0 0 0  1]]
```

(g) system has a unique solution: $x = -2, y = 4, z = 3$.

```
rref([A])
  [[1 0 0 -2]
   [0 1 0 4 ]
   [0 0 1 3 ]]
```

(h) System has a unique solution: $x = 4, y = -2, z = 1$.

```
rref([A])
  [[1 0 0 4 ]
   [0 1 0 -2]
   [0 0 1 1 ]]
```

5. **(a)** When the determinant is non-zero, then the matrix will be not be singular.

$$\det(\boldsymbol{A}) = \begin{vmatrix} 1 & 1 & k-1 \\ k & 0 & -1 \\ 6 & 2 & -3 \end{vmatrix} = 2k^2 + k - 4 \Rightarrow \det(\boldsymbol{A}) = 0 \Rightarrow k = \frac{-1 \pm \sqrt{33}}{4}$$

So, the matrix is **not** singular for all $k \neq \frac{-1 \pm \sqrt{33}}{4}$.

(b) If A is the inverse of matrix $\boldsymbol{B}$, then $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$ must be satisfied. So, we have:

$$\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$$

$$\begin{pmatrix} 1 & 1 & k-1 \\ k & 0 & -1 \\ 6 & 2 & -3 \end{pmatrix}\begin{pmatrix} k-3 & -3 & k \\ 3 & k+2 & -1 \\ -2 & -4 & 1 \end{pmatrix} = \begin{pmatrix} k-3 & -3 & k \\ 3 & k+2 & -1 \\ -2 & -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & k-1 \\ k & 0 & -1 \\ 6 & 2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -k+2 & -3k+3 & 2k-2 \\ k^2-3k+2 & -3k+4 & k^2-1 \\ 6k-6 & 2k-2 & 6k-5 \end{pmatrix} = \begin{pmatrix} 4k-3 & 3k-3 & k^2-7k+6 \\ k^2+2k-3 & 1 & 2k-2 \\ 4-4k & 0 & -2k+3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By comparing the corresponding elements, we find the solution $k = 1$ satisfies all the conditions.

(c) For $k = 1$ we have:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & 2 & -3 & 0 & 0 & 1 \end{array}\right) \quad \begin{cases} R_1 \\ R_1 - R_2 \\ 6R_1 - R_3 \end{cases} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 4 & 3 & 6 & 0 & -1 \end{array}\right) \quad \{4R_2 - R_3$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right) \quad \{R_2 - R_3 \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 3 & 3 & -1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right) \quad \{R_1 - R_2$$

$$\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & 3 & 3 & -1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right)$$

The square matrix on the right-hand side of this augmented matrix is exactly $\boldsymbol{B} = \boldsymbol{A}^{-1}$.

6. (a) When the determinant is non-zero, then the matrix will be not be singular.

$$\det(A) = \begin{vmatrix} \frac{2}{5} & \frac{-17}{5} & \frac{k+9}{5} \\ \frac{-1}{5} & \frac{21}{5} & \frac{-13}{5} \\ k-2 & 3 & -2 \end{vmatrix} = \frac{1}{25}\left(-21k^2 + 71k - 63\right)$$

$$\Rightarrow 21k^2 - 71k + 63 = 0 \Rightarrow k = \frac{71 \pm \sqrt{-251}}{42}$$

Because there are no real solutions for k, $\boldsymbol{A}$ is regular (not singular) for all real numbers k.

(b) If A is the inverse of matrix $\boldsymbol{B}$, then $\boldsymbol{AB}=\boldsymbol{BA}=\boldsymbol{I}$ must be satisfied. So, we have:

$$\boldsymbol{AB}=\begin{pmatrix} \frac{2}{5} & \frac{-17}{5} & \frac{k+9}{5} \\ \frac{-1}{5} & \frac{21}{5} & \frac{-13}{5} \\ k-2 & 3 & -2 \end{pmatrix}\begin{pmatrix} k+1 & 1 & k \\ 2 & k+2 & -3 \\ 3 & 6 & -5 \end{pmatrix}$$

$$=\begin{pmatrix} k-1 & \frac{-11k+22}{5} & \frac{-3k+6}{5} \\ \frac{-k+2}{5} & \frac{21k-37}{5} & \frac{-k+2}{5} \\ (k-2)(k+1) & 4k-8 & k^2-2k+1 \end{pmatrix}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By comparing the corresponding elements, we see that the solution $k=2$ satisfies all the equations.

(c) For $k=2$:

$$\left(\begin{array}{ccc|ccc} 2 & -17 & 11 & 1 & 0 & 0 \\ -1 & 21 & -13 & 0 & 1 & 0 \\ 0 & 15 & -10 & 0 & 0 & 1 \end{array}\right) \begin{cases} R_2+R_1 \\ R_1+R_2 \end{cases} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 4 & -2 & 1 & 1 & 0 \\ 0 & 25 & -15 & 1 & 2 & 0 \\ 0 & 15 & -10 & 0 & 0 & 1 \end{array}\right) \begin{cases} \frac{1}{25}R_2 \\ \frac{1}{15}R_3 \end{cases}$$

$$\left(\begin{array}{ccc|ccc} 1 & 4 & -2 & 1 & 1 & 0 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{25} & \frac{2}{25} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 & 0 & \frac{1}{15} \end{array}\right) \begin{cases} -R_2+R_3 \\ -4R_2+R_1 \end{cases} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 4 & \frac{2}{5} & \frac{21}{25} & \frac{17}{25} & 0 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{25} & \frac{2}{25} & 0 \\ 0 & 0 & -\frac{1}{15} & -\frac{1}{25} & -\frac{2}{25} & \frac{1}{15} \end{array}\right) \{-15R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{2}{5} & \frac{21}{25} & \frac{17}{25} & 0 \\ 0 & 1 & -\frac{3}{5} & \frac{1}{25} & \frac{2}{25} & 0 \\ 0 & 0 & 1 & \frac{3}{5} & \frac{6}{5} & -1 \end{array}\right) \begin{cases} -\frac{2}{5}R_3+R_1 \\ \frac{3}{5}R_3+R_2 \end{cases} \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{5} & \frac{1}{5} & \frac{2}{5} \\ 0 & 1 & 0 & \frac{2}{5} & \frac{4}{5} & -\frac{3}{5} \\ 0 & 0 & 1 & \frac{3}{5} & \frac{6}{5} & -1 \end{array}\right)$$

7. **(a)** Use your GDC with "rref" command:

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 \end{array}\right) \begin{cases} R_1 + 2R_2 \\ R_3 - R_1 \end{cases} \Rightarrow \left(\begin{array}{ccc|ccc} 2 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & 5 & 1 & 2 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right)$$

$$\cdots \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{2}{3} & -\frac{5}{6} \\ 0 & 0 & 1 & 0 & \frac{2}{3} & \frac{1}{3} \end{array}\right)$$

B is the inverse of ***A***.

(b) Use your GDC with "rref" command:

$$\left(\begin{array}{ccc|ccc} 1 & 4 & 5 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 & 1 & 0 \\ -1 & 8 & 6 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -\frac{16}{13} & -\frac{19}{13} \\ 0 & 1 & 0 & 1 & -\frac{11}{13} & -\frac{9}{13} \\ 0 & 0 & 1 & -1 & \frac{12}{13} & \frac{11}{13} \end{array}\right)$$

B is the inverse of ***A***.

8. **(a)** For $f(x) = ax^2 + bx + c$ to contain the given points, then $f(-1) = 5, f(2) = -1, f(4) = 35$ and we obtain the following system of equations:

$a - b + c = 5$

$4a + 2b + c = -1$

$16a + 4b + c = 35$

For the augmented matrix of the system, we solve using a GDC:

```
[A]
   [[1   -1  1  5 ]
    [4   2   1  -1]
    [16  4   1  35]]
```

```
rref([A]
         [[1 0 0  4 ]
          [0 1 0  -6]
          [0 0 1  -5]]
```

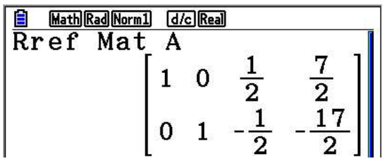
So, the function is $f(x) = 4x^2 - 6x - 5$.

(b) Similarly, $f(-1)=12, f(2)=-3$ we have the following system of equations:

$a-b+c=12$

$4a+2b+c=-3$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 12 \\ 4 & 2 & 1 & -3 \end{array}\right) \cdots \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{7}{2} \\ 0 & 1 & -\frac{1}{2} & -\frac{17}{2} \end{array}\right)$$



So, there is an infinite number of solutions:

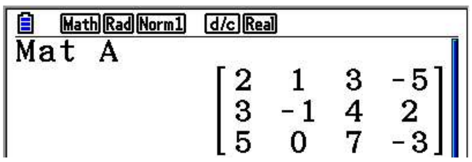
$$c=m \Rightarrow a=\frac{1}{2}(7-m), b=\frac{1}{2}(m-17) \Rightarrow$$

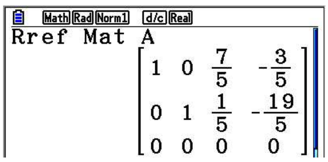
$$f(x)=\frac{1}{2}(7-m)x^2+\frac{1}{2}(m-17)+m$$

9. We use row operations to reduce the system into echelon form

$$\left(\begin{array}{ccc|c} 2 & 1 & 3 & -5 \\ 3 & -1 & 4 & 2 \\ 5 & 0 & 7 & m-5 \end{array}\right) \sim \cdots \sim \left(\begin{array}{ccc|c} 2 & 1 & 3 & -5 \\ 0 & 5 & 1 & -19 \\ 0 & 0 & 0 & 2m-4 \end{array}\right)$$

The system is consistent if last row is all zeros, thus, $2m-4=0 \Rightarrow m=2$.





The general solution is $x=-7t-\frac{3}{5}, y=-t-\frac{19}{5}, z=5t$.

10. We use row operations to reduce the system into echelon form

$$\left(\begin{array}{ccc|c} -3 & 2 & 3 & 1 \\ 4 & -1 & -5 & -5 \\ 1 & 1 & -2 & m-3 \end{array}\right) \sim \cdots \sim \left(\begin{array}{ccc|c} -3 & 2 & 3 & 1 \\ 0 & 5 & -3 & -11 \\ 0 & 0 & 0 & 3m+3 \end{array}\right)$$

The system is consistent if $3m+3=0 \Rightarrow m=-1$.

$$\left(\begin{array}{ccc|c} -3 & 2 & 3 & 1 \\ 0 & 5 & -3 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 15 & 0 & -21 & -27 \\ 0 & 5 & -3 & -11 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -\frac{7}{5} & -\frac{9}{5} \\ 0 & 1 & -\frac{3}{5} & -\frac{11}{5} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The general solution is $x=7t-\frac{9}{5}, y=3t-\frac{11}{5}, z=5t$.

11. **(a)** $\det(\boldsymbol{A}) = \begin{vmatrix} 3 & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{vmatrix} = 3(20-21)+4(-32+35)-6(-24+25)=3$

(b) here is a sample

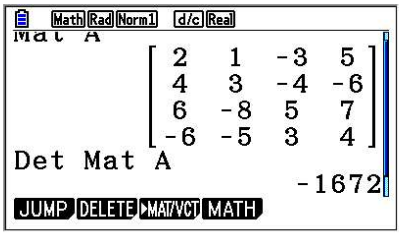
$$\begin{pmatrix} 3 & -4 & -6 \\ -8 & 5 & 7 \\ -5 & 3 & 4 \end{pmatrix} \begin{cases} \frac{8}{3}R_1 + R_2 \\ \frac{5}{3}R_1 + R_3 \end{cases} \Rightarrow \begin{pmatrix} 3 & -4 & -6 \\ 0 & -\frac{17}{3} & -9 \\ 0 & -\frac{11}{3} & -6 \end{pmatrix} \left\{ -\frac{11}{17}R_2 + R_3 \right.$$

$$\Rightarrow \begin{pmatrix} 3 & -4 & -6 \\ 0 & -\frac{17}{3} & -9 \\ 0 & 0 & -\frac{3}{17} \end{pmatrix}$$

(c) $\det(\boldsymbol{B}) = 3 \cdot \left(-\frac{17}{3}\right) \cdot \left(-\frac{3}{17}\right) = 3$

Note that this is the same determinant as for $\boldsymbol{A}$. Also, the determinant of a triangular matrix is the product of entries on the main diagonal.

(d) Here is a GDC's output



(e) A sample

$$\begin{pmatrix} 2 & 1 & -3 & 5 \\ 4 & 3 & -4 & -6 \\ 6 & -8 & 5 & 7 \\ -6 & -5 & 3 & 4 \end{pmatrix} \begin{cases} -2r_1 + r_2 \\ -3r_1 + r_3 \\ 3r_1 + r_4 \end{cases} \sim \begin{pmatrix} 2 & 1 & -3 & 5 \\ 0 & 1 & 2 & -16 \\ 0 & -14 & 14 & -8 \\ 0 & -2 & -6 & 19 \end{pmatrix} \sim \cdots \sim \begin{pmatrix} 2 & 1 & -3 & 5 \\ 0 & 1 & 2 & -16 \\ 0 & 0 & 36 & -184 \\ 0 & 0 & 0 & -\frac{209}{9} \end{pmatrix}$$

$$\det(\boldsymbol{D}) = 2 \cdot 1 \cdot 36 \cdot \left(-\frac{209}{9}\right) = -1672$$

Exercise 7.4

1. In this exercise you need to recall that finding an eigenvector and eigenvalue for a matrix is to find λ and $\vec{v}$ such that $\boldsymbol{A}\vec{v} = \lambda\vec{v}$. To that end we need to solve the equation $(\boldsymbol{A} - \lambda\boldsymbol{I})\vec{v} = \vec{0}$, which leads to solving the equation $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$.

Thus, the steps followed in all parts are:

- form the matrix $\boldsymbol{A} - \lambda\boldsymbol{I}$
- solve the equation $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$; the real solutions are the eigenvalues of A
- for each eigenvalue λ_0, form the matrix $\boldsymbol{A} - \lambda_0\boldsymbol{I}$ and solve the homogeneous system $(\boldsymbol{A} - \lambda_0\boldsymbol{I})\overrightarrow{X} = \vec{0}$.
- To diagonalize a matrix A, we write as $\boldsymbol{A} = \boldsymbol{PDP}^{-1}$, where P is the matrix whose columns are the eigenvectors and $\boldsymbol{D}$ is the diagonal matrix whose entries are the eigenvalues.

We will show details in one question, but the rest is repetitive, and thus, we will give the end result.

(a) $(\boldsymbol{A}-\lambda \boldsymbol{I})=\begin{pmatrix}3 & -1\\ 2 & 0\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}3-\lambda & -1\\ 2 & -\lambda\end{pmatrix}$

$\det(\boldsymbol{A}-\lambda \boldsymbol{I})=(3-\lambda)(-\lambda)+2=\lambda^2-3\lambda+2\Rightarrow \lambda=1$ or $\lambda=2$.

For $\lambda=1$, we solve

$(\boldsymbol{A}-1\cdot \boldsymbol{I})\vec{v}=\vec{0}\Rightarrow\begin{pmatrix}2 & -1\\ 2 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}2a-b=0\\ 2a-b=0\end{cases}\Rightarrow\vec{v}=\begin{pmatrix}t\\ 2t\end{pmatrix}$

For $\lambda=2$, we solve

$(\boldsymbol{A}-2\cdot \boldsymbol{I})\vec{v}=\vec{0}\Rightarrow\begin{pmatrix}1 & -1\\ 2 & -2\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}a-b=0\\ 2a-2b=0\end{cases}\Rightarrow\vec{v}=\begin{pmatrix}t\\ t\end{pmatrix}$

Finally, $\begin{pmatrix}3 & -1\\ 2 & 0\end{pmatrix}=\begin{pmatrix}1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}-1 & 1\\ 2 & -1\end{pmatrix}$

(b) $\lambda^2-9=0;\lambda=\pm3$; $\begin{pmatrix}2t\\ t\end{pmatrix},\begin{pmatrix}-t\\ t\end{pmatrix}$; $\begin{pmatrix}2 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}3 & 0\\ 0 & -3\end{pmatrix}\frac{1}{3}\begin{pmatrix}1 & 1\\ -1 & 2\end{pmatrix}$

(c) $\lambda^2-5\lambda+6;\lambda=2$ or $\lambda=3$; $\begin{pmatrix}t\\ t\end{pmatrix},\begin{pmatrix}0\\ t\end{pmatrix}$; $\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}2 & 0\\ 0 & 3\end{pmatrix}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$

(d) $\lambda^2-2\lambda-3;\lambda=-1$ or $\lambda=3$; $\begin{pmatrix}t\\ t\end{pmatrix},\begin{pmatrix}-t\\ 3t\end{pmatrix}$; $\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 3\end{pmatrix}\frac{1}{4}\begin{pmatrix}3 & 1\\ -1 & 1\end{pmatrix}$

(e) $\lambda^2-2\lambda-3;\lambda=-1$ or $\lambda=3$; $\begin{pmatrix}t\\ -t\end{pmatrix},\begin{pmatrix}t\\ t\end{pmatrix}$; $\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 3\end{pmatrix}\frac{1}{2}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}$

(f) $\lambda^2-3\lambda+2;\lambda=1$ or $\lambda=2$; $\begin{pmatrix}t\\ 0\end{pmatrix},\begin{pmatrix}0\\ t\end{pmatrix}$; $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

(g) $\lambda^2-6\lambda+5;\lambda=1$ or $\lambda=5$; $\begin{pmatrix}3t\\ -t\end{pmatrix},\begin{pmatrix}t\\ t\end{pmatrix}$; $\begin{pmatrix}3 & 1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 5\end{pmatrix}\frac{1}{4}\begin{pmatrix}1 & -1\\ 1 & 3\end{pmatrix}$

(h) $\lambda^2-9\lambda+20;\lambda=4$ or $\lambda=5$; $\begin{pmatrix}t\\ 3t\end{pmatrix},\begin{pmatrix}t\\ 2t\end{pmatrix}$; $\begin{pmatrix}1 & 1\\ 3 & 2\end{pmatrix}\begin{pmatrix}4 & 0\\ 0 & 5\end{pmatrix}\begin{pmatrix}-2 & 1\\ 3 & -1\end{pmatrix}$

(i) $\lambda^2-11\lambda+10;\lambda=1$ or $\lambda=10$; $\begin{pmatrix}-3t\\ t\end{pmatrix},\begin{pmatrix}3t\\ 2t\end{pmatrix}$; $\begin{pmatrix}-3 & 3\\ 1 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 10\end{pmatrix}\frac{1}{9}\begin{pmatrix}-2 & 3\\ 1 & 3\end{pmatrix}$

(j) $\lambda^2-\lambda-1; \lambda=\frac{1+\sqrt{5}}{2}$ or $\lambda=\frac{1-\sqrt{5}}{2}$; $\begin{pmatrix}\frac{1+\sqrt{5}}{2}\\1\end{pmatrix},\begin{pmatrix}\frac{1-\sqrt{5}}{2}\\1\end{pmatrix}$;

$$\begin{pmatrix}\frac{1+\sqrt{5}}{2} & \frac{1-\sqrt{5}}{2}\\1 & 1\end{pmatrix}\begin{pmatrix}\frac{1+\sqrt{5}}{2} & 0\\0 & \frac{1-\sqrt{5}}{2}\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}1 & \frac{\sqrt{5}-1}{2}\\-1 & \frac{1+\sqrt{5}}{2}\end{pmatrix}$$

(k) $\lambda^2-(a+b)\lambda+ab; \lambda=a$ or $\lambda=b$; $\begin{pmatrix}t\\0\end{pmatrix},\begin{pmatrix}0\\t\end{pmatrix}$; $\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}a&0\\0&b\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}$

(l) $\lambda^2-6\lambda+5; \lambda=5$ or $\lambda=1$; $\begin{pmatrix}t\\t\end{pmatrix},\begin{pmatrix}-t\\t\end{pmatrix}$; $\begin{pmatrix}1&-1\\1&1\end{pmatrix}\begin{pmatrix}5&0\\0&1\end{pmatrix}\frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}$

(m) $\lambda^2-a^2b^2; \lambda=ab$ or $\lambda=-ab$; $\begin{pmatrix}a\\b\end{pmatrix},\begin{pmatrix}-a\\b\end{pmatrix}$; $\begin{pmatrix}a&-a\\b&b\end{pmatrix}\begin{pmatrix}ab&0\\0&-ab\end{pmatrix}\frac{1}{2ab}\begin{pmatrix}b&a\\-b&a\end{pmatrix}$

(n) $\lambda^2-3\lambda-10; \lambda=5$ or $\lambda=-2$; $\begin{pmatrix}3t\\4t\end{pmatrix},\begin{pmatrix}t\\-t\end{pmatrix}$; $\begin{pmatrix}3&1\\4&-1\end{pmatrix}\begin{pmatrix}5&0\\0&-2\end{pmatrix}\frac{1}{7}\begin{pmatrix}1&1\\4&-3\end{pmatrix}$

(o) $\lambda^2-13\lambda-5; \lambda=\frac{13+3\sqrt{21}}{2}$ or $\lambda=\frac{13-3\sqrt{21}}{2}$; $\begin{pmatrix}\frac{3\sqrt{21}-7}{10}\\1\end{pmatrix},\begin{pmatrix}\frac{-7-3\sqrt{21}}{10}\\1\end{pmatrix}$;

$$\begin{pmatrix}\frac{3\sqrt{21}-7}{10} & \frac{-7-3\sqrt{21}}{10}\\1 & 1\end{pmatrix}\begin{pmatrix}\frac{13+3\sqrt{21}}{2} & 0\\0 & \frac{13-3\sqrt{21}}{2}\end{pmatrix}\frac{5}{3\sqrt{21}}\begin{pmatrix}1 & \frac{7+3\sqrt{21}}{10}\\-1 & \frac{3\sqrt{21}-7}{10}\end{pmatrix}$$

(p) $\lambda^2-81; \lambda=9$ or $\lambda=-9$; $\begin{pmatrix}4t\\3t\end{pmatrix},\begin{pmatrix}t\\3t\end{pmatrix}$; $\begin{pmatrix}4&1\\3&3\end{pmatrix}\begin{pmatrix}9&0\\0&-9\end{pmatrix}\frac{1}{9}\begin{pmatrix}3&-1\\-3&4\end{pmatrix}$

(q) $\lambda^2-2\lambda-3; \lambda=-1$ or $\lambda=3$; $\begin{pmatrix}t\\t\end{pmatrix},\begin{pmatrix}t\\2t\end{pmatrix}$; $\begin{pmatrix}1&1\\1&2\end{pmatrix}\begin{pmatrix}-1&0\\0&3\end{pmatrix}\begin{pmatrix}2&-1\\-1&1\end{pmatrix}$

(r) $\lambda^2+3\lambda+2; \lambda=-2$ or $\lambda=-1$; $\begin{pmatrix}-3t\\5t\end{pmatrix},\begin{pmatrix}-2t\\3t\end{pmatrix}$; $\begin{pmatrix}-3&-2\\5&3\end{pmatrix}\begin{pmatrix}-2&0\\0&-1\end{pmatrix}\begin{pmatrix}3&2\\-5&-3\end{pmatrix}$

2. **(a)** Given that state 1 has occurred, the chance that it happens again is 30%.

(b) $\boldsymbol{TX}_0 = \begin{pmatrix} 0.3 & 0.6 \\ 0.7 & 0.4 \end{pmatrix}\begin{pmatrix} 0.4 \\ 0.6 \end{pmatrix} = \begin{pmatrix} 0.48 \\ 0.52 \end{pmatrix}$

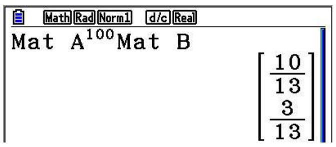
3. The transition matrix is $\begin{pmatrix} \frac{4}{5} & \frac{2}{3} \\ \frac{1}{5} & \frac{1}{3} \end{pmatrix}$

Assume that Kevin was happy on one day, then the chances for next day are

$\begin{pmatrix} \frac{4}{5} & \frac{2}{3} \\ \frac{1}{5} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{1}{5} \end{pmatrix}$, a day after that $\begin{pmatrix} \frac{4}{5} & \frac{2}{3} \\ \frac{1}{5} & \frac{1}{3} \end{pmatrix}\begin{pmatrix} \frac{4}{5} \\ \frac{1}{5} \end{pmatrix} = \begin{pmatrix} \frac{58}{75} \\ \frac{17}{75} \end{pmatrix}$, and in the long term

$\lim_{n\to\infty} \begin{pmatrix} \frac{4}{5} & \frac{2}{3} \\ \frac{1}{5} & \frac{1}{3} \end{pmatrix}^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \approx \begin{pmatrix} \frac{10}{13} \\ \frac{3}{13} \end{pmatrix}$, thus Kevin's chance being happy on a given day is 10/13

Use GDC and replace infinity with a large number. Here is a sample



4. **(a)** The matrix is given below. We chose columns to represent how each grocery customers are distributed. Thus, every column has to have a total of 1.

$$\boldsymbol{T} = \begin{pmatrix} 0.80 & 0.05 & 0.10 \\ 0.05 & 0.90 & 0.15 \\ 0.15 & 0.05 & 0.75 \end{pmatrix}$$

(b) $\boldsymbol{X}_1 = \boldsymbol{TX}_0 = \begin{pmatrix} 0.80 & 0.05 & 0.10 \\ 0.05 & 0.90 & 0.15 \\ 0.15 & 0.05 & 0.75 \end{pmatrix}\begin{pmatrix} 0.4 \\ 0.3 \\ 0.3 \end{pmatrix} = \begin{pmatrix} 0.365 \\ 0.335 \\ 0.300 \end{pmatrix}$

(c) $\boldsymbol{X}_2 = \boldsymbol{TX}_1 = \boldsymbol{T}^2\boldsymbol{X}_0 = \begin{pmatrix} 0.3388 \\ 0.3648 \\ 0.2965 \end{pmatrix}$

5. **(a)** $T = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}; X_0 = \begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} \Rightarrow X_1 = TX_0 = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}\begin{pmatrix} 0.6 \\ 0.4 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$

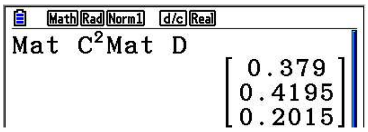
vote is evenly split.

(b) $X_2 = TX_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix}\begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.45 \\ 0.55 \end{pmatrix}$

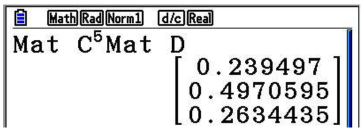
Liberal with 55%.

6. **(a)** $T = \begin{pmatrix} 0.75 & 0.05 & 0.05 \\ 0.15 & 0.90 & 0.10 \\ 0.10 & 0.05 & 0.85 \end{pmatrix}; X_0 = \begin{pmatrix} 0.6 \\ 0.3 \\ 0.1 \end{pmatrix}$

$$\Rightarrow X_2 = \begin{pmatrix} 0.75 & 0.05 & 0.05 \\ 0.15 & 0.90 & 0.10 \\ 0.10 & 0.05 & 0.85 \end{pmatrix}^2 \begin{pmatrix} 0.6 \\ 0.3 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.3790 \\ 0.4195 \\ 0.2015 \end{pmatrix} \Rightarrow \begin{cases} A & 0.3790 \\ B & 0.4195 \\ C & 0.2015 \end{cases}$$



(b) $$\Rightarrow X_5 = \begin{pmatrix} 0.75 & 0.05 & 0.05 \\ 0.15 & 0.90 & 0.10 \\ 0.10 & 0.05 & 0.85 \end{pmatrix}^5 \begin{pmatrix} 0.6 \\ 0.3 \\ 0.1 \end{pmatrix} = \begin{pmatrix} 0.2395 \\ 0.4971 \\ 0.2634 \end{pmatrix} \Rightarrow \begin{cases} A & 0.2395 \\ B & 0.4971 \\ C & 0.2634 \end{cases}$$



7. **(a)** $X_1 = TX_0 = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}; \ X_2 = TX_1 = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}\begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix} = \begin{pmatrix} 0.45 \\ 0.55 \end{pmatrix}$

$X_3 = TX_2 = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}\begin{pmatrix} 0.45 \\ 0.55 \end{pmatrix} = \begin{pmatrix} 0.525 \\ 0.475 \end{pmatrix}$

(b) $X_{11} = T^{11}X_0 = \begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}^{11}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.600 \\ 0.400 \end{pmatrix}$; long term will stabilize around 60% donation and 40% no donations.

8. $X_1 = TX_0 = \begin{pmatrix} 0.8 & 0.3 & 0.2 \\ 0.1 & 0.2 & 0.6 \\ 0.1 & 0.5 & 0.2 \end{pmatrix}\begin{pmatrix} 0.35 \\ 0.40 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0.45 \\ 0.265 \\ 0.285 \end{pmatrix} \Rightarrow X_n = T^n X_0 \approx \begin{pmatrix} 0.56 \\ 0.23 \\ 0.21 \end{pmatrix}$

9. **(a)** $\boldsymbol{T} = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix}; \boldsymbol{X}_0 = \begin{pmatrix} 200000 \\ 25000 \end{pmatrix}$

$\Rightarrow \boldsymbol{X}_1 = \boldsymbol{T}\boldsymbol{X}_0 = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix}\begin{pmatrix} 200000 \\ 25000 \end{pmatrix} = \begin{pmatrix} 190750 \\ 34250 \end{pmatrix}, \dots$

	Now	1 year	2 years	3 years	4 years	5 years
City	200000	190750	182240	174411	167208	160581
Suburbs	25000	34250	42760	50589	57792	64419

(b) $\Rightarrow \boldsymbol{X}_n = \boldsymbol{T}^n\boldsymbol{X}_0 = \begin{pmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{pmatrix}^n \begin{pmatrix} 200000 \\ 25000 \end{pmatrix} \approx \begin{pmatrix} 84375 \\ 140625 \end{pmatrix}.$

City population will approach 84375 and the suburbs population will approach 140625.

Exercise 7.5

1. **(a)** y-axis reflection. (0, 0), (−3, 0), (−3, 1)

(b) dilation in both directions of magnitude 2. (0, 0), (6, 0), (6, 2)

(c) x-axis reflection. (0, 0), (3, 0), (3, −1)

(d) reflection in $y = x$. (0, 0), (0, 3), (1, 3)

(e) dilation in y-direction of magnitude 3. (0, 0), (3, 0), (3, 3)

(f) Composition: reflection in $y = x$ and reflection in y-axis since

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, (0, 0), (0, 3), (−1, 3)

2. Since $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, then it is a dilation followed by reflection in $y = x$

3. A cannot represent a rotation since there is no angle with $\sin\theta = \frac{1}{2}$ and $\sin\theta = -\frac{1}{2}$ at the same time!

4. **(a)** $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x_1}{3} \\ \frac{y_1}{3} \end{pmatrix}$, now substitute the values into the original equation: $3\left(\frac{x_1}{3}\right) + 2\left(\frac{y_1}{3}\right) = 6 \Rightarrow 3x_1 + 2y_1 = 18$, which can be written as $3x + 2y = 18$

(b) $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3y \\ 3x \end{pmatrix} \Rightarrow \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} \frac{x_1}{3} \\ \frac{y_1}{3} \end{pmatrix}$, now substitute the values into the original equation: $3\left(\frac{y_1}{3}\right) + 2\left(\frac{x_1}{3}\right) = 6 \Rightarrow 3y_1 + 2x_1 = 18$, which can be written as $2x + 3y = 18$

5. The composition of transformations is achieved with matrix multiplication keeping in mind that the order is reversed! $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

6. **(a)** $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x_1}{2} \\ \frac{y_1}{2} \end{pmatrix}$; now substitute the values into the original equation: $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{2} - 1\right)^2 = 9 \Rightarrow x^2 + (y-2)^2 = 36$; a circle with centre at (0, 1) and radius 3 is transformed into a circle with centre at (0, 2) and radius 6.

(b) $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ -y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{x_1}{2} \\ -y_1 \end{pmatrix}$; now substitute the values into the original equation: $3x - 4y = 12$; A line with slope -1.5 and y-intercept 3 is transformed into a line with slope 0.75 and y-intercept -3.

7. **(a)** As in question 5, product of matrices = $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos 90 & -\sin 90 \\ \sin 90 & \cos 90 \end{pmatrix}$

(b) Product of matrices = $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(-90) & -\sin(-90) \\ \sin(-90) & \cos(-90) \end{pmatrix}$

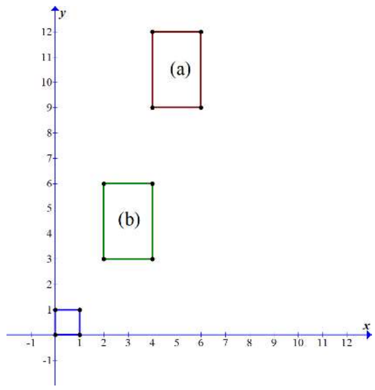
8. **(a)** Translation first:

$\begin{pmatrix}2\\3\end{pmatrix},\begin{pmatrix}3\\3\end{pmatrix},\begin{pmatrix}3\\4\end{pmatrix},\begin{pmatrix}2\\4\end{pmatrix}$

Then dilation

$$\begin{pmatrix}2&0\\0&3\end{pmatrix}\begin{pmatrix}2&3&3&2\\3&3&4&4\end{pmatrix}$$

$$=\begin{pmatrix}4&6&6&4\\9&9&12&12\end{pmatrix}$$



(b) Dilation first: $\begin{pmatrix}2&0\\0&3\end{pmatrix}\begin{pmatrix}0&1&1&0\\0&0&1&1\end{pmatrix}=\begin{pmatrix}0&2&2&0\\0&0&3&3\end{pmatrix}$

Then translation: $\begin{pmatrix}0&2&2&0\\0&0&3&3\end{pmatrix}\Rightarrow\begin{pmatrix}2&4&4&2\\3&3&6&6\end{pmatrix}$

9. $\begin{pmatrix}x_1\\y_1\end{pmatrix}=\begin{pmatrix}1&0\\3&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x\\3x+y\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}x_1\\y_1-3x_1\end{pmatrix}$, then substitute into the original equation: $3x+y=6\Rightarrow 3x_1+y_1-3x_1=6\Rightarrow y=6$.

For example, here are the images of three points $\begin{pmatrix}2&1&0\\0&3&6\end{pmatrix}\mapsto\begin{pmatrix}2&1&0\\6&6&6\end{pmatrix}$

10. **(a)** Eigenvectors: $\begin{pmatrix}k_1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\k_2\end{pmatrix}$. Images of vectors along the x- and y-axes are multiples of the original vectors.

(b) Eigenvectors: $\begin{pmatrix}k\\k\end{pmatrix}$ and $\begin{pmatrix}k\\-k\end{pmatrix}$. Images of vectors along the y = x and $y=-x$ are multiples of the original vectors.

(c) Eigenvectors: $\begin{pmatrix}-k\\k\end{pmatrix}$ and $\begin{pmatrix}k\\k\end{pmatrix}$. Images of vectors along the y = x and $y=-x$ are multiples of the original vectors.

(d) Eigenvectors: $\begin{pmatrix}-2k\\3k\end{pmatrix}$ and $\begin{pmatrix}k\\2k\end{pmatrix}$. Images of vectors along $y=-\frac{3}{2}x$ and $y=2x$ are multiples of the original vectors.

11. **(a)** $\begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$.

(b)
$$\boldsymbol{M}(\beta)\boldsymbol{M}(\alpha) = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$
$$= \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta & -\sin\alpha\cos\beta - \cos\alpha\sin\beta \\ \cos\alpha\sin\beta + \sin\alpha\cos\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{pmatrix}$$

(c) By comparing corresponding entries we recognise the formulas for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$.

12. **(a)** The first iteration transforms *AD* into "broken" line *ABECD*, which is $\frac{1}{3}$ unit longer, i.e., it is $\frac{4}{3}$ of the original. This process is repeated and at each stage the previous line is extended to $\frac{4}{3}$ of the previous line. Thus the length of the *n*th iterated line is $\left(\frac{4}{3}\right)^n$.

(b) $\lim\limits_{n\to\infty}\left(\frac{4}{3}\right)^n = \infty$

(c) Each iterated side is made into 4 smaller sides and thus the number on each side is 4^n. For the whole triangle it is $3\cdot 4^n$.

(d) Every smaller side is $\frac{1}{3}$ the previous one, and so, on the *n*th iteration it is $\left(\frac{1}{3}\right)^n$

(e) $3\cdot 4^n \cdot \left(\frac{1}{3}\right)^n = 3\left(\frac{4}{3}\right)^n$; $\lim\limits_{n\to\infty} 3\left(\frac{4}{3}\right)^n = \infty$

(f) Except for the original triangle, each side is made into 4 smaller sides, each of which creates a triangle, and so, we have $3\cdot 4^{n-1}$ triangles.

(g) Each smaller triangle is similar to the previous one and has a side $\frac{1}{3}$ of its side, thus, the ratio of the areas is the square of the ratio of similarity, $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$.

Thus the area of each "iterated" triangle is

$a_0 \cdot \left(\frac{1}{9}\right)^n, a_0 =$ area of original triangle $= \frac{\sqrt{3}}{4}$

(h) $$a_0 + a_0\left(\sum_1^n 3\cdot 4^{i-1}\cdot\left(\frac{1}{9}\right)^i\right) = a_0\left(1+\sum_1^n \frac{3}{4}\cdot\left(\frac{4}{9}\right)^i\right)$$

$$\Rightarrow \lim_{n\to\infty} = a_0\left(1+\frac{3}{4}\cdot\frac{\frac{4}{9}}{1-\frac{4}{9}}\right) = \frac{8}{5}\cdot\frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{5}$$

Chapter 7 Practice questions

1. $\det(\boldsymbol{A}) = \begin{vmatrix} 2x & 3 \\ -4x & x \end{vmatrix} = 2x\cdot x - 3(-4x) = 2x^2 + 12x$

$\det(\boldsymbol{A}) = 14 \Rightarrow 2x^2 + 12x = 14 \Rightarrow x^2 + 6x - 7 = 0 \Rightarrow x_1 = -7,\ x_2 = 1$

2. **(a)** $\boldsymbol{M}^2 = \begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix}^2 = \begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} a & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} a^2+4 & 2a-2 \\ 2a-2 & 5 \end{pmatrix}$

(b) $\boldsymbol{M}^2 = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} \Rightarrow \begin{pmatrix} a^2+4 & 2a-2 \\ 2a-2 & 5 \end{pmatrix} = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix} \Rightarrow \begin{cases} a^2+4=5 \\ 2a-2=-4 \end{cases}$

The solution $a = -1$ satisfies both equations.

(c) For $a = -1$, we have $\boldsymbol{M} = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$

$\boldsymbol{M}^{-1} = \frac{1}{\det(\boldsymbol{M})}\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}$

The system of equations can be written as:
$\boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{M}^{-1}\begin{pmatrix} -3 \\ 3 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix}\begin{pmatrix} -3 \\ 3 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$

Therefore, the solution is $x = 1, y = -1$.

3. $\boldsymbol{BA} = \begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix} \Rightarrow \boldsymbol{B} = \begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}\boldsymbol{A}^{-1}.$

$\boldsymbol{A} = \begin{pmatrix} 5 & 2 \\ 2 & 0 \end{pmatrix} \Rightarrow \boldsymbol{A}^{-1} = -\frac{1}{4}\begin{pmatrix} 0 & -2 \\ -2 & 5 \end{pmatrix}$

$\Rightarrow \boldsymbol{B} = -\frac{1}{4}\begin{pmatrix} 11 & 2 \\ 44 & 8 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 5 \end{pmatrix} = -\frac{1}{4}\begin{pmatrix} -4 & -12 \\ -16 & -48 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$

4. $\boldsymbol{AX}+\boldsymbol{X}=\boldsymbol{B} \Rightarrow \begin{pmatrix} 3 & 1 \\ -5 & 6 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}+\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 4 & 8 \\ 0 & -3 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 4a+c & 4b+d \\ -5a+7c & -5b+7d \end{pmatrix}=\begin{pmatrix} 4 & 8 \\ 0 & -3 \end{pmatrix}$

By comparing the corresponding elements, we obtain two systems of linear equations:

$4a+c=4$	$4b+d=8$
$\underline{-5a+7c=0}$	$\underline{-5b+7d=-3}$
$a=\dfrac{28}{33},\ c=\dfrac{20}{33}$	$b=\dfrac{59}{33},\ d=\dfrac{28}{33}$

5. **(a)** $\boldsymbol{A}^{-1}=\dfrac{1}{5+14}\begin{pmatrix} 1 & 2 \\ -7 & 5 \end{pmatrix}=\dfrac{1}{19}\begin{pmatrix} 1 & 2 \\ -7 & 5 \end{pmatrix}=\begin{pmatrix} \frac{1}{19} & \frac{2}{19} \\ \frac{-7}{19} & \frac{5}{19} \end{pmatrix}$

(b) **(i)** $\boldsymbol{XA}+\boldsymbol{B}=\boldsymbol{C} \Rightarrow \boldsymbol{XA}=\boldsymbol{C}-\boldsymbol{B} \Rightarrow \boldsymbol{X}=(\boldsymbol{C}-\boldsymbol{B})\boldsymbol{A}^{-1}$

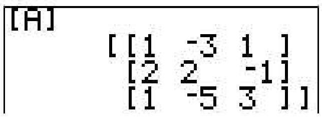
(ii) $X=\left(\begin{pmatrix} -5 & 0 \\ -8 & 7 \end{pmatrix}-\begin{pmatrix} 6 & 7 \\ 5 & -2 \end{pmatrix}\right)\begin{pmatrix} \frac{1}{19} & \frac{2}{19} \\ \frac{-7}{19} & \frac{5}{19} \end{pmatrix}$

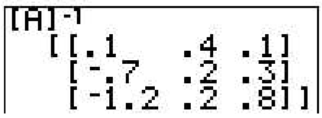
$=\dfrac{1}{19}\begin{pmatrix} -11 & -7 \\ -13 & 9 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -7 & 5 \end{pmatrix}=\begin{pmatrix} 2 & -3 \\ -4 & 1 \end{pmatrix}$

6. **(a)** $\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix} a & b \\ c & 1 \end{pmatrix}+\begin{pmatrix} 1 & 2 \\ d & c \end{pmatrix}=\begin{pmatrix} a+1 & b+2 \\ c+d & c+1 \end{pmatrix}$

(b) $\boldsymbol{AB}=\begin{pmatrix} a & b \\ c & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ d & c \end{pmatrix}=\begin{pmatrix} a+bd & 2a+bc \\ c+d & 3c \end{pmatrix}$

7. **(a)** Using a GDC:





(b) The system of equations can be written as:

$$A\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\2\\3\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=A^{-1}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

For A as above and

```
[B]
                [[1]
                 [2]
                 [3]]
```

```
[A]⁻¹[B]
                [[1.2]
                 [.6 ]
                 [1.6]]
```

The solution is $x=\frac{6}{5}, y=\frac{3}{5}, z=\frac{8}{5}$.

8. Given $\boldsymbol{C}=\begin{pmatrix}-2&4\\1&7\end{pmatrix}$, $\boldsymbol{D}=\begin{pmatrix}5&2\\-1&a\end{pmatrix}$, and $3\boldsymbol{Q}=2\boldsymbol{C}-\boldsymbol{D}$, we have:

(a) $$\boldsymbol{Q}=\frac{1}{3}(2\boldsymbol{C}-\boldsymbol{D})=\frac{1}{3}\left[2\begin{pmatrix}-2&4\\1&7\end{pmatrix}-\begin{pmatrix}5&2\\-1&a\end{pmatrix}\right]=\frac{1}{3}\begin{pmatrix}-9&6\\3&14-a\end{pmatrix}=\begin{pmatrix}-3&2\\1&\frac{14-a}{3}\end{pmatrix}$$

(b) $$\boldsymbol{CD}=\begin{pmatrix}-2&4\\1&7\end{pmatrix}\begin{pmatrix}5&2\\-1&a\end{pmatrix}=\begin{pmatrix}-10-4&-4+4a\\5-7&2+7a\end{pmatrix}=\begin{pmatrix}-14&4a-4\\-2&7a+2\end{pmatrix}$$

(c) $$\boldsymbol{D}^{-1}=\begin{pmatrix}5&2\\-1&a\end{pmatrix}=\frac{1}{5a+2}\begin{pmatrix}a&-2\\1&5\end{pmatrix}$$

9. **(a)** If $\boldsymbol{A}$ is the inverse of matrix $\boldsymbol{B}$, then $\boldsymbol{AB}=\boldsymbol{BA}=\boldsymbol{I}$ must be satisfied. So, we have:

$$\boldsymbol{AB}=\begin{pmatrix}a&-4&-6\\-8&5&7\\-5&3&4\end{pmatrix}\begin{pmatrix}1&2&-2\\3&b&1\\-1&1&-3\end{pmatrix}=\begin{pmatrix}a-6&2a-4b-6&-2a+14\\0&5b-9&0\\0&3b-6&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\boldsymbol{BA}=\begin{pmatrix}1&2&-2\\3&b&1\\-1&1&-3\end{pmatrix}\begin{pmatrix}a&-4&-6\\-8&5&7\\-5&3&4\end{pmatrix}=\begin{pmatrix}a-6&0&0\\3a-8b-5&5b-9&7b-14\\-a+7&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

By comparing the corresponding elements, we can see that the solution $a=7, b=2$ satisfies all the equations.

(b) The system can be written as:

$$B\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\0\\6\end{pmatrix}\Rightarrow\begin{pmatrix}1&2&-2\\3&2&1\\-1&1&-3\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\0\\6\end{pmatrix}\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&2&-2\\3&2&1\\-1&1&-3\end{pmatrix}^{-1}\begin{pmatrix}5\\0\\6\end{pmatrix}$$

Since $\boldsymbol{AB}=\boldsymbol{BA}=\boldsymbol{I}\ \Rightarrow\boldsymbol{B}^{-1}=\boldsymbol{A}$, then:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&2&-2\\3&2&1\\-1&1&-3\end{pmatrix}^{-1}\begin{pmatrix}5\\0\\6\end{pmatrix}=\begin{pmatrix}7&-4&-6\\-8&5&7\\-5&3&4\end{pmatrix}\begin{pmatrix}5\\0\\6\end{pmatrix}=\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

The solution is: $x=-1, y=2, z=-1$.

10. **(a)** $\boldsymbol{AB}=\boldsymbol{C}\ \Rightarrow\ \boldsymbol{B}=\boldsymbol{A}^{-1}\boldsymbol{C}$.

(b) **(i)** For $\boldsymbol{DA}$:

```
[D][A]
        [[1 0 0]
         [0 1 0]
         [0 0 1]]
```

(ii) $\boldsymbol{B}=\boldsymbol{A}^{-1}\boldsymbol{C}$.

```
[A]⁻¹[C]
        [[1 ]
         [-1]
         [2 ]]
```

(c) The coordinates of the point of intersection of the planes are given as the solution of the system of equations that can be written as:

$$\boldsymbol{A}\begin{pmatrix}x\\y\\z\end{pmatrix}=\boldsymbol{C}\ \Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\boldsymbol{A}^{-1}\boldsymbol{C}=\boldsymbol{B}$$

The point has coordinates $(1, -1, 2)$.

11. **(a)** $\det(\boldsymbol{A})=\begin{vmatrix}1&1&2\\1&2&1\\2&1&5\end{vmatrix}=1(10-1)-1(5-2)+2(1-4)=0$

(b) We transform the augmented matrix of the system:

$$\left(\begin{array}{ccc|c}1&1&2&3\\1&2&1&4\\2&1&5&\lambda\end{array}\right)\begin{cases}R_2-R_1\\R_3-2R_1\end{cases}\sim\left(\begin{array}{ccc|c}1&1&2&3\\0&1&-1&1\\0&-1&1&\lambda-6\end{array}\right)\{R_3+R_2$$

$$\sim\left(\begin{array}{ccc|c}1&1&2&3\\0&1&-1&1\\0&0&0&\lambda-5\end{array}\right)$$

The general solution of the system exists if $\lambda-5=0\Rightarrow\lambda=5$.

(c) For $\lambda=5$ we have:

$$\left(\begin{array}{ccc|c}1&1&2&3\\0&1&-1&1\\0&0&0&0\end{array}\right)\{-R_2+R_1\sim\left(\begin{array}{ccc|c}1&0&3&2\\0&1&-1&1\\0&0&0&0\end{array}\right)$$

For $z=t\Rightarrow x=2-3t,\ y=1+t.$

12. Eigenvalues:

$$|A-\lambda I|=\left|\begin{pmatrix}2&2\\5&-1\end{pmatrix}-\begin{pmatrix}\lambda&0\\0&\lambda\end{pmatrix}\right|=\lambda^2-\lambda-12=0\Rightarrow\lambda=-3\text{ or }\lambda=4;$$

Eigenvectors:

$$\lambda=-3\Rightarrow\begin{pmatrix}5&2\\5&2\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow5a+2b=0\Rightarrow v=\begin{pmatrix}2t\\-5t\end{pmatrix}$$

$$\lambda=4\Rightarrow\begin{pmatrix}-2&2\\5&-5\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow a-b=0\Rightarrow v=\begin{pmatrix}t\\t\end{pmatrix}$$

13. **(a)** Transition matrix: $\boldsymbol{T}=\begin{pmatrix}0.86&0.62&0.17\\0.08&0.29&0.37\\0.06&0.09&0.46\end{pmatrix}$. Initial state: $\boldsymbol{X}_0=\begin{pmatrix}0.80\\0.11\\0.09\end{pmatrix}$

$\boldsymbol{X}_1=\boldsymbol{T}\boldsymbol{X}_0=\begin{pmatrix}0.772\\0.139\\0.099\end{pmatrix}$; D: 0.772; L: 0.129, P: 0.099

(b) $\boldsymbol{X}_3=\boldsymbol{T}^3\boldsymbol{X}_0=\begin{pmatrix}0.755\\0.139\\0.106\end{pmatrix}$; D: 0.755; L: 0.139; P: 0.106

14. **(a)** Eigenvalues: $\begin{vmatrix} 1-\lambda & 0 \\ 6 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1 \text{ or } \lambda = -1$

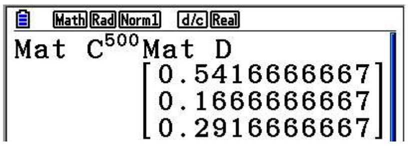
Eigenvectors: $\begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow P = \begin{pmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{pmatrix} \Rightarrow D = P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) $\boldsymbol{A}^8 = \boldsymbol{P}\boldsymbol{D}^8\boldsymbol{P}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

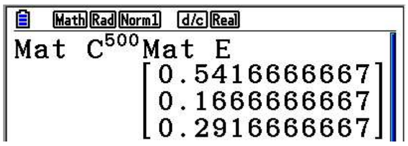
15. Transition matrix: $\boldsymbol{T} = \begin{pmatrix} 0.90 & 0.15 & 0.10 \\ 0.05 & 0.75 & 0.05 \\ 0.05 & 0.10 & 0.85 \end{pmatrix}$

Now, in the long run, regardless of the initial state, the distribution will stabilize around $54.1\overline{6}$ in region 1, $16.\overline{6}$ in region 2, and $29.1\overline{6}$ in region 3. See 2 scenarios for starting initial states below.

With initial state of 80% 10% and 10% respectively, we have



With initial state of 40%, 30%, and 30%, have



16. **(a)** $\boldsymbol{X}(t) = \begin{pmatrix} 0.80 & 0.90 \\ 0.20 & 0.10 \end{pmatrix} \boldsymbol{X}(t-1) \Rightarrow \boldsymbol{X}(1) = \begin{pmatrix} 0.80 & 0.10 \\ 0.20 & 0.90 \end{pmatrix} \begin{pmatrix} 0.50 \\ 0.50 \end{pmatrix} = \begin{pmatrix} 0.45 \\ 0.55 \end{pmatrix}$

(b)

	1	2	3	4	5
Channel 1	0.45	0.415	0.391	0.373	0.361
Channel 2	0.55	0.585	0.609	0.627	0.639

(c) $\lim_{t\to\infty} \boldsymbol{X}(t) = \begin{pmatrix} 0.80 & 0.10 \\ 0.20 & 0.90 \end{pmatrix}^t \begin{pmatrix} 0.50 \\ 0.50 \end{pmatrix} = \begin{pmatrix} 0.\overline{3} \\ 0.\overline{6} \end{pmatrix}$

Channel $1 \approx 0.\overline{3}$

Channel $2 \approx 0.\overline{6}$

17. **(a)** Stage 2 has 9, stage 3 has 27, stage 4 has $81 = 3^4$, and stage has $243 = 3^5$.

(b) 3^n

(c) $\frac{3}{4}a, \frac{9}{16}a, \frac{27}{64}a, \cdots, \left(\frac{3}{4}\right)^n a$, area tends to zero.

(d) $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}; \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}; \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ \frac{\sqrt{3}}{4} \end{pmatrix}$

18. Transition matrix: $\boldsymbol{T} = \begin{pmatrix} 0.60 & 0.26 & 0.14 \\ 0.29 & 0.37 & 0.34 \\ 0.16 & 0.27 & 0.57 \end{pmatrix}$, Initial state: $\boldsymbol{X}_0 = (0.12, 0.32, 0.56)$

Notice that this is a "horizontal" arrangement, and hence the next stage is

$$\boldsymbol{X}_0\boldsymbol{T} = (0.12, 0.32, 0.56)\begin{pmatrix} 0.60 & 0.26 & 0.14 \\ 0.29 & 0.37 & 0.34 \\ 0.16 & 0.27 & 0.57 \end{pmatrix} = (0.2544, 0.3008, 0.4448)$$

$N = 0$	**$N = 1$**	**$N = 2$**	**$N = 3$**	**$N = 4$**	**$N = 5$**
0.12	0.254	0.311	0.336	0.346	0.351
0.32	0.301	0.298	0.297	0.296	0.296
0.56	0.425	0.391	0.368	0.357	0.353

Table produced with GDC. Some minor discrepancies are due to rounding.

Exercise 8.1

1. **(a)** $\boldsymbol{v} = 2\begin{pmatrix}5\\-2\end{pmatrix} = \begin{pmatrix}10\\-4\end{pmatrix}$

(b) $\boldsymbol{v} = -3\begin{pmatrix}5\\-2\end{pmatrix} = \begin{pmatrix}-15\\6\end{pmatrix}$

(c) $\boldsymbol{u} = \frac{\boldsymbol{v}}{3} \Rightarrow \boldsymbol{v} = 3\boldsymbol{u} = 3\begin{pmatrix}5\\-2\end{pmatrix} = \begin{pmatrix}15\\-6\end{pmatrix}$

2. **(a)** $\boldsymbol{w} = \begin{pmatrix}6\\0\\-3\end{pmatrix} - \begin{pmatrix}2\\-1\\2\end{pmatrix} = \begin{pmatrix}4\\1\\-5\end{pmatrix}$

(b) $\boldsymbol{w} = \begin{pmatrix}6\\0\\-3\end{pmatrix} + 2\begin{pmatrix}2\\-1\\2\end{pmatrix} = \begin{pmatrix}10\\-2\\1\end{pmatrix}$

(c) $\boldsymbol{w} = 3\begin{pmatrix}6\\0\\-3\end{pmatrix} - 4\begin{pmatrix}2\\-1\\2\end{pmatrix} = \begin{pmatrix}10\\4\\-17\end{pmatrix}$

3. Unit vector $= \frac{1}{|\boldsymbol{u}|} \cdot \boldsymbol{u}$

(a) $|\boldsymbol{u}| = \sqrt{12^2 + 5^2} = 13 \Rightarrow$ unit vector $= \begin{pmatrix}\frac{12}{13}\\ \frac{-5}{13}\end{pmatrix}$

(b) $|\boldsymbol{u}| = \sqrt{1^2 + 3^2} = \sqrt{10} \Rightarrow$ unit vector $= \begin{pmatrix}\frac{-1}{\sqrt{10}}\\ \frac{3}{\sqrt{10}}\end{pmatrix}$

(c) $|\boldsymbol{u}| = \sqrt{1^2 + 2^2 + 2^2} = 3 \Rightarrow$ unit vector $= \begin{pmatrix}\frac{1}{3}\\ \frac{2}{3}\\ \frac{2}{3}\end{pmatrix}$

4. $\begin{pmatrix} x-2 \\ y \end{pmatrix} = \begin{pmatrix} 12-8 \\ 12-4 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

5. $\begin{pmatrix} 2-5 \\ y-4 \end{pmatrix} = \begin{pmatrix} 0-x \\ 3-2 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$

6. $\begin{pmatrix} 8 \\ 46 \end{pmatrix} = \begin{pmatrix} r \\ 9r \end{pmatrix} + \begin{pmatrix} s \\ -4s \end{pmatrix} = \begin{pmatrix} r+s \\ 9r-4s \end{pmatrix} \Rightarrow r = 6,\ s = 2$

7. $\begin{pmatrix} 4 \\ 7 \end{pmatrix} = r\begin{pmatrix} 2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2r+2s \\ 3r+s \end{pmatrix} \Rightarrow r = \frac{5}{2}, s = -\frac{1}{2}$

$\Rightarrow \begin{pmatrix} 4 \\ 7 \end{pmatrix} = \frac{5}{2}\begin{pmatrix} 2 \\ 3 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

8. $\begin{pmatrix} 5 \\ -5 \end{pmatrix} = r\begin{pmatrix} 1 \\ -1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} r-s \\ -r+s \end{pmatrix} \Rightarrow r-s=5 \Rightarrow s = r-5$

$\Rightarrow \begin{pmatrix} 5 \\ -5 \end{pmatrix} = r\begin{pmatrix} 1 \\ -1 \end{pmatrix} + (r-5)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

9. $\begin{pmatrix} -11 \\ 0 \end{pmatrix} = r\begin{pmatrix} 2 \\ 5 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2r+3s \\ 5r+2s \end{pmatrix} \Rightarrow r = 2, s = -5$

$\Rightarrow \begin{pmatrix} -11 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 5 \end{pmatrix} - 5\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

10. A vector $\boldsymbol{v}$ of magnitude 10 and parallel to vector $\boldsymbol{u}$ is of the form $\boldsymbol{v} = 10\cdot$ unit vector $\boldsymbol{u} = 10\cdot\frac{1}{|\boldsymbol{u}|}\cdot\boldsymbol{u}$

(a) $\boldsymbol{v} = 10\cdot\frac{1}{\left|\sqrt{4^2+3^2}\right|}\cdot\begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \end{pmatrix}$

(b) $\boldsymbol{v} = 10\cdot\frac{1}{\left|\sqrt{5^2+12^2}\right|}\cdot\begin{pmatrix} -5 \\ -12 \end{pmatrix} = \begin{pmatrix} \frac{-50}{13} \\ \frac{-120}{13} \end{pmatrix}$

(c) $\boldsymbol{v} = 10\cdot\frac{1}{\left|\sqrt{2^2+1^2+2^2}\right|}\cdot\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{20}{3} \\ \frac{-10}{3} \\ \frac{20}{3} \end{pmatrix}$

Exercise 8.2

1. For a line with gradient $m = \frac{q}{p}$, a direction vector for the line is of the form $\boldsymbol{v} = \begin{pmatrix} p \\ q \end{pmatrix}$.

An equation of such a line passing through a point (h, k) is of the form: $\frac{x-h}{p} = \frac{y-k}{q}$

(a) $\frac{x-1}{4} = \frac{y+3}{3}$

(b) $\frac{x-6}{4} = \frac{y+1}{1}$

(c) $\frac{x+9}{2} = \frac{y-5}{-3}$

(d) $\frac{x-7}{5} = \frac{y-1}{-2}$

2. A line with direction vector $\boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, through point $P(x_0, y_0, z_0)$ has a cartesian equation of the form $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$, with a, b, and c as non-zero real numbers.

(a) $\frac{x-6}{1} = \frac{y-7}{-2} = \frac{z-0}{2} \Rightarrow x-6 = \frac{y-7}{-2} = \frac{z}{2}$

(b) $\frac{x+3}{-3} = \frac{y+2}{3} = \frac{z-9}{6}$

3. A line with direction vector $\boldsymbol{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, through point $P(x_0, y_0, z_0)$ has a vector equation of the form $r = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + k\begin{pmatrix} a \\ b \\ c \end{pmatrix}$, with a, b, and c as real numbers.

(a) $\boldsymbol{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + k\begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$

(b) $\boldsymbol{r} = \begin{pmatrix} -2 \\ 3 \\ 0 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

4. **(a)** If $x = 6$, then a point on this line is of the form

$$r=\begin{pmatrix}6\\y\end{pmatrix}=\begin{pmatrix}3\\2\end{pmatrix}+k\begin{pmatrix}1\\3\end{pmatrix}\Rightarrow 6=3+k\Rightarrow k=3\Rightarrow y=2+3k=11$$

$$\Rightarrow\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}6\\11\end{pmatrix}$$

(b) If $x = 6$, then a point on this line is of the form

$$r=\begin{pmatrix}6\\y\end{pmatrix}=\begin{pmatrix}-2\\4\end{pmatrix}+k\begin{pmatrix}2\\-1\end{pmatrix}\Rightarrow 6=-2+2k\Rightarrow k=4\Rightarrow y=4-4=0$$

$$\Rightarrow\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}6\\0\end{pmatrix}$$

(c) If $x = 6$, then a point on this line is of the form

$$r=\begin{pmatrix}6\\y\\z\end{pmatrix}=\begin{pmatrix}8\\0\\-1\end{pmatrix}+k\begin{pmatrix}2\\2\\-3\end{pmatrix}\Rightarrow 6=8+2k\Rightarrow k=-1\Rightarrow y=-2, z=2$$

$$\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\-2\\2\end{pmatrix}$$

(d) If $x = 6$, then a point on this line is of the form

$$r=\begin{pmatrix}6\\y\\z\end{pmatrix}=\begin{pmatrix}5\\-2\\7\end{pmatrix}+k\begin{pmatrix}2\\2\\-4\end{pmatrix}\Rightarrow 6=5+2k\Rightarrow k=\frac{1}{2}\Rightarrow y=-1, z=5$$

$$\Rightarrow\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}6\\-1\\5\end{pmatrix}$$

5. A line with direction vector $\boldsymbol{u}=\begin{pmatrix}a\\b\\c\end{pmatrix}$, through point $P(x_0, y_0, z_0)$ has a vector equation of the form $r=\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}+k\begin{pmatrix}a\\b\\c\end{pmatrix}$, with a, b, and c as real numbers.

(a) $\boldsymbol{r}=\begin{pmatrix}3\\-2\\1\end{pmatrix}+k\begin{pmatrix}2\\-2\\4\end{pmatrix}$

(b) $\boldsymbol{r}=\begin{pmatrix}0\\3\\-1\end{pmatrix}+k\begin{pmatrix}3\\-2\\1\end{pmatrix}$

(c) $\boldsymbol{r}=\begin{pmatrix}-3\\0\\-2\end{pmatrix}+k\begin{pmatrix}\frac{1}{2}\\5\\-2\end{pmatrix}$

(d) $\boldsymbol{r}=\begin{pmatrix}a\\b\\c\end{pmatrix}+k\begin{pmatrix}m\\n\\p\end{pmatrix}$

6. **(a)** The line has the same direction vector as the one given, but the point $P(x_0, y_0, z_0)$ is different $\Rightarrow \boldsymbol{r}=\begin{pmatrix}3\\-1\\-5\end{pmatrix}+k\begin{pmatrix}2\\-2\\4\end{pmatrix}$

(b) Similar to **(a)**: $\boldsymbol{r}=\begin{pmatrix}3\\-1\\-5\end{pmatrix}+k\begin{pmatrix}3\\2\\-4\end{pmatrix}$

(c) Direction vector $\boldsymbol{u}=\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}3\\-1\\5\end{pmatrix}$, and $P(x_0, y_0, z_0)=(3,-1,-5)$,

$$\boldsymbol{r}=\begin{pmatrix}3\\-1\\-5\end{pmatrix}+k\begin{pmatrix}3\\-1\\5\end{pmatrix}$$

(d) Direction vector $\boldsymbol{u}=\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}7\\1\\-2\end{pmatrix}$, and $P(x_0, y_0, z_0)=(3,-1,-5)$

$$\boldsymbol{r}=\begin{pmatrix}3\\-1\\-5\end{pmatrix}+k\begin{pmatrix}7\\1\\-2\end{pmatrix}$$

7. $\boldsymbol{a} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}$, and $\boldsymbol{c} = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$

8. **(a)** A direction vector can be $\overrightarrow{AB} = \begin{pmatrix} -1+3 \\ 11-7 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \Rightarrow \boldsymbol{r} = \begin{pmatrix} -3 \\ 7 \end{pmatrix} + k\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

(b) A direction vector can be $\overrightarrow{CD} = \begin{pmatrix} -2-2 \\ 1+5 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} \Rightarrow \boldsymbol{r} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} + k\begin{pmatrix} -4 \\ 6 \end{pmatrix}$

(c) A direction vector can be $\overrightarrow{EF} = \begin{pmatrix} -3-8 \\ 10+2 \\ 7-1 \end{pmatrix} = \begin{pmatrix} -11 \\ 12 \\ 6 \end{pmatrix} \Rightarrow \boldsymbol{r} = \begin{pmatrix} 8 \\ -2 \\ 1 \end{pmatrix} + k\begin{pmatrix} -11 \\ 12 \\ 6 \end{pmatrix}$

(d) A direction vector can be $\overrightarrow{GH} = \begin{pmatrix} 7-0 \\ -1+6 \\ 0+3 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix} \Rightarrow \boldsymbol{r} = \begin{pmatrix} 0 \\ -6 \\ -3 \end{pmatrix} + k\begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix}$

(e) A direction vector can be $\overrightarrow{JK} = \begin{pmatrix} 5-3 \\ -2+4 \\ 5-5 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{r} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix} + k\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$

(f) A direction vector can be $\overrightarrow{LM} = \begin{pmatrix} 2+7 \\ -4+4 \\ 12-2 \end{pmatrix} = \begin{pmatrix} 9 \\ 0 \\ 10 \end{pmatrix} \Rightarrow \boldsymbol{r} = \begin{pmatrix} -7 \\ -4 \\ 2 \end{pmatrix} + k\begin{pmatrix} 9 \\ 0 \\ 10 \end{pmatrix}$

9. Apply the cartesian form: $\dfrac{x-x_0}{a} = \dfrac{y-y_0}{b} = \dfrac{z-z_0}{c}$

(a) $\dfrac{x+3}{2} = \dfrac{y-7}{4}$

(b) $\dfrac{x-2}{-4} = \dfrac{y+5}{6}$

(c) $\dfrac{x-8}{-11} = \dfrac{y+2}{12} = \dfrac{z-1}{6}$

(d) $\dfrac{x}{7} = \dfrac{y+6}{5} = \dfrac{z+3}{3}$

(e) $\dfrac{x-3}{2} = \dfrac{y+4}{2}$, $z = 5$

(f) $\dfrac{x+7}{9} = \dfrac{z-2}{10}$, $y = -4$

10. $r_1 = r_2 \Rightarrow \begin{pmatrix} 5+3k \\ 1-2k \end{pmatrix} = \begin{pmatrix} -2+4t \\ 2+t \end{pmatrix} \Rightarrow \begin{cases} 5+3k = -2+4t \\ 1-2k = 2+t \end{cases} \Rightarrow k=-1; t=1$

When we substitute the values into their respective equations, we get the intersection at $(2,3)$

11. If we write the equations in parametric form, then equate the respective coordinates

$$\begin{pmatrix} 2+t \\ 2+3t \\ 3+t \end{pmatrix} = \begin{pmatrix} 2+s \\ 3+4s \\ 4+2s \end{pmatrix} \Rightarrow \begin{cases} 2+t = 2+s \\ 2+3t = 3+4s \Rightarrow s = t = -1 \\ 3+t = 4+2s \end{cases}$$

Substitute these values into the equations we find that $(x, y, z) = (1, -1, 2)$

Exercise 8.3

1. The distance travelled is the magnitude of the direction vector multiplied by time.

(a) $|\boldsymbol{u}| = \sqrt{1.2^2 + 2^2} \approx 2.33$, after 1 minute the distance is $2.33 \times 1 = 2.33$ m.

(b) after 2 minutes the distance is $2.33 \times 2 = 4.66$ m

(c) after 10 minutes the distance is $2.33 \times 10 = 23.3$ m

(d) after t minutes the distance is $2.33 \times t = 2.33t$ m

2. If we consider the starting position as O(0, 0), then the observer is at the point (10,0)

(a) After 1 minute the robot is at (1.2, 2) and the distance will be

$$d = \sqrt{(1.2-12)^2 + 2^2} \approx 11.0 \text{ m}$$

(b) After 2 minutes the robot is at (2.4, 4) and the distance will be

$$d = \sqrt{(2.4-12)^2 + 4^2} \approx 10.4 \text{ m}$$

3. The new robot is at $\begin{pmatrix} 22 \\ 0 \end{pmatrix}$ and moving in the direction $\begin{pmatrix} -2 \\ 2 \end{pmatrix}$, and thus its position t minutes after the start at $\begin{pmatrix} 22 \\ 0 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 \end{pmatrix} t$. The position vector of the first robot is $\begin{pmatrix} 1.2 \\ 2 \end{pmatrix} t$. Distance between the two robots is given by

$$D = \sqrt{(22-2t-1.2t)^2 + (2t-2t)^2} = 22 - 3.2t$$

Since this is a linear function and as a distance has to be non-negative, the minimum is 0 and it happens at $t = \frac{22}{3.2} = 6.875$ minutes.

4. **(a)** At the point $P(24,16,48)$, the distance from $O(0,0,0)$ is

$$D=\sqrt{24^2+16^2+48^2}=56\,\text{m}$$

(b) in 10 s it will move $\sqrt{2^2+1^2+2^2}\times 10=30\,\text{m}$.

(c) in 10 s it will be at $\begin{pmatrix}24\\16\\48\end{pmatrix}+\begin{pmatrix}20\\10\\-20\end{pmatrix}=\begin{pmatrix}44\\26\\28\end{pmatrix}$ and thus its distance from O is

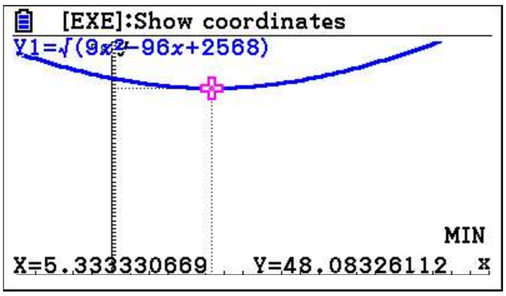
$$D=\sqrt{44^2+26^2+28^2}\approx 58.3\,\text{m}.$$

5. **(a)** at time t the drone is at $\begin{pmatrix}24\\16\\48\end{pmatrix}+\begin{pmatrix}2\\1\\-2\end{pmatrix}t=\begin{pmatrix}24+2t\\16+t\\48-2t\end{pmatrix}$ and thus, the displacement from the sensor is $d=\begin{pmatrix}24+2t\\16+t\\48-2t\end{pmatrix}-\begin{pmatrix}10\\0\\2\end{pmatrix}=\begin{pmatrix}14+2t\\16+t\\46-2t\end{pmatrix}$

(b) The minimum distance is the minimum of

$$\sqrt{(14+2t)^2+(16+t)^2+(46-2t)^2}=\sqrt{9t^2-96t+2568}$$

Using your GDC this minimum is approximately 48.1 m



6. Positions of the drones are given by

$$\begin{pmatrix}1300\\2800\\1000\end{pmatrix}+\begin{pmatrix}20\\20\\20\end{pmatrix}t \text{ and } \begin{pmatrix}1000\\4000\\2000\end{pmatrix}+\begin{pmatrix}30\\-20\\-10\end{pmatrix}t$$

(a) One way to look at this, is to find the equations of the paths and find their point of intersection.

The gradient of the first path is $\frac{20}{20}=1$ and it contains the point (1300, 2800) and so its equation is $y-2800=x-1300\Rightarrow y=x+1500$

The gradient of the second path is $\frac{-20}{30}=-\frac{2}{3}$ and it contains the point (1000, 400) and so its equation is $y-4000=-\frac{2}{3}(x-1000)\Rightarrow y=-\frac{2}{3}x+\frac{14000}{3}$

At the point of intersection $x+1500=-\frac{2}{3}x+\frac{14000}{3}\Rightarrow x=1900\Rightarrow y=3400$

Thus, the point of intersection is (1900, 3400).

(b) The time taken corresponds to

$$\begin{pmatrix}1300\\2800\end{pmatrix}+\begin{pmatrix}20\\20\end{pmatrix}t=\begin{pmatrix}1900\\3400\end{pmatrix}\Rightarrow t=30 \text{ and also}$$

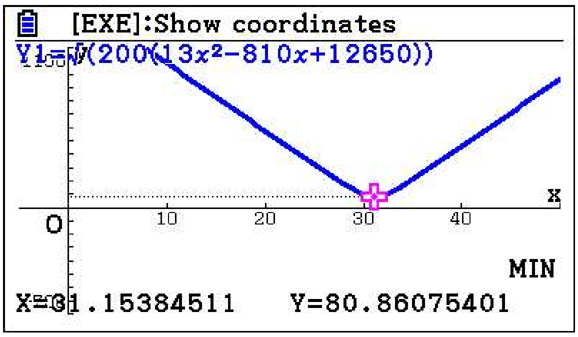
$$\begin{pmatrix}1000\\4000\end{pmatrix}+\begin{pmatrix}30\\-20\end{pmatrix}t=\begin{pmatrix}1900\\3400\end{pmatrix}, t=30.$$

(c) From appearance, the answer is yes since they arrive at the point at the same time. (in fact, they don't collide since at time t = 30, they will be at different altitudes – the first at 1600 and the second at 1700)

(d, e) The distance between the drones is

$$d=\sqrt{(300-10t)^2+(1200-40t)^2+(1000-30t)^2}$$

Using GDC, the minimum distance is 80.9 at 31.2 seconds.



Exercise 8.4

1. **(a)** $\boldsymbol{u}\cdot\boldsymbol{v}=4\times3-3\times3=3$

(b) $\boldsymbol{u}\cdot\boldsymbol{v}=6\times-2-6\times-2=0$

(c) $\boldsymbol{u}\cdot\boldsymbol{v}=3\times-1-1\times5+4\times2=0$

(d) $\boldsymbol{u}\cdot\boldsymbol{v}=-7\times-1+4\times-4+3\times3=0$

2. **(a)** $\boldsymbol{u}\cdot\boldsymbol{v}=6\cdot9\cdot\cos30°=27\sqrt{3}\approx46.8$

(b) $\boldsymbol{u}\cdot\boldsymbol{v}=12\cdot8\cdot\cos45°=48\sqrt{2}\approx67.9$

(c) $\boldsymbol{u}\cdot\boldsymbol{v}=12\cdot3\cdot\cos23°\approx33.1$

(d) $\boldsymbol{u}\cdot\boldsymbol{v}=10\cdot13\cdot\cos13°\approx127$

3. $\overrightarrow{AB} = \begin{pmatrix} 15 \\ 40 \end{pmatrix} \Rightarrow \text{work} = \begin{pmatrix} 30 \\ 150 \end{pmatrix} \cdot \begin{pmatrix} 15 \\ 40 \end{pmatrix} = 6450 \text{ joules}$

4. By formula or GDC:

(a) $\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{v} \times \boldsymbol{u} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

(b) $\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v} \times \boldsymbol{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$

(c) $\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} \times \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 14 \\ -20 \\ -19 \end{pmatrix}, \boldsymbol{v} \times \boldsymbol{u} = \begin{pmatrix} 4 \\ -1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -14 \\ 20 \\ 19 \end{pmatrix}$

(d) $\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} -7 \\ 2 \\ -3 \end{pmatrix} \times \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 48 \\ 18 \\ -10 \end{pmatrix}, \boldsymbol{v} \times \boldsymbol{u} = \begin{pmatrix} -2 \\ 2 \\ 6 \end{pmatrix} \times \begin{pmatrix} -7 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -48 \\ -18 \\ 10 \end{pmatrix}$

Exercise 8.5

1. **(a)** $\theta = \cos^{-1} \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \cos^{-1} \frac{-6+6}{\sqrt{9+1}\sqrt{4+36}} = 90°$

(b) $\theta = \cos^{-1} \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \cos^{-1} \frac{-10+12}{\sqrt{25+16}\sqrt{4+9}} = 85°$

(c) $\theta = \cos^{-1} \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \cos^{-1} \frac{-6-2+4}{\sqrt{4+1+4}\sqrt{9+4+4}} = 109°$

(d) $\theta = \cos^{-1} \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \cos^{-1} \frac{12+6+18}{\sqrt{9+4+4}\sqrt{16+9+81}} = 32°$

2. If two vectors are orthogonal, then $\boldsymbol{u} \cdot \boldsymbol{v} = 0$ and if they are parallel then their components are proportional.

(a) components are not proportional, and $\boldsymbol{u} \cdot \boldsymbol{v} = 1+1 = 2 \neq 0$. So, neither parallel nor orthogonal.

(b) $\boldsymbol{u} \cdot \boldsymbol{v} = 48 - 48 = 0$. So, orthogonal

(c) $\boldsymbol{u} \cdot \boldsymbol{v} = 2\sqrt{3} - 2\sqrt{3} = 0$. So, orthogonal

3. Find the vectors corresponding to the three sides and then find the angles between these vectors.

(a) $\overrightarrow{AB}=\begin{pmatrix}2\\2\end{pmatrix}, \overrightarrow{AC}=\begin{pmatrix}1\\3\end{pmatrix}, \overrightarrow{BC}=\begin{pmatrix}2\\-2\end{pmatrix}$

$\overrightarrow{AB}\cdot\overrightarrow{BC}=4-4=0 \Rightarrow \measuredangle B=90°$

$\measuredangle C=\cos^{-1}\dfrac{4}{\sqrt{1+9}\sqrt{4+4}}\approx 63.4°$ ($\measuredangle C$ is between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.)

$\measuredangle A=180-90-63.4\approx 26.6°$

(b) $\overrightarrow{AB}=\begin{pmatrix}-4\\-11\end{pmatrix}, \overrightarrow{AC}=\begin{pmatrix}-11\\-6\end{pmatrix}, \overrightarrow{BC}=\begin{pmatrix}-7\\5\end{pmatrix}$

$\measuredangle A=\cos^{-1}\dfrac{44+66}{\sqrt{16+121}\sqrt{121+36}}\approx 41.4°$

$\measuredangle B=\cos^{-1}\dfrac{-28+55}{\sqrt{137}\sqrt{49+25}}\approx 74.4°$

$\measuredangle C=180-41.4-74.4\approx 64.2°$

4. Let the vector be $\begin{pmatrix}a\\b\end{pmatrix}$, then $\begin{pmatrix}a\\b\end{pmatrix}\cdot\begin{pmatrix}3\\5\end{pmatrix}=3a+5b=0\Rightarrow b=-\dfrac{3}{5}a$

Let $a=5t\Rightarrow b=-3t$, thus, any vector of the form $\begin{pmatrix}5t\\-3t\end{pmatrix}$ will be perpendicular to $\begin{pmatrix}3\\5\end{pmatrix}$.

5. Find vectors representing the sides of the triangle and then find the scalar products of these vectors.

$\overrightarrow{AB}=\begin{pmatrix}1\\3\end{pmatrix}, \overrightarrow{AC}=\begin{pmatrix}5\\1\end{pmatrix}, \overrightarrow{BC}=\begin{pmatrix}4\\-2\end{pmatrix}$

$\Rightarrow \overrightarrow{AB}\cdot\overrightarrow{AC}=8; \overrightarrow{AB}\cdot\overrightarrow{BC}=-2; \overrightarrow{AC}\cdot\overrightarrow{BC}=18$

Since none of these products is zero, then there is no right angle in this triangle.

6. for the vectors to be perpendicular, then

$\begin{pmatrix}-6\\b\end{pmatrix}\cdot\begin{pmatrix}b\\b^2\end{pmatrix}=0\Rightarrow b^3-6b=0\Rightarrow b\left(b^2-6\right)\Rightarrow b=0, b=\pm\sqrt{6}$

(a zero vector is perpendicular to all vectors!)

7. $\cos 30 = \dfrac{3x+4}{\sqrt{3^2+4^2}\sqrt{x^2+1}} = \dfrac{\sqrt{3}}{2} \Rightarrow 6x+8 = 5\sqrt{3\left(x^2+1\right)}$

By squaring both sides and simplifying, we get

$x = \dfrac{48 \pm 25\sqrt{3}}{39}$

8. A rhombus is a parallelogram with adjacent sides *equal* in length. Thus, if we denote the vectors for the adjacent sides by $\boldsymbol{a}$ and $\boldsymbol{b}$, then the diagonals of this parallelogram are $\boldsymbol{d}_1 = \boldsymbol{a}+\boldsymbol{b}$ and $\boldsymbol{d}_2 = \boldsymbol{a}-\boldsymbol{b}$. Now,

$$\begin{aligned}\boldsymbol{d}_1 \cdot \boldsymbol{d}_2 &= (\boldsymbol{a}+\boldsymbol{b})(\boldsymbol{a}-\boldsymbol{b}) = \boldsymbol{a}^2 - \boldsymbol{ab} + \boldsymbol{ab} - \boldsymbol{b}^2 \\ &= \boldsymbol{a}^2 - \boldsymbol{b}^2 = |\boldsymbol{a}|^2 - |\boldsymbol{b}|^2 = 0\end{aligned}$$

Therefore, $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ are perpendicular to each other.

9. Let $\boldsymbol{M}(x, y)$ be any point on the circle. A geometric key fact is that $A\hat{M}B$ is a right angle.

(a) $\overrightarrow{AM} = \begin{pmatrix} x-1 \\ y-2 \end{pmatrix}, \overrightarrow{BM} = \begin{pmatrix} x-3 \\ y-4 \end{pmatrix} \Rightarrow \overrightarrow{AM} \cdot \overrightarrow{BM} = 0$

$\Rightarrow (x-1)(x-3)+(y-2)(y-4) = x^2-4x+y^2-6y+17 = 0$

(b) $\overrightarrow{AM} = \begin{pmatrix} x-3 \\ y-4 \end{pmatrix}, \overrightarrow{BM} = \begin{pmatrix} x+1 \\ y+7 \end{pmatrix} \Rightarrow \overrightarrow{AM} \cdot \overrightarrow{BM} = 0$

$\Rightarrow (x-3)(x+1)+(y-4)(y+7) = x^2-2x+y^2+3y-31 = 0$

Chapter 8 Practice questions

1. **(a)** $\boldsymbol{v} = \begin{pmatrix} -8 \\ 2 \end{pmatrix}$

(b) $\boldsymbol{v} = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

(c) $\boldsymbol{v} = \begin{pmatrix} 12 \\ -3 \end{pmatrix}$

2. **(a)** $w = \begin{pmatrix} -8 \\ 10 \\ -10 \end{pmatrix}$

(b) $w = \begin{pmatrix} 8 \\ -10 \\ 10 \end{pmatrix}$

(c) $w = \begin{pmatrix} 34 \\ -26 \\ 41 \end{pmatrix}$

3. **(a)** Direction vector is $\begin{pmatrix} 5 \\ 3 \end{pmatrix} \Rightarrow \frac{x+2}{5} = \frac{y-2}{3}$

(b) Direction vector is $\begin{pmatrix} 3 \\ -2 \end{pmatrix} \Rightarrow \frac{x-4}{3} = \frac{y+1}{-2}$

(c) Direction vector is $\begin{pmatrix} 2 \\ 3 \end{pmatrix} \Rightarrow \frac{x}{2} = \frac{y-4}{3}$

(d) Direction vector is $\begin{pmatrix} 3 \\ -4 \end{pmatrix} \Rightarrow \frac{x-11}{3} = \frac{y-7}{-4}$

4. **(a)** $\frac{x-3}{2} = \frac{y}{-3} = \frac{z+2}{-6}$

(b) $\frac{x+4}{5} = \frac{z}{-2},\ y = 4$

5. **(a)** $r = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix} + k\begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$

(b) $r = \begin{pmatrix} -9 \\ 3 \\ -3 \end{pmatrix} + k\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$

6. The boat's motion is described by $\boldsymbol{u}t$ where $\boldsymbol{u}$ is the velocity vector.

(a) Distance covered in 1 minute $= |\boldsymbol{u}| \cdot 1 = \sqrt{1^2 + 2^2} = \sqrt{5} \approx 2.24$

(b) Distance covered in 2.25 minutes $= |\boldsymbol{u}| \cdot 2.25 = \sqrt{5} \times 2.25 \approx 5.03$

(c) Distance covered in 10 minutes $= |\boldsymbol{u}| \cdot 10 = \sqrt{5} \times 10 \approx 22.4$

(d) Distance covered in t minutes $= |\boldsymbol{u}| \cdot t = \sqrt{5}t \approx 2.24t$

7. **(a)** Boat's position is $\begin{pmatrix}-1\\2\end{pmatrix}t$ and the controller is 20 m west of the boat at $\begin{pmatrix}-20\\0\end{pmatrix}$.

In 1 minute, the boat is at $\begin{pmatrix}-1\\2\end{pmatrix}$ and the distance between the two is

$\sqrt{(1-20)^2+(2-0)^2}\approx 19.1\,\text{m}$

(b) In t minutes, the boat is at $\begin{pmatrix}-1\\2\end{pmatrix}t$ and the distance between the two is

$\sqrt{(t-20)^2+(2t)^2}=\sqrt{5t^2-40t+400}$

8. **(a)** **(i)** The second boat's position is $\begin{pmatrix}-25\\0\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}t$ and after 1 minute it is at $\begin{pmatrix}-23\\1\end{pmatrix}$, While the first boat is at $\begin{pmatrix}-1\\2\end{pmatrix}$.

Thus, the distance between them is $\sqrt{(-23+1)^2+1}\approx 22$

(ii) The second boat's position is $\begin{pmatrix}-25\\0\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}t$ While the first boat is at $\begin{pmatrix}-1\\2\end{pmatrix}t$. Thus, the distance between them is

$\sqrt{(-25+3t)^2+t^2}=\sqrt{10t^2-150t+625}$.

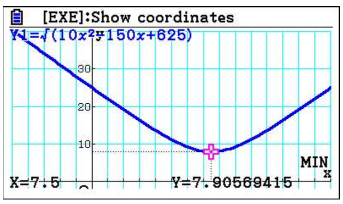
(b) Two ways of looking at this:

One method is to look at their positions if they were to collide.

At the point of collision, $\begin{pmatrix}-25\\0\end{pmatrix}+\begin{pmatrix}2\\1\end{pmatrix}t=\begin{pmatrix}-1\\2\end{pmatrix}t\Rightarrow\begin{cases}-25+2t=-t\\t=2t\end{cases}$

This system is inconsistent, and hence it has no solution, and thus, no collision.

Alternatively, the minimum distance between the two boats is 7.9 m. Thus, the distance will not reach zero and consequently they will not collide.



9. **(a)** $\boldsymbol{u}\cdot\boldsymbol{v} = -5\times 2 + 2\times 1 = -8$

(b) $\boldsymbol{u}\cdot\boldsymbol{v} = -3\times -6 - 6\times 3 = 0$

(c) $\boldsymbol{u}\cdot\boldsymbol{v} = 8\times -1 + 2\times 4 - 7\times 0 = 0$

(d) $\boldsymbol{u}\cdot\boldsymbol{v} = -2\times 2 + 2\times -3 - 1\times 6 = -16$

10. **(a)** $\boldsymbol{u}\cdot\boldsymbol{v} = 7\times 11\times \cos 60 = \frac{77}{2}$

(b) $\boldsymbol{u}\cdot\boldsymbol{v} = 11.2\times 5\times \cos 120 = -28$

(c) $\boldsymbol{u}\cdot\boldsymbol{v} = 9\times 9\times \cos 45 = \frac{81\sqrt{2}}{2}$

(d) $\boldsymbol{u}\cdot\boldsymbol{v} = 13\times 6\times \cos 23 \approx 71.8$

11. Use GDC or formula.

(a) $\boldsymbol{u}\times\boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$`

(b) $\boldsymbol{u}\times\boldsymbol{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(c) $\boldsymbol{u}\times\boldsymbol{v} = \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix}$

(d) $\boldsymbol{u}\times\boldsymbol{v} = \begin{pmatrix} 14 \\ -34 \\ 1 \end{pmatrix}$

12. **(a)** $\theta = \cos^{-1}\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \cos^{-1}\frac{-36+36}{\sqrt{81+144}\sqrt{16+9}} = 90°$

(b) $\theta = \cos^{-1}\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \cos^{-1}\frac{-195+96}{\sqrt{169+144}\sqrt{225+64}} \approx 109°$

(c) $\theta = \cos^{-1}\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \cos^{-1}\frac{2-2-4}{\sqrt{4+1+4}\sqrt{1+4+4}} \approx 116°$

(d) $\theta = \cos^{-1}\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|} = \cos^{-1}\frac{-3+12-4}{\sqrt{1+4+4}\sqrt{9+36+4}} \approx 76°$

13. **(a)** speed of Ryan's plane $= |\boldsymbol{v}| = \sqrt{4^2 + 2^2 + 4^2} = 6\text{ m s}^{-1}$

(b) $r = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + 2\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \\ 8 \end{pmatrix}$, the height is the third component, i.e., 8 m.

(c) Ryan's direction $= \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix}$, and Jack's direction $= \begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix}$.

$\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix} = -16 - 12 + 28 = 0$, thus, the paths are perpendicular.

(d) At collision:

Rayan's plane: $r = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix} + t\begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -23 \\ 20 \\ 28 \end{pmatrix} \Rightarrow \begin{cases} 5 - 4t = -23 \\ 6 + 2t = 20 \\ 4t = 28 \end{cases} \Rightarrow t = 7$

Jack's plane: $r = \begin{pmatrix} -39 \\ 44 \\ 0 \end{pmatrix} + s\begin{pmatrix} 4 \\ -6 \\ 7 \end{pmatrix} = \begin{pmatrix} -23 \\ 20 \\ 28 \end{pmatrix} \Rightarrow \begin{cases} -39 + 4s = -23 \\ 44 - 6s = 20 \\ 7s = 28 \end{cases} \Rightarrow s = 4$

Therefore, Jack's plane took off 7 – 4 = 3 seconds after Ryan's.

14. **(a)** **(i)** $\overrightarrow{AB} = \begin{pmatrix} -1 - (-2) \\ 3 - 4 \\ 1 - 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

(ii) $\left|\overrightarrow{AB}\right| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6}$

(b) $\boldsymbol{r} = \boldsymbol{r}_0 + k\boldsymbol{u} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

(c) $\begin{pmatrix} 0 \\ y \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 + k \\ 4 - k \\ 3 - 2k \end{pmatrix} \Rightarrow k = 2 \Rightarrow y = 4 - 2 = 2$

(d) **(i)** $\overrightarrow{OC} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \Rightarrow \overrightarrow{OC} \cdot \overrightarrow{AB} = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 0$

(ii) The lines are perpendicular, the angle is a right angle.

(e) From **(d)** OC is the height of the triangle, BC is the base, and hence, the area is

$$\frac{1}{2} \cdot \left|\overrightarrow{AB}\right| \cdot \left|\overrightarrow{OC}\right| = \frac{1}{2} \cdot \sqrt{6} \cdot \sqrt{2^2 + 1} = \frac{\sqrt{30}}{2}$$

15. $\boldsymbol{v} \perp \boldsymbol{u} \Rightarrow \boldsymbol{v}.\boldsymbol{u} = 0 \Rightarrow -3 \times 0 + 1 \times m + 1 \times n \Rightarrow m + n = 0 \Rightarrow n = -m$

$\boldsymbol{v}$ is a unit vector $\Rightarrow \sqrt{m^2 + n^2} = 1 \Rightarrow 2m^2 = 1 \Rightarrow m = \pm \frac{1}{\sqrt{2}}$

$(m, n) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right)$ or $(m, n) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$

16. **(a)** $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 6 - (-3) \\ 4 - (-2) \\ -1 - 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ -3 \end{pmatrix}$

(b) Let $\overrightarrow{OC} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\overrightarrow{AC} = 2\overrightarrow{CB} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix} = 2\left(\begin{pmatrix} 6 \\ 4 \\ -1 \end{pmatrix} - \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) \Rightarrow 3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} 6 \\ 4 \\ -1 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$$

$$\Rightarrow 3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} 6 \\ 4 \\ -1 \end{pmatrix} + \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

(c) $\theta = \cos^{-1} \frac{\overrightarrow{AB} \cdot \overrightarrow{OC}}{\left|\overrightarrow{AB}\right| \left|\overrightarrow{OC}\right|} = \cos^{-1} \frac{27 + 12 - 0}{\sqrt{81 + 36 + 9}\sqrt{9 + 4}} = 15.5°$

(d) **(i)** $\sin\theta = \frac{\boldsymbol{DE}}{\boldsymbol{CD}}$.

Also, since $\overrightarrow{\boldsymbol{OD}} = k\overrightarrow{\boldsymbol{OC}}$, then $\overrightarrow{\boldsymbol{CD}} = \overrightarrow{\boldsymbol{OD}} - \overrightarrow{\boldsymbol{OC}} = (k-1)\overrightarrow{\boldsymbol{OC}}$

Consequently $\boldsymbol{DE} = |\overrightarrow{\boldsymbol{DE}}| = \boldsymbol{CD}\sin\theta = (k-1)|\overrightarrow{\boldsymbol{OC}}|\sin\theta$

(ii) $|\overrightarrow{\boldsymbol{DE}}| < 3 \Rightarrow (k-1)|\overrightarrow{\boldsymbol{OC}}|\sin\theta < 3 \Rightarrow (k-1)\sqrt{9+4}\sin 15.5° < 3$

$\Rightarrow (k-1) < \frac{3}{\sqrt{13}\sin 15.5°} = 3.11 \Rightarrow k < 4.11$

Thus, considering the initial condition, $1 < k < 4.11$.

Exercise 9.1

1. **(a)** The distance travelled is $900t$, so the distance remaining to destination is $d(t) = 3000 - 900t$

(b) The *d*-intercept is $d(0)$, so the distance from the initial position to the final destination, in this case 3000 km

(c) The *t*-intercept is the solution to $d(t) = 0$, so the time required to reach destination. In this case $0 = 3000 - 900t \rightarrow t = \dfrac{3000}{900} \approx 3.33$ hours.

(d) A reasonable domain is the duration of the trip, so $0 \le t \le 3.33$ hours. The range is the set of distances to destination, so $0 \le d \le 3000$ km.

2. **(a)** The average speed of the plane in the first 1.5 hours of its trip is $v = \dfrac{5000 - 3800}{1.5} = 800 \text{ km h}^{-1}$. Assuming it remains constant for the whole duration of the trip, $d(t) = 5000 - 800t$

(b) The slope is the negative of the speed of the airplane

(c) The *d*-intercept is $d(0)$, so the distance from the initial position to the final destination, in this case 5000 km

(d) The *t*-intercept is the solution to $d(t) = 0$, so the time required to reach the destination. In this case $0 = 5000 - 800t \rightarrow t = \dfrac{5000}{800} \approx 6.25$ hours.

(e) A reasonable domain is the duration of the trip, so $0 \le t \le 6.25$ hours. The range is the set of distances to destination, so $0 \le d \le 5000$ km.

3. **(a)** The European size is equal to the US size + 33

(b) Writing the model above as EU=USA + 33 and replacing USA=12 gives EU = 45.

(c) Setting EU = 44 and solving for USA gives USA = 11.

(d) The gradient of our model, which is equal to one, is the number of EU size steps within one USA size step.

(e) Domain is the set of reasonable USA sizes, so $6 \le \text{USA} \le 16$. The range is the set of equivalent EU sizes, so $39 \le \text{EU} \le 49$.

4. **(a)** The three points are collinear since $\begin{pmatrix}340\\490\end{pmatrix}-\begin{pmatrix}290\\420\end{pmatrix}\propto\begin{pmatrix}290\\420\end{pmatrix}-\begin{pmatrix}500\\714\end{pmatrix}$.

The line through the first two points has equation $\frac{t-490}{n-340}=\frac{490-420}{340-290}$

so $t=490+\frac{7}{5}(n-340)$ or $t(n)=14+\frac{7}{5}n$

(b) This is $t(1000)=14+\frac{7}{5}\cdot 1000=1414\approx 1410$ minutes

(c) The gradient is $\frac{7}{5}$ minutes per page, so it is the time required to read one page.

The *t*-intercept (14 minutes) is the time required to read a book of zero pages, so it can be interpreted as the time required for additional activities included in the reading of book, as e.g. retrieving it from the shelf etc. etc.

(d) We could argue that in order for a book to be a book it has to have at least one page, so a reasonable domain could be $n\in\mathbb{N}, n\geq 1$. The range could therefore be $t\in\mathbb{R}, t\geq 14+\frac{7}{5}=15.4$.

5. **(a)** The equation of the line between the two given points is $\frac{F-68}{C-20}=\frac{212-68}{100-20}$,

or $F-68=\frac{9}{5}\cdot(C-20)$

$F(C)=32+\frac{9}{5}C$.

(b) The meaning of the gradient is how many degrees Fahrenheit fit in one degree Celsius. The size of one degree Celsius is 1.8 times the size of one degree Fahrenheit.

(c) The *F*-intercept is $F(0)=32$, so the freezing temperature of water in degrees Fahrenheit.

(d) The *C*-intercept is the solution to $F(C)=0$, so

$0=32+\frac{9}{5}C\rightarrow C=-\frac{5\cdot 32}{9}=-17.8$.

A temperature of 0 degrees Fahrenheit corresponds to a temperature of −17.8 degrees Celsius.

(e) $F(10)=32+\frac{9}{5}\cdot 10=50$ degrees Fahrenheit

(f) Solving the system $\begin{cases} F = 32 + \frac{9}{5}C \\ F = C \end{cases}$

yields $C = 32 + \frac{9}{5}C \rightarrow 5C = 160 + 9C \rightarrow 4C = -160$ so

$C = -40$ degree Celsius. A temperature of -40 degrees Celsius corresponds to a temperature of -40 degrees Fahrenheit.

(g) A reasonable domain is $C > -273$. The corresponding range is $F > F(-273) = -459.4$.

6. **(a)** $C(n) = 350 + 8.50n$

(b) Domain is $n > 0$, range is $C > 350$. Any reasonable order would consist in at least one cup, so more specifically we could have $n \geq 1, C \geq 350 - 8.5 = 358.5$.

(c) **(i)** $C(100) = 350 + 8.50 \cdot 100 = 1200\ \text{ZAR}$

(ii) $C(200) = 350 + 8.50 \cdot 200 = 2050\ \text{ZAR}$

(iii) $C(400) = 350 + 8.50 \cdot 400 = 3750\ \text{ZAR}$

(d) **(i)** $\frac{C(100)}{100} = \frac{1200}{100} = 12\ \text{ZAR cup}^{-1}$

(ii) $\frac{C(200)}{200} = \frac{2050}{200} = 10.25\ \text{ZAR cup}^{-1}$

(iii) $\frac{C(400)}{400} = \frac{3750}{400} = 9.375\ \text{ZAR cup}^{-1}$

(e) Because for larger orders the fixed cost of 350 ZAR is spread over a larger number of cups, thus affecting the cost per cup less.

(f) A linear model would be given by the equation of the straight line through the two given points, $(200, 2150)$ and $(400, 3750)$. This gives

$$\frac{D-2150}{n-200}=\frac{3750-2150}{400-200}$$

$$D-2150=(n-200)\frac{1600}{200}$$

$$D=2150+(n-200)\cdot 8 \rightarrow D(n)=550+8n$$

(g) Domain is the same as for C, $n>0$. The range starts for the fixed cost for this manufacturer, $D>550$.

(h) The cost per cup for this manufacturer is 8 ZAR cup^{-1}.

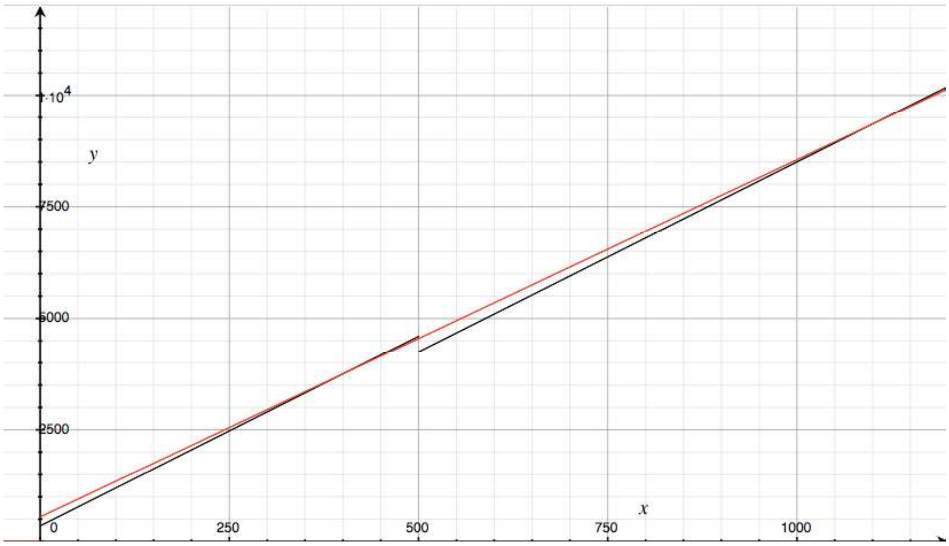
(i) $D(600)=550+8\cdot 600=5350$ ZAR

(j) For 400 cups, the total cost is 3750 for both manufacturers. Since the cost per additional cup is less for Cupomatic, 8 ZAR instead of 8.50 ZAR, from $x=400$ it is more convenient to order from Cupomatic.

(k) This new fee scheme is described by a piecewise model

$$C(n)=\begin{cases}350+8.50n & 0<n<500\\ 8.50n & n\geq 500\end{cases}.$$

(l) Graphing both schemes (black for $C(n)$ and red for $D(n)$) yields



which shows two intersections, one at $550+8n=350+8.5n \rightarrow n=400$ and another one at $550+8n=8.5n \rightarrow n=1100$. It follows that is less expensive ($D<C$) to order from Cupomatic if the number of cups to be ordered is between 401 and 499 inclusive, and then from 1101 onwards. Matching this information to the required format $a\leq x<b$ or $x>k$, we get $a=401$, $b=500$ and $k=1100$.

7. **(a)** Putting together all the conditions, we build the following piecewise model for cost as a function of distance travelled. The cost is constant for the first 234.8 metres, and equal to £2.60. The other intervals are from 234.8 to 9656.1 metres, and more that 9656.1.

$$C(m)=\begin{cases} 2.60 & 0 \le m \le 234.8 \\ ? & 234.8 < m \le 9656.1 \\ ? & m > 9656.1 \end{cases}$$

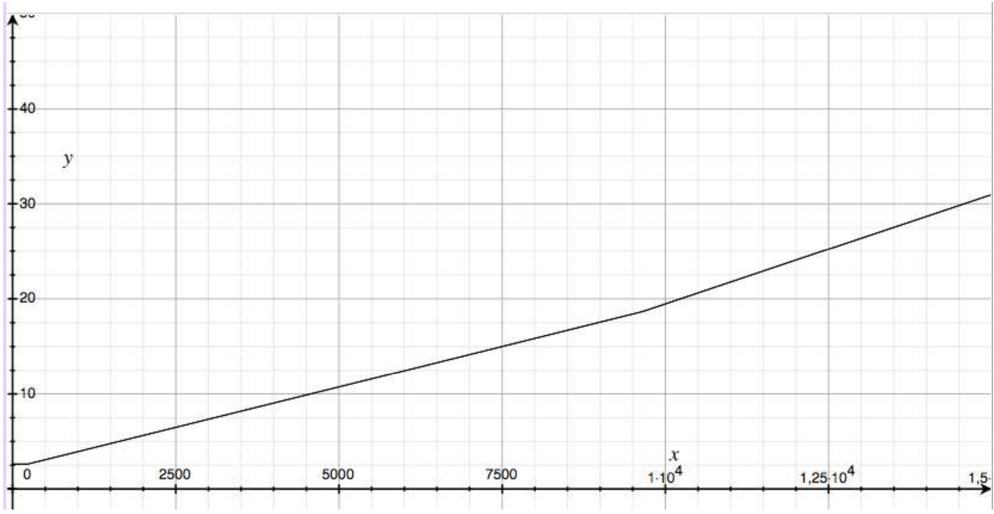
If the distance travelled exceeds 234.8, then the cost increases above £2.60 by £0.20 for every 117.4 metres in excess of 234.8, so to account for this we need to count the excess distance $m-234.8$, figure out how many blocks of 117.4 metres are there $\frac{m-234.8}{117.4}$, and finally multiply this number by the rate £0.20. This yields

$$C(m)=\begin{cases} 2.60 & 0 \le m \le 234.8 \\ 2.60+0.20\frac{m-234.8}{117.4} & 234.8 < m \le 9656.1 \\ ? & m > 9656.1 \end{cases}$$

for the second interval. For the third interval, we first calculate the fee for a trip of exactly 9656.1 metres. This gives $2.60+0.20\frac{9565.1-234.8}{117.4} \approx £18.65$.
For every 86.9 metres in excess of 9656.1, the cost increases by an additional £0.20. To account for this we need to count the excess distance $m-9565.1$, figure out how many blocks of 86.9 metres are there $\frac{m-9656.1}{86.9}$, and finally multiply this number by the rate £0.20. This yields

$$C(m)=\begin{cases} 2.60 & 0 \le m \le 234.8 \\ 2.60+0.20\frac{m-234.8}{117.4} & 234.8 < m \le 9656.1 \\ 18.65+0.20\frac{m-9656.1}{86.9} & m > 9656.1 \end{cases}$$

as shown by the graph below where the three sections are visible.



(b) $C(0.2\text{ km}) = C(200) = £2.60$

$$C(5\text{ km}) = C(5000) = 2.60 + 0.20\frac{5000-234.8}{117.4} \approx £10.72$$

$$C(15\text{ km}) = C(15000) = 18.65 + 0.20\frac{15000-9656.1}{86.9} = £30.95$$

(c) Repeating the same process as in a), we have

$$D(t) = \begin{cases} 2.60 & 0 \le t \le 50.4 \\ 2.60 + 0.20\dfrac{t-50.4}{25.2} & 50.4 < t < \bar{t} \\ 2.60 + 0.20\dfrac{\bar{t}-50.4}{25.2} + 0.20\dfrac{t-\bar{t}}{18.7} & t > \bar{t} \end{cases}$$

where $\bar{t}$ is the time needed to travel 9565.1 metres which depends on the taxi average speed, so on traffic conditions, weather, etc. etc. This value would be accessible to the taxi on-board taximeter. Since the question says "ignoring distance", we can disregard the last interval and have only

$$D(t) = \begin{cases} 2.60 & 0 \le t \le 50.4 \\ 2.60 + 0.20\dfrac{t-50.4}{25.2} & t > 50.4 \end{cases}$$

(d) $D(0.5\,\text{min}) = D(30) = £2.60$

$$D(5\,\text{min}) = D(300) = 2.60 + 0.20\frac{300-54.1}{25.2} \approx £4.58$$

$$D(15\,\text{min}) = D(900) = 2.60 + 0.20\frac{900-54.1}{25.2} \approx £9.34$$

(e) **(i)** $D(20\,\text{min}) = D(1200) = £11.72$

$C(4\,\text{km}) = C(4000) = £9.01$ so the fare would be £11.72

(ii) $D(5\,\text{min}) = D(300) = £4.58$ so the fare would be £9.01.

8. **(a)** If the ball is thrown upwards the initial velocity is positive. The initial height is the height of the building. This gives $h(t) = -4.9t^2 + 5t + 60$.

(b) **(i)** The maximum height of the ball is the h-coordinate of the vertex of the parabola $h(t)$. The time when the ball is at its highest is given by

$t = -\frac{b}{2a} = -\frac{5}{-9.8} \approx 0.51$ seconds, so maximum height is

$h(0.51) \approx 61.3$ metres.

(ii) Since heights are measured with respect to ground, the height of the ground itself is $h = 0$. Solving the equation $-4.9t^2 + 5t + 60 = 0$ yields the solutions $t = \frac{-5 \pm \sqrt{1201}}{-9.8} \approx 4.05$ or -3.03. Discarding the negative solution since the domain for the model is $t > 0$ gives $t = 4.05$ seconds.

(iii) The condition is satisfied from the moment the ball leaves the roof of the building till it falls below 50 metres of height. Solving the inequality $-4.9t^2 + 5t + 60 > 50$ with a GDC yields $-1.01 < t < 2.03$. Putting this together with the domain of the model gives $0 \le t < 2.03$ to three significant figures.

9. **(a)** The height of the cliff is the same as the initial height of the rock, so $f(0) = 106$ metres.

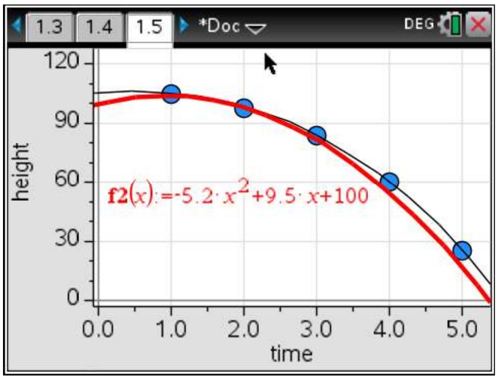
(b) $f(4.5) = -0.25 \cdot 4.5^3 - 2.32 \cdot 4.5^2 + 1.93 \cdot 4.5 + 106 \approx 44.9$ metres.

(c) Solving the equation $f(t) = 30$ with the GDC yields $t = 4.91$ seconds

(d) Solving the equation $g(t) = 0$ with the GDC yields $t = -3.57$ or $t = 5.39$.

The only acceptable solutions are positive, so $t = 5.39$ seconds.

(e) Plotting the data points and the two models yields the following graph.



From the graph above, it is clear that the black curve – Jane's model – is a better fit for the data points.

An issue both models share is that they start with a positive slope, which in this context would imply that a falling rock starts its motion by going upwards.

10. **(a)** At 13.00, ship A is where ship B was at noon and ship B has travelled 11 km east. The distance between the ships is 11 km.

(b) At 14.00, ship A is 15 km below the dashed horizontal line and ship B is 22 km to the right of the dashed vertical line. The distance between the ships is $\sqrt{22^2+15^2}\approx 26.6$ km.

(c) By setting a cartesian frame of reference with origin in the initial position of ship B and axes aligned with the dashed lines in the diagram, the position of ship A is given by $\boldsymbol{A}(t)=\begin{pmatrix}0\\15-15t\end{pmatrix}$

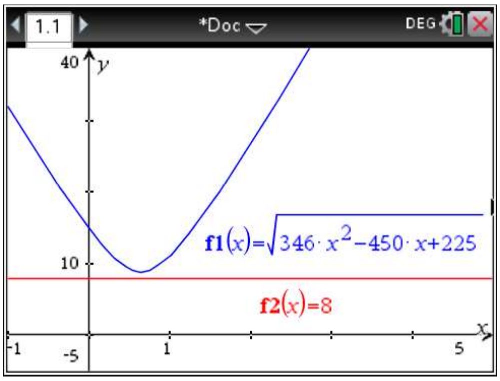
while the position of ship B is given by $\boldsymbol{B}(t)=\begin{pmatrix}11t\\0\end{pmatrix}$

The distance between the ships is the magnitude of the vector $\boldsymbol{A}-\boldsymbol{B}$, so

$$\begin{aligned}s(t)&=\left|\begin{pmatrix}11t\\15t-15\end{pmatrix}\right|\\&=\sqrt{(11t)^2+(15t-15)^2}\\&=\sqrt{121t^2+225t^2-450t+225}\\&=\sqrt{346t^2-450t+225}\end{aligned}$$

.

(d) The graph is given by the blue curve below.



(e) The ships cannot see each other because the distance between them is always larger than 8 km – see graph above. The minimum value for the function $s(t)$ is in fact 8.87 km.

11. **(a)** profit is given by the difference between income and cost, so evaluating $I(x)-C(x)=150x-0.6x^2-(2600+0.4x^2)=-x^2+150x-2600$ we obtain the desired function $P(x)$.

(b) The vertex of the parabola $P(x)$ has x-coordinate $\frac{-b}{2a}=\frac{-150}{-2}=75$. Maximum profit occurs for 75 machines.

(c) For this value of *x*, the total income is $I(75)=150\cdot 75-0.6\cdot 75^2=7875$ dollars. Dividing this by 75 gives the selling price of each machine, 105 dollars.

(d) The profit is zero at $x=20$ and $x=130$, and it is positive for $20<x<130$. The smallest number of machines for which profit is positive is therefore $x=21$.

12. **(a)** The height of the bridge is $s(0)=300$ metres.

(b) **(i)** Evaluating $s(4)$ yields $s(4)=-4.9\cdot 4^2+300=221.6b$ metres

(ii) t_0 is the boundary between free fall and constant velocity motion, so in this case $t_0=4$ seconds

(iii) In order for the height function to be continuous at $t=t_0$, the value of $h(4)$ has to be the same as $s(4)$.

This gives $-7\cdot 4+k=221.6\to k=249.6$ metres.

(iv) $h(1)=-4.9\cdot 1^2+300=295.4\text{ m}$

$h(2)=-4.9\cdot 2^2+300=280.4\text{ m}$

$h(8)=-7\cdot 8+249.6=193.6\text{ m}$

(v) At $t=4$ the height is 221.6, so using the second part of the piecewise model gives $-7t+249.6>100$. Solving for t yields $t<\frac{249.6-100}{7}\approx 21.4$ seconds.

(vi) Setting $h(t)=0$ and solving for time gives $-7t+249.6=0 \rightarrow t=\frac{249.6}{7}\approx 35.7$ seconds.

(vii) Domain is $0\le t\le 35.7$, range is $0\le h\le 300$.

(c) **(i)** During free fall height is given by $s(t)$. Setting $s(x)=100$ gives $-4.9x^2+300=100 \rightarrow x=\pm\sqrt{\frac{300-100}{4.9}}\approx \pm 6.39$ s. Keeping only the positive solution, $x\approx 6.39$ s.

(ii) At $t=6.39$ the function has to be continuous, so setting $-7\cdot 6.39+k=100 \rightarrow k=100+44.7\approx 145$ m. The piecewise model is

$$g(t)=\begin{cases}-4.9t^2+300 & t<6.39\\ -7t+145 & t<6.39\end{cases}$$

(iii) Setting $g(t)=0$ and knowing that this will be after 6.39 seconds, we obtain $-7t+145=0 \rightarrow t=\frac{145}{7}\approx 20.7$ s.

(d) **(i)** The jumper in free fall would hit the ground when $s(t)=0$, so setting $-4.9t^2+300=0$ we obtain $t=\pm\sqrt{\frac{300}{4.9}}\approx \pm 7.82$ and we keep the positive solution only. So the jumper open her parachute at $t=7.82-2=5.82$ s.

(ii) $d=s(5.82)=-4.9\cdot 5.82^2+300\approx 134$ m

(iii) At $t=5.82$ the function has to be continuous, so setting $-7\cdot 5.82+k=134 \rightarrow k=134+40.47\approx 175$ m.

The piecewise model is $g(t)=\begin{cases}-4.9t^2+300 & t<5.82\\ -7t+175 & t\ge 5.82\end{cases}$

(iv) Solving $-7t+175=0 \rightarrow t=\frac{175}{7}=25.0$ s

Exercise 9.2

1. **(a)** Assuming exponential behaviour, we can fit our data to a decreasing exponential function $d(t) = d_0 \cdot \mathrm{e}^{-kt}$. Replacing the first and the last data point gives $d(0) = d_0 = 56$ and

$$\begin{aligned} d(20) &= 56\mathrm{e}^{-k\cdot 20} \\ &= 1.71 \to -20k \\ &= \ln\frac{1.71}{56} \to k \quad \mathrm{s}^{-1} \\ &= -\frac{1}{20}\ln\frac{1.71}{56} \\ &\approx -.174 \end{aligned}$$

The exponential model is $d(t) = 56\mathrm{e}^{-0.174t}$. Observing that $\mathrm{e}^{-0.174} \approx 0.840$, the model can also be written as $d(t) = 56 \cdot 0.840^t$.

(b) According to our model, $d(t) = 56 \cdot 0.840^7 \approx 16.5 \ \mathrm{m\,s^{-1}}$

(c) Setting the inequality $d(t) < 5$ and solving it gives

$$56 \cdot 0.840^t < 5 \to 0.840^t < \frac{5}{56} \to t > \log_{0.840}\frac{5}{56} \approx 13.9 \ \mathrm{s}$$

(d) Setting the inequality $d(t) < 1$ and solving it gives

$$56 \cdot 0.840^t < 1 \to 0.840^t < \frac{1}{56} \to t > \log_{0.840}\frac{1}{56} \approx 23.1 \ \mathrm{s}$$

2. **(a)** $A(4) = 12\mathrm{e}^{0.4\cdot 4} \approx 59$

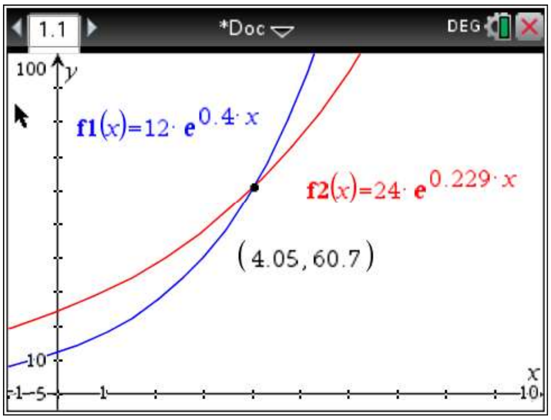
(b) Setting the equation $A(t) = 400$ and solving it yields

$$12\mathrm{e}^{0.4t} = 400 \to \mathrm{e}^{0.4t} = \frac{400}{12} \to 0.4t = \ln\frac{400}{12} \to t = \frac{1}{0.4}\ln\frac{400}{12} \approx 8.77 \text{ hours}$$

(c) Setting $B(4) = 60$ and solving for k gives

$$24\mathrm{e}^{k\cdot 4} = 60 \to \mathrm{e}^{4k} = \frac{60}{24} \to 4k = \ln\frac{60}{24} \to k = \frac{1}{4}\ln\frac{60}{24} \approx 0.229 \ \mathrm{s}^{-1}$$

(d) Setting $A(n) > B(n)$ and solving for n with a GDC gives



so the first time population A is larger than population B is when $n = 5$ hours, since the smallest integer larger than 4.05 is 5.

3. **(a)** $A(0) = 10(0.5)^0 = 10$

(b) $A(50) = 10 \cdot 0.5^{0.014 \cdot 50} \approx 6.16 \text{ mg } l^{-1}$

(c) Setting the equation $A(t) = .395$ and solving it for t yields

$$10 \cdot 0.5^{0.014t} = 0.395 \rightarrow 0.5^{0.014t}$$
$$= \frac{0.395}{10} \rightarrow 0.014t$$
$$= \log_{0.5} 0.0395 \rightarrow t$$
$$= \frac{1}{0.014} \log_{0.5} 0.0395 \approx 333$$

minutes, so Jose can take his medication again after 5 hours and 33 minutes, at 18.33

4. **(a)** $L(t) = 16 \cdot (1 + 8\%)^t = 16 \cdot 1.08^t$

(b) $L(3) = 16 \cdot 1.08^3 \approx 20.2 \text{ cm}$

(c) Setting the equation $L(t) = 25$ and solving it for t yields

$$16 \cdot 1.08^t = 25 \rightarrow 1.08^t = \frac{25}{16} \rightarrow t = \log_{1.08} \frac{25}{16} \approx 5.80 \text{ years.}$$

5. **(a)** Half-life is time after which only half of the initial amount remains. Setting the equation $A(t)=\frac{A_0}{2}$ and solving it for k yields

$$e^{k\cdot 15}=\frac{1}{2}\rightarrow 15k=\ln\frac{1}{2}\rightarrow k=\frac{\ln\frac{1}{2}}{15}\approx -0.0462\ \text{s}^{-1}$$

(b) $A(2)=50\cdot e^{-0.462\cdot 2}\approx 45.6$ units

(c) Setting the equation $A(20)=35=A_0e^{-0.0462\cdot 20}$ and solving it for A_0 yields

$$35=A_0e^{-0.0462\cdot 20}\rightarrow A_0=35e^{0.0462\cdot 20}\approx 88.2\text{ units}$$

(d) Setting the equation $A(t)=120e^{-0.0462t}=40$ and solving it for t yields

$$120e^{-0.0462t}=40\rightarrow e^{-0.0462t}$$
$$=\frac{1}{3}\rightarrow -0.0462t$$
$$=\ln\frac{1}{3}\rightarrow t$$
$$=\frac{\ln\frac{1}{3}}{-0.0462}\approx 23.8\text{ years}$$

6. **(a)** Assuming that the data describe the amount of pesticide in the soil, $A(0)=500\cdot 0.5^0=500$

represents the initial amount of the pesticide glyphosate.

(b) Given that the base of the exponential is $0.5=\frac{1}{2}$, and that the half-life of the pesticide is 45 days, t must represent the number of 45-day intervals.

(c) $A(1)=250$ represents the remaining amount of glyphosate after one 45-day interval, so after 45 days.

7. **(a)** Given that the temperature of the cake tends to $150\,^{\circ}\mathrm{C}$, this value must be the set temperature of the oven.

(b) Setting the equation $T(0)=18$ and solving it for a yields

$150-a\cdot 1.1^{0}=18\rightarrow a=150-18=132\ ^{\circ}C$

(c) Setting the equation $T(t)=130$ and solving it for t yields

$$150-132\cdot 1.1^{-t}=130\rightarrow 132\cdot 1.1^{-t}$$
$$=150-130\rightarrow 1.1^{-t}$$
$$=\frac{20}{132}\rightarrow t$$
$$=-\log_{1.1}\frac{20}{132}\approx 19.8 \text{ minutes.}$$

Adding 15 to this value gives the total time in the oven as 34.8 minutes.

8. **(a)** In order to find the two parameters a and b we need two independent equations in a and b. These are $N(0)=850$ from the January data point, and $N(4)=100$ from the May data point. Replacing these values yields

$$\begin{cases} 850=a\cdot b^{0}+40 \\ 100=a\cdot b^{-4}+40 \end{cases}.$$

From the first equation we obtain $850=a+40\rightarrow a=810$. Replacing in the second equation gives $100=810\cdot b^{-4}+40\rightarrow b=\dfrac{1}{\sqrt[4]{\dfrac{100-40}{810}}}\approx 1.917$.

(b) Just like May was the fourth month after January, September is the eighth. Evaluating $N(8)=810\cdot 1.917^{-8}+40\approx 44$ we estimate the number of fish in September.

(c) Setting the inequality $N(t)<50$ and solving it we have $810\cdot 1.917^{-t}+40<50$

$1.917^{-t}<\dfrac{50-40}{810}\rightarrow -t<\log_{1.917}\dfrac{10}{810}\rightarrow t>-\log_{1.917}\dfrac{10}{810}\approx 6.75$ months. The first time the number of fish decreases below 50 is therefore 7 months after January, hence August.

(d) As $t\rightarrow +\infty$, $1.917^{-t}\rightarrow 0$ so that $N(t)\rightarrow 40$.

9. **(a)** As $m \to +\infty$, $k^{-m} \to 0$ so that $T(m) \to a$. Since the temperature of the water in the cup tends to the temperature of the room, and the temperature of the room is 20, we have $a = 20\ ^\circ\text{C}$.

(b) In order to find the two parameters *b* and *k* we need two independent equations in *b* and *k*. These are $T(0) = 100$, and $T(1) = 85$ from the May data point. Replacing these values yields

$$\begin{cases} 100 = 20 + b \cdot k^{-0} \\ 85 = 20 + b \cdot k^{-1} \end{cases}.$$

From the first equation we obtain $100 = 20 + b \to b = 80\ ^\circ\text{C}$. Replacing in the second equation gives $85 = 20 + 80 \cdot k^{-1} \to k^{-1} = \dfrac{85-20}{80} \to k = \dfrac{80}{65} \approx 1.23$.

(c) Evaluating $T(5)$ gives $20 + 80 \cdot 1.23^{-5} \approx 48.3\ ^\circ\text{C}$

(d) Setting the equation $T(m) = 35$ and solving for *m* gives $20 + 80 \cdot 1.23^{-m} = 35$

$1.23^{-m} = \dfrac{35-20}{80} \to -m = \log_{1.23} \dfrac{15}{80} \to m = -\log_{1.23} \dfrac{15}{80} \approx 8.086$ minutes, or 8 minutes and 5 seconds

Note: Answers may differ by a second due to rounding of intermediate answers.

10. **(a)** $p(0) = 200 - 190 \cdot 0.97^{0} = 10\ ^\circ\text{C}$

(b) $p(30) = 200 - 190 \cdot 0.97^{30} \approx 124\ ^\circ\text{C}$

(c) Setting the equation $p(k) = 40$ and solving it for *k* gives

$200 - 190 \cdot 0.97^{k} = 40$

$0.97^{k} = \dfrac{200-40}{190} \to k = \log_{0.97} \dfrac{160}{190} \approx 5.64\ \text{min}$.

11. **(a)** $C(0) = 2.5 - 2^{0} = 1.5$ C (Coulomb)

(b) as $t \to +\infty, 2^{-t} \to 0$ so $C(t) \to 2.5$. The equation of the asymptote is $C = 2.5$.

(c) Setting the equation $C(t) = 2.4$ and solving it for *t* yields
$2.4 = 2.5 - 2^{-t} \to 2^{-t} = 2.5 - 2.4 \to -t = \log_2 0.1 \to t = -\log_2 0.1 \approx 3.32$ hours, or 3 hours and 19 minutes.

12. In order to find the two parameters p and q we need two independent equations in p and q. These are $C(0)=47$ from the January data point, and $C(4)=53$ from the May data point. Replacing these values yields

(a)

$$\begin{cases} p\cdot 2^{0}+q=47 \\ p\cdot 2^{0.5\cdot 4}+q=53 \end{cases}$$

$$\begin{cases} p+q=47 \\ 4p+q=53 \end{cases}$$

(b) Taking the first equation from the second gives $3p=6\to p=2$. Replacing this value in the first equation gives $q=45$.

(c) Evaluating $C(10)=2\cdot 2^{0.5\cdot 10}+45=64+45=109$ gives the number of cells at 22.00 on Monday

Exercise 9.3

1. **(a)** This is given by the difference in the time coordinates of A and B, so $12.5-6.25=6.25$ hours, or 6 hours 15 minutes.

(b) This is given by the difference in the height coordinates of A and B, so $1.5-0.6=0.9$ metres.

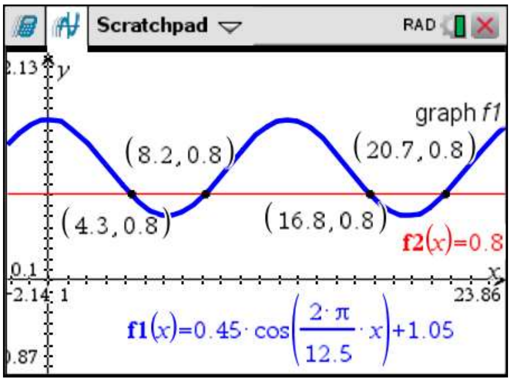
(c) The parameter p gives the amplitude of the oscillation, so half the difference between maximum and minimum: $p=\frac{0.9}{2}=0.45$ m.

(d) Assuming the argument of the cosine function is in radians, the parameter q relates to the period T as $T=\frac{2\pi}{q}$. The period is $T=2\cdot 6.25=12.5$ hours, so $q=\frac{2\pi}{T}=\frac{2\pi}{12.5}\approx 0.503$ rad h^{-1}.

(e) The parameter r gives the principal axis of the oscillation, which is halfway through the maximum and the minimum. In our case $r=\frac{1.5+0.6}{2}=1.05$.

(f) Since the period is 12.5 hours, high tides occur every 12.5 hours. The first high tide is at $t_0 = 0$, so the next are at $t_1 = 12.5, t_2 = 25, t_3 = 37.5, t_4 = 50, t_5 = 62.5, t_6 = 75, \ldots$ Of these, t_0 is on the 10th of December at 21.00, t_1 and t_2 are on the 11th of December, t_3 and t_4 are on the 12th December, etc. In particular the second high tide of Dec. 12th occurs 50 hours after t_0, so at 23.00.

(g) Graphing $h(t)$ on a GDC and finding intersections with $h = 0.8$ gives this result.



The height of the water is at most 0.8 m, or $h(t) \le 0.8$, from 4.3 to 8.2 hours after 21:00, and again from 16.8 to 20.7 hours after 21:00, for a total of $(8.2 - 4.3) + (20.7 - 16.8) = 7.8$ hours or 468 minutes.

Note: Answers may differ by a minute due to rounding

2. **(a)** The parameter p gives the amplitude of the oscillation, so half the difference between maximum and minimum: $p = \dfrac{9.7 - 5.3}{2} = 2.2$ m.

(b) Assuming the argument of the cosine function is in radians, the parameter q relates to the period T as $T = \dfrac{2\pi}{q}$. The period is $T = 2 \cdot 7 = 14$ hours, so $q = \dfrac{2\pi}{14} = \dfrac{\pi}{7}$ rad h^{-1}.

(c) Evaluating $d(10) = 2.2\cos\left(\dfrac{\pi}{7}10\right) + 7.5 = 7.01$ m gives the depth of water 10 hours after high tide.

3. **(a)** $h(0) = -15\cos(0) + 17 = 2$ m

(b) Setting the equation $h(k)=20$ and solving it for k gives

$-15\cos(1.2k)+17=20$

$-15\cos(1.2k)=3 \to \cos(1.2k)=-\frac{3}{15} \to 1.2k=\cos^{-1}\left(-\frac{3}{15}\right)$

$k=\frac{1}{1.2}\cos^{-1}\left(-\frac{3}{15}\right)\approx 1.48$ minutes.

(c) Assuming the argument of the cosine function is in radians, the parameter 1.2 relates to the period T as $T=\frac{2\pi}{1.2}$. The period is therefore $T=\frac{2\pi}{1.2}\approx 5.2\text{ min}$.

4. (a) (i) Since 10 minutes is half a period, the wheel has completed half a revolution and P is at the top. The height of point P is 100 metres.

(ii) Since 15 minutes is $\frac{3}{4}$ of the period, the wheel has completed $\frac{3}{4}$ of a revolution and P is halfway through the descent. The height of point P is 50 metres.

(b) (i) c is the principal axis, so $c=50$ m.

Assuming the argument of the cosine function is in radians, b is $\frac{2\pi}{\text{period}}$, so $b=\frac{2\pi}{20}=\frac{\pi}{10}$ rad min^{-1}.

$|a|$ is the amplitude, so $|a|=50$ m. At $t=0$ the seat is 50 metres *below* the principal axis, so $a=-50$ m.

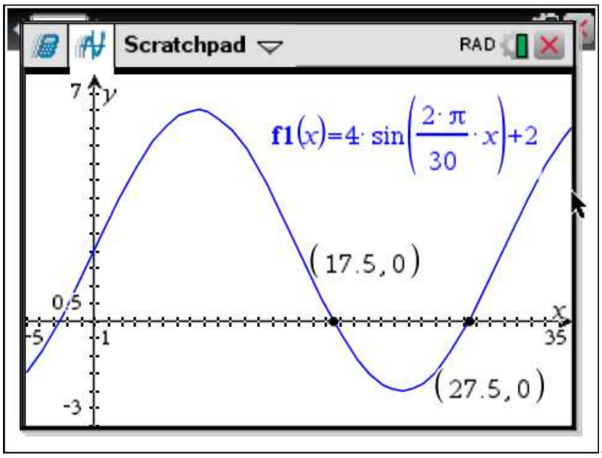
(ii) expressing the model in terms of a sine function requires a horizontal shift *to the left* by a quarter of a period, so $d=-5$ minutes.

5. (a) The parameter a gives the amplitude of the oscillation, so half the difference between maximum and minimum: $a=\frac{6-(-2)}{2}=4$ m.

(b) Assuming the argument of the sine function is in radians, the parameter b relates to the period T as $T=\frac{2\pi}{b}$. The period is $T=30$ seconds, so

$b=\frac{2\pi}{30}=\frac{\pi}{15}$ s^{-1}.

(c) Graphing the model with a GDC and looking for the solutions to $h(t) < 0$ gives



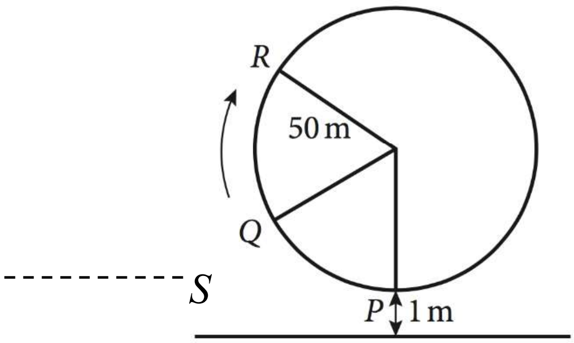
,

from which we derive that the bucket was underwater for $27.5 - 17.5 = 10$ s.

6. **Note:** point R is not needed for the question

(a) Since 15 minutes is half a period, the seat is at the top of the Ferris wheel, so $2 \times 50 + 1 = 101$ metres above the ground.

(b) Since 6 minutes are $\frac{1}{5}$ of a period, the seat has swept an angle of $\frac{2\pi}{5}$ radians from its initial position. Point *S* in the diagram below



is $50\cos\left(\frac{2\pi}{5}\right) \approx 15.5$ metres below the centre of the wheel, so it is $50 + 1 - 15.5 = 35.5$ metres above the ground. The same is true for point *Q*.

(c) d is the principal axis, which is halfway through the maximum and the minimum, so $d = \frac{1+101}{2} = 51$ metres.

Assuming the argument of the sine function is in radians, *b* relates to the period T as $T = \frac{2\pi}{b}$. The period is $T = 30$ minutes, so $b = \frac{2\pi}{30} = \frac{\pi}{15}$ rad min^{-1}.

$|a|$ is the amplitude, so $|a| = 50$ metres. The seat starts below the principal axis, so we choose $a = -50$ m.

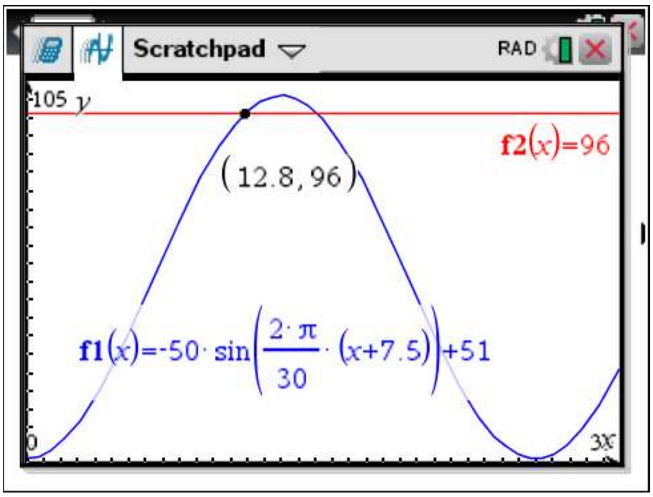
The sine function starts from zero, while the seat starts from its maximum displacement from the principal axis (having chosen a negative *a*) which is what the cosine function does. To change a sine into a cosine, a horizontal shift of a quarter of a period to the right is needed, so $c = -7.5$.

Finally, $h(t) = -50\sin\left(\frac{\pi}{15}(t+7.5)\right)+51$.

Equivalently, a can be chosen as positive, which means that the seat starts from its minimum displacement from the principal axis which is what the negative cosine function does. To change a sine into a negative cosine, a horizontal shift of a quarter of a period *to the left* is needed, so $c = 7.5$ and

$h(t) = 50\sin\left(\frac{\pi}{15}(t-7.5)\right)+51$. In fact, graphing the two functions produces exactly the same curve, which can also be proven with the compound angle formulae.

(d) Setting the equation $h(t) = 96$ and solving it for *t* gives with the GDC gives $t = 12.8$ min - see below.



7. **(a)** The parameter *a* gives the amplitude of the oscillation, so half the difference between maximum and minimum: $a = \frac{-1-(-5)}{2} = 2\,^\circ\mathrm{C}$.

(b) Since the argument of the sine function is in degrees, b is $\frac{360^\circ}{\text{period}}$, so

$b = \frac{360}{24} = 15$ degrees h^{-1}.

(c) c is the principal axis, which is halfway through the maximum and the minimum, so $c = \frac{-1+(-5)}{2} = -3\,°\text{C}$

(d) From the graph, this occurs for $16 \le t \le 20$.

8. **(a)** From the graph, $h_{\max} = 35$ cm.

(b) From the graph, $h_{\min} = 5$ cm.

(c) The amplitude is given by $\frac{h_{\max} - h_{\min}}{2} = \frac{35-5}{2} = 15$ cm.

(d) A represents the amplitude, so $A = 15$ cm. C is the principal axis, so $C = \frac{h_{\max} + h_{\min}}{2} = \frac{35+5}{2} = 20$ cm.

(e) From the graph, $T = 4$ s.

(f) $b = \frac{360°}{T} = \frac{360°}{4} = 90$ degrees s^{-1}.

(g) Setting the equation $h(t) = 30$ and solving it for t gives

$$15\cos(90t) + 20 = 30$$

$$15\cos(90t) = 10 \to \cos(90t) = \frac{10}{15} \to 90t = \cos^{-1}\left(\frac{2}{3}\right)$$

$$t = \frac{1}{90}\cos^{-1}\left(\frac{2}{3}\right) \approx 0.535 \text{ seconds.}$$

(h) This is given by $\frac{1\,\text{min}}{T} = \frac{60}{4} = 15$ times.

9. **(a)** the amplitude is 4 metres.

(b) The maximum value is one amplitude above the principal axis, so $d_{\max} = 7 + 4 = 11$ m.

(c) Assuming the argument of the sine function is in degrees, the period is related to the parameter 0.5 by $T = \frac{360°}{0.5°\,\text{min}^{-1}} = 720\,\text{min}$ or 12 hours.

(d) In a sinusoidal function, a maximum occurs half a period after a minimum, so the next maximum after 14:00 occurs $\frac{12}{2} = 6$ hours later, at 20:00.

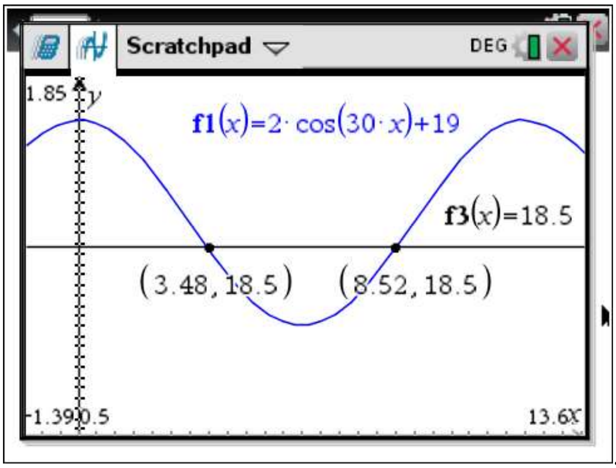
10. **(a)** a is the principal axis, which is halfway through the maximum and the minimum, so $a = \frac{21+17}{2} = 19\,°\text{C}$.

b gives the amplitude of the oscillation, so half the difference between maximum and minimum: $b = \frac{21-17}{2} = 2\,°\text{C}$.

(b) Assuming the argument of the cosine function is in degrees, c relates to the period because

period $= \frac{360°}{c}$. The period is 12 days, so $c = \frac{360°}{12} = 30$ degrees day^{-1}.

(c) Using a GDC, we obtain this graph.



The solution to the inequality $T(x) < 18.5$ is $3.48 < x < 8.52$.

Exercise 9.4

1. **(a)** Using the Logistic Regression features of the GDC, we obtain $L = 97.2\,\text{m}\,\text{s}^{-1}, C = 91.2, k = 1.17\,\text{s}^{-1}$.

(b) The maximum, asymptotic velocity is obtained for $t \to +\infty$, when $e^{-kt} \to 0$ and therefore $P(t) \to L = 97.2\ \text{m}\,\text{s}^{-1}$

(c) This corresponds to setting the equation $P(t) = 99\%L$, or equivalently

$\frac{1}{1+91.2e^{-1.17t}} = 0.99$. Solving it for t gives

$$\frac{1}{0.99} = 1 + 91.2e^{-1.17t} \to 91.2e^{-1.17t} = \frac{1}{0.99} - 1 \to 91.2e^{-1.17t} = \frac{1-0.99}{0.99} = \frac{1}{99}$$

$$e^{-1.17t} = \frac{1}{91.2}\frac{1}{99} \to -1.17t = \ln\frac{1}{91.2\cdot 99} \to t = -\frac{1}{1.17}\ln\frac{1}{91.2\cdot 99} \approx 7.78 \text{ seconds.}$$

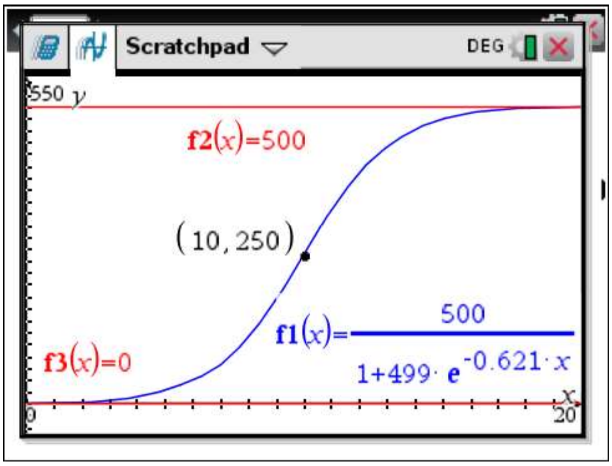
2. **(a)** L is the maximum asymptotic number of infected cows, so $L = 500$.

(b) Assuming d is the number of days after discovering the first infected cow, this means $B(0) = 1$. It follows that $1 = \frac{L}{1+C}$, which gives $C = L - 1 = 499$.

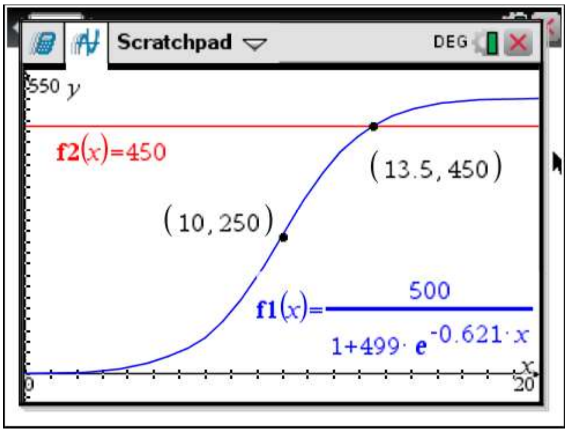
(c) This means $B(10) = 250$, or equivalently $\frac{1}{1+499e^{-10k}} = \frac{1}{2}$. Solving for k gives

$$1 + 499\text{e}^{-10k} = 2 \rightarrow \text{e}^{-10k} = \frac{1}{499} \rightarrow -10k = \ln\frac{1}{499} \rightarrow k = -\frac{1}{10}\ln\frac{1}{499} \approx 0.621\,\text{d}^{-1}.$$

(d) The graph is shown below, with asymptotes $B = 500$, $B = 0$ and inflection point at $(10, 250)$.



(e) Setting up the equation $B(t) = 450$ and solving it with the GDC as shown below gives $t = 13.5$ days.



3. In order to find the two parameters C and L we need two independent equations in C and L. Assuming t is the number of years after 2000, these equations are $P(0)=1000$ and $P(5)=2500$. Replacing these values yields

(a)

$$\begin{cases} \dfrac{L}{1+C}=1000 \\ \dfrac{L}{1+Ce^{-0.2\cdot 5}}=2500 \end{cases}$$

$$\begin{cases} L=1000(1+C) \\ L=2500\left(1+\dfrac{C}{e}\right) \end{cases}$$

$$\begin{cases} L-1000C=1000 \\ L-\dfrac{2500}{e}C=2500 \end{cases}$$

solving the system of linear equations with the GDC yields $L=19679.6\approx 19680$ and $C=18.6796\approx 18.7$.

(b) The maximum, aymptotic velocity is obtained for $t\to+\infty$, when $e^{-kt}\to 0$ and therefore $P(t)\to L=19680$ rats.

(c) This is given by $\dfrac{\ln C}{k}=\dfrac{\ln 18.7}{0.2y^{-1}}=14.6$ years.

Exercise 9.5

1. **(a)** Always

(b) Always (direct variation is a polynomial)

(c) Sometimes

(d) Never

2. **(a)** Never

(b) Never

(c) Never

(d) Never

3. y varies directly with x means $y = kx$, so for any two pairs $(x_1, y_1), (x_2, y_2)$ we have $\frac{y_1}{x_1} = \frac{y_2}{x_2}(=k)$. So,

(a) $y_2 = \frac{x_2}{x_1} y_1 = \frac{5}{11} 462 = 210$

(b) $x_2 = \frac{y_2}{y_1} x_1 = \frac{672}{462} 11 = 16$

4. y varies directly with the square of x means $y = kx^2$, so for any two pairs $(x_1, y_1), (x_2, y_2)$ we have $\frac{y_1}{x_1^{\ 2}} = \frac{y_2}{x_2^{\ 2}}(=k)$. So,

(a) $y_2 = \frac{x_2^{\ 2}}{x_1^{\ 2}} y_1 = \left(\frac{20}{5}\right)^2 10 = 160$

(b) $x_2^{\ 2} = \frac{y_2}{y_1} x_1^{\ 2} \rightarrow x_2 = \sqrt{\frac{y_2}{y_1}} x_1 = \sqrt{\frac{40}{10}} 5 = 10$

5. y varies directly with the cube of x means $y = kx^3$, so for any two pairs $(x_1, y_1), (x_2, y_2)$ we have $\frac{y_1}{x_1^{\ 3}} = \frac{y_2}{x_2^{\ 3}}(=k)$. So,

(a) $y_2 = \frac{x_2^{\ 3}}{x_1^{\ 3}} y_1 = \left(\frac{8}{5}\right)^3 250 = 1024$

(b) $x_2^{\ 3} = \frac{y_2}{y_1} x_1^{\ 3} \rightarrow x_2 = \sqrt[3]{\frac{y_2}{y_1}} x_1 = \sqrt[3]{\frac{128}{250}} 5 = 4$

6. y varies inversely with x means $y = \frac{k}{x}$, so for any two pairs $(x_1, y_1), (x_2, y_2)$ we have $x_1 y_1 = x_2 y_2 (=k)$. So,

(a) $y_2 = \frac{x_1}{x_2} y_1 = \frac{5}{20} 10 = 2.5$

(b) $x_2 = \frac{y_1}{y_2} x_1 = \frac{10}{0.5} 5 = 100$

7. y varies inversely with the square of x means $y=\frac{k}{x^2}$, so for any two pairs $(x_1,y_1),(x_2,y_2)$ we have $x_1^{\,2}y_1=x_2^{\,2}y_2\,(=k)$. So,

(a) $y_2=\frac{x_1^{\,2}}{x_2^{\,2}}y_1=\left(\frac{5}{20}\right)^2 10=\frac{10}{16}=\frac{5}{8}=0.625$

(b) $x_2^{\,2}=\frac{y_1}{y_2}x_1^{\,2}\rightarrow x_2=\sqrt{\frac{y_1}{y_2}}x_1=\sqrt{\frac{10}{2.5}}5=10$

8. y varies inversely with the cube of x means $y=\frac{k}{x^3}$, so for any two pairs $(x_1,y_1),(x_2,y_2)$ we have $x_1^{\,3}y_1=x_2^{\,3}y_2\,(=k)$. So,

(a) $y_2=\frac{x_1^{\,3}}{x_2^{\,3}}y_1=\left(\frac{5}{15}\right)^3 54=2$

(b) $x_2^{\,3}=\frac{y_1}{y_2}x_1^{\,3}\rightarrow x_2=\sqrt[3]{\frac{y_1}{y_2}}x_1=\sqrt[3]{\frac{54}{250}}5=3$

9. **(a)** $S=kd$, and $1.5=k\cdot 2.5$ so that $k=\frac{1.5}{2}=0.75$. Finally, $S(d)=0.75d$.

(b) $S(7)=0.75\cdot 7=5.25$ m.

(c) $S(d)=10\rightarrow 0.75d=10\rightarrow d=\frac{10}{0.75}\approx 13.3$ m

10. **(a)** $v(t)=kt$, and $19.6=k\cdot 2$ so that $k=\frac{19.6}{2}=9.8\ \text{m s}^{-2}$

(b) $v(4)=9.8\cdot 4=39.2$ m s^{-1}

(c) $200\ \text{km h}^{-1}=200\cdot\frac{1000}{3600}\approx 55.6\ \text{m s}^{-1}$

so $v(t)=55.6\rightarrow 9.8t=55.6\rightarrow t=\frac{55.6}{9.8}\approx 5.7$ seconds

11. Distance travelled varies directly with the square of time, so $s(t) = kt^2$.

(a) $s(10) = 162 \rightarrow k \cdot 10^2 = 162 \rightarrow k = \dfrac{162}{10^2} = 1.62 \text{ m s}^{-2}$

(b) $s(5) = \dfrac{1}{4}s(10) = \dfrac{162}{4} = 40.5 \text{ m}$

(c) $s(t) = 200 \rightarrow 1.62t^2 = 200 \rightarrow t^2 = \dfrac{200}{1.62} \rightarrow t = \sqrt{\dfrac{200}{1.62}} \approx 11.1 \text{ seconds}$

12. Radius varies inversely with the square of velocity, so $r(v) = \dfrac{k}{v^2}$.

(a)

$$\begin{aligned} r(7700) = 6.75\times10^6 &\rightarrow \frac{k}{(7700)^2} \\ &= 6.75\times10^6 \rightarrow k \\ &= (7700)^2 \cdot 6.75\times10^6 \\ &= 4.00\times10^{14} \text{ m}^3\text{s}^{-2} \end{aligned}$$

(b) $r(v) = 7.0\times10^6 \rightarrow \dfrac{4.00\times10^{14}}{v^2} = 7.0\times10^6 \rightarrow v^2 = \dfrac{4.00\times10^{14}}{7.0\times10^6} \rightarrow v = \sqrt{\dfrac{4.00\times10^{14}}{7.0\times10^6}}$

$v = 7559.29 \approx 7560 \text{ m s}^{-1}$

(c) $r(8000) = \dfrac{4.00\times10^{14}}{8000^2} = 6.25\times10^6 \text{ m}$

13. Volume varies directly with cube of edge length, so $V(a) = ka^3$.

(a) assuming the edge length is measured in cm,

$V(5) = 958 \rightarrow k \cdot 5^3 = 958 \rightarrow k = \dfrac{958}{5^3} = 7.664$

(b) $V(8) = 7.664 \cdot 8^3 = 3923.97 \approx 3920 \text{ cm}^3$

(c) $V(a) = 100 \rightarrow 7.664a^3 = 100 \rightarrow a^3 = \dfrac{100}{7.664} \rightarrow a = \sqrt[3]{\dfrac{100}{7.664}} \approx 2.35 \text{ cm}$

14. Power varies directly with cube of wind speed, so $P(v) = kv^3$.

(a) $P(8) = 314 \rightarrow k \cdot 8^3 = 314 \rightarrow k = \dfrac{314}{8^3} \approx 0.613 \text{ W m}^{-3}\text{s}^3$

(b) $P(12) = 0.613 \cdot 12^3 = 1059.26 \approx 1060 \text{ W}$

(c) $P(v) = 2000 \rightarrow 0.613v^3 = 2000 \rightarrow v^3 = \dfrac{2000}{0.613} \rightarrow v = \sqrt[3]{\dfrac{2000}{0.613}} \approx 14.8 \text{ m s}^{-1}$

Exercise 9.6

1. **(a)** Growth with an asymptote, so exponential $M = M_{\text{max}} - c\text{e}^{-kt}$ or logistic depending on when the growth is fastest (exponential is faster first, then slows down, while logistic is fastest halfway through).

(b) Large mass $\to$ small acceleration, so a decreasing function. As a matter of fact acceleration varies inversely as mass

(c) Double the amount of fuel $\to$ double the cost, so the cost varies directly with the amount of fuel.

(d) Growth with an asymptote, and fastest halfway through, so logistic.

(e) Double the width $\to$ quadruple the area, so the area varies directly with the square of the width

(f) Cyclic phenomenon so trigonometric

(g) Volume varies directly with the cube of the side length

(h) Growth with an asymptote, fastest at first and then slower and slower as velocity increases, so exponential $v = v_{\text{max}} - c\text{e}^{-kt}$

(i) Value increases by a constant factor over equal intervals of time, so exponential growth

(j) Cyclic phenomenon so trigonometric

(k) Value decrease by a constant factor over equal intervals of time, so exponential decay

(l) The cost per person of hiring a bus varies inversely as the number of people

(m) The cost varies linearly with the number of guests but not directly, because even a wedding with no guests would have a cost!

2. Interpolation requires values between the smallest and largest data points, so

(a) $2 \le t \le 6$

(b) $11 \le x \le 100$

(c) $1.3 \le n \le 8.5$

3. **(a)** Rate of change is constant. Graph is a straight line.

(b) Rate of change increases linearly. Graph is a parabola.

(c) Periodic/cyclical behaviour.

(d) Dependent variable increases/decreases by a constant factor over equal intervals of time/independent variable.

(e) Rate of change is small, then large, then small again as asymptote is approached.

(f) Graph is hyperbolic.

Chapter 9 practice questions

1. **(a)** $T(5) = 280 \cdot 1.12^5 \approx 493$

(b) Setting the equation $T(n) = 2T(0)$ gives $1.12^n = 2$, and solving it for n yields $n = \log_{1.12} 2 \approx 6.12$ years so for the first time in 2007.

(c) $P(5) = \dfrac{2560000}{10 + 90e^{-0.1 \cdot 5}} \approx 39636$ people

(d) $P(7) = \dfrac{2560000}{10 + 90\mathrm{e}^{-0.1 \cdot 7}} \approx 46807$, so less than twice the number at the end of 2000, i.e. 26500.

(e) $R(n) = \dfrac{P(n)}{T(n)}$, so $R(0) = \dfrac{25600}{280} 91.4$.

(f) Setting the inequality $R(n) < 70$ gives $\dfrac{\dfrac{2560000}{10 + 90e^{-0.1n}}}{280 \cdot 1.12^n} < 70$. Graphing the left hand side on a GDC and finding the intersection with $R = 70$ yields $n > 9.31$, so after 10 complete years.

2. **(a)** $P(0) = -30$

$P(20) = 50$

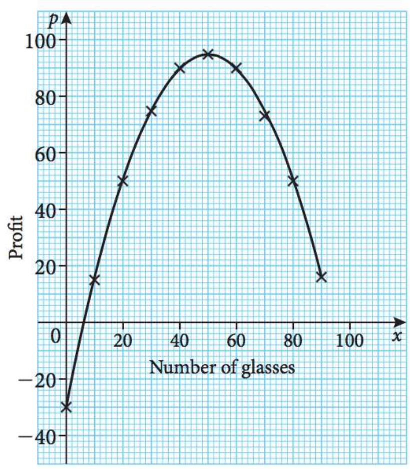
$P(30) = 75$

$P(50) = 95$

$P(60) = 90$

$P(90) = 15$

(b)



(c)

(i) maximum profit = y-coordinate of vertex = 95 Swiss Francs

(ii) number of glasses for maximum profit = x-coordinate of vertex = 50 glasses

(iii) x-coordinates of points with y-coordinate = 80, so approximately 33 and 67 glasses.

(iv) Selling zero glasses would result in a negative profit, so in a loss, of 30 Swiss Francs. This is the amount they initially invested.

(v) The profit for 40 glasses is 90 Swiss Francs. Fiona earns as much as Baljeet and Jane together, so she earns a half of 90 Swiss Francs or 45 Swiss Francs.

3. **(a)** "Immediately before" has the same meaning as "initially", so $Q(0)=25$ energy units.

(b) $Q(20)=0.003\cdot 20^2-0.625\cdot 20+25\approx 13.7$ energy units

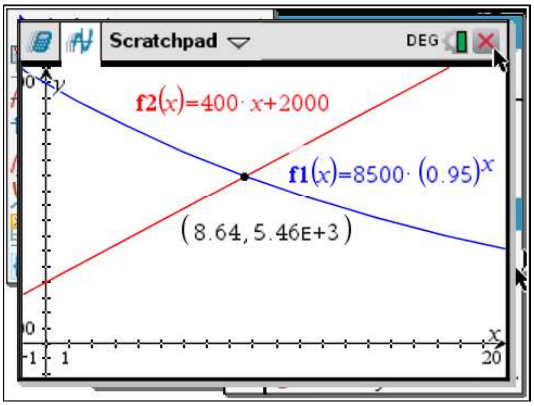
(c) energy lost per minute =

$$\frac{\text{total energy lost}}{\text{total time}}=\frac{Q(10)-Q(20)}{20-10}=\frac{19.05-13.7}{10}\approx 0.535 \text{ energy units min}^{-1}$$

(d) Setting $Q(t)=0$ and solving for t gives $t\approx 54$ or 154. Of course, the domain of validity of this model is $0\le t\le 54$, so only the solution $t\approx 54$ is acceptable.

4. **(a)** The amount added every month is the slope of the straight line $S(t)$, so 400 USD month^{-1}.

(b) Graphing both $S(t)$ and $P(t)$



and finding their intersection, gives 8.64 months so about 9 months.

(c) $P(2)-S(2)=8500\cdot 0.95^2-(400\cdot 2+2000)=4871.25$ USD.

5. **(a)** $A(20)=20(200-20)=3600$ m^2

(b) Setting the equation $A(x) = 3600$ and solving it for x gives

$$x(200-x) = 3600 \rightarrow x^2 - 200x + 3600 = 0 \rightarrow x = \frac{200 \pm \sqrt{200^2 - 4 \cdot 3600}}{2} = 20 \text{ or } 180.$$

Since site T is different from site S, the answer is 180 metres.

(c) **(i)** Maximum area when $x = -\frac{b}{2a} = -\frac{-200}{2} = 100$ metres.

(ii) Maximum area = $A(100) = 100(200-100) = 10000\ \text{m}^2$

(d) The area goes from a minimum of 3600 m^2 for $x = 20$ or 180 to a maximum of 10000. So $3600 \le A \le 10000$.

6. **(a)** $T(0) = 100\,°\text{C}$

(b) $T(1.37km) = T(1370) = -0.0034 \cdot 1370 + 100 \approx 95.3\,°C$

(c) Setting the equation $T(h) = 70$ and solving it for h gives

$$-0.0034h + 100 = 70 \rightarrow h = \frac{70-100}{-0.0034} \approx 8820 \text{ m.}$$

7. **(a)** $T(0) = 20 + 70 \cdot 2.72^{-0.4\times 0} = 20 + 70 = 90\,°\text{C}$

(b) $T(10) = 20 + 70 \cdot 2.72^{-0.4\times 10} \approx 21.3\,°\text{C}$

(c) Setting the equation $T(m) = 56$ and solving it for m gives

$$20 + 70 \cdot 2.72^{-0.4m} = 56 \rightarrow 70 \cdot 2.72^{-0.4m} = 36 \rightarrow 2.72^{-0.4m} = \frac{36}{70} \rightarrow -0.4m = \log_{2.72}\frac{36}{70}$$

$$m = \frac{1}{-0.4}\log_{2.72}\frac{36}{70} \approx 1.66 \text{ minutes}$$

(d) As $m \rightarrow +\infty, 2.72^{-0.4m} \rightarrow 0$, so $T \rightarrow 20 + 0 = 20\,°\text{C}$. This is the asymptotic value for the temperature.

(e) It represents the rate at which the temperature of the soup is increasing in $°\text{C min}^{-1}$.

(f) Setting the equation $20m - 40 = 20 + 70 \cdot 2.72^{-0.4m}$ and solving it for m with a GDC gives $m \approx 3.8$ minutes. This value represents the time in minutes after the water has been removed from the cooker at which the water and the soup have the same temperature.

(g) Since the water is cooling down and the soup is heating up, the soup has a larger temperature than water for $m > 3.8$. On the other hand, the domain of $S(m)$ is $2 \le m \le 6$, so the solution to $S(m) > T(m)$ is $3.8 < m \le 6$.

8. **(a)** $V(0) = 25000 \cdot 1.5^{-0.2\times 0} = 25000$ USD

(b) $V(3) = 25000 \cdot 1.5^{-0.2\times 3} \approx 19601.32$ USD

(c) Setting the equation $V(t) = \frac{V(0)}{2}$ and solving it for *t* gives

$1.5^{-0.2t} = 0.5 \rightarrow -0.2t = \log_{1.5} 0.5 \rightarrow t = \frac{1}{-0.2}\log_{1.5} 0.5 \approx 8.55$ years

9. **(a)** In 8 hours, the number of bacteria triples twice, so $200 \cdot 3 \cdot 3 = 1800$ bacteria.

(b) In 24 hours, the number of bacteria triples 6 times, so $200 \cdot 3^6 = 145800$ bacteria.

(c) Setting the equation $200 \cdot 3^t = 2000000$ and solving it for *t*, the number of 4-hour intervals, gives $3^t = \frac{2000000}{200} \rightarrow t = \log_3 10000 \approx 8.38361$, which in turn gives $8.38361 \cdot 4 \approx 33.5$ hours.

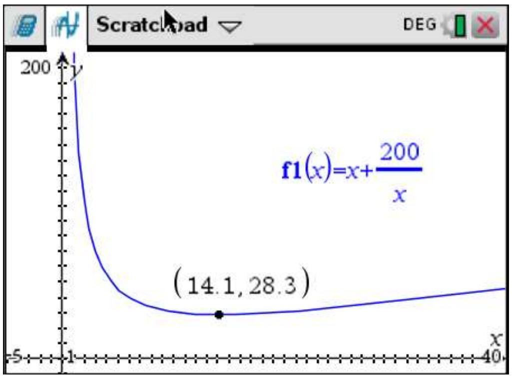
10. **(a)** $C(40) = 40 + \frac{200}{40} = 45$ euros

(b) Setting the equation $C(x) = 33$ and solving it for *x* gives

$x + \frac{200}{x} = 33 \rightarrow x^2 - 33x + 200 = 0 \rightarrow x = \frac{33 \pm \sqrt{33^2 - 4\cdot 200}}{2} = 25$ or 8 people,

so e.g. $a = 8$ and $b = 25$ (or vice versa).

(c) Graphing $C(x)$ and finding its minimum on a GDC gives



(i) $n = 14.1$, so the number of people is either 14.

(ii) the minimum cost per person is $C(14) \approx 28.29$ euros.

11. **(a)** The maximum height is a diameter plus the height of the bottom of the wheel above the ground, so 122 + 12 = 135 metres

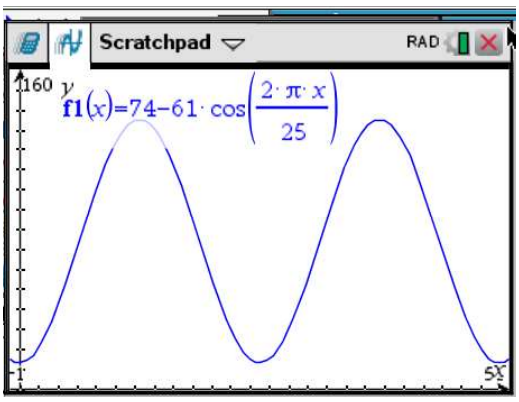
(b) The wheel completes 2.4 rotations in an hour, so the time for one rotation is

$$T = \frac{1 \text{ hour}}{2.4} = \frac{60\,\text{min}}{2.4} = 25\,\text{min}.$$

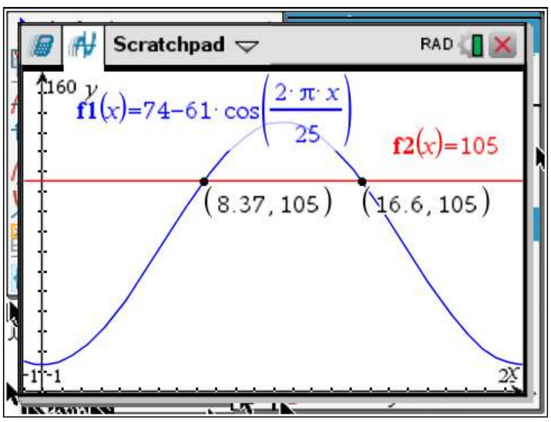
(c) The parameter b relates to the period as $b = \frac{2\pi}{T}$, so $b = \frac{2\pi}{25}$ rad min^{-1}.

(d) The magnitude of the parameter a is the amplitude, so in this case $|a| =$ radius of wheel = 61 metres. At $t = 0$, when the seat is at the bottom of the wheel, the cosine function is equal to one, so the prefactor a has to be negative in order for the height of the seat to be smaller than 74. Therefore $a = -61$ metres.

(e)



(f) The amount of time any given seats spends at a height larger than or equal to 105 m is given by the difference between the time coordinates of the intersections of $h(t)$ and $h = 105$. Finding these intersections with the GDC gives



$t = 8.37$ and $t = 16.6$. The fraction of time spent at $h \geq 105$, and therefore the probability during one rotation, is given by $P = \frac{16.6 - 8.37}{25} = 0.329$.

Note: there may be some *slight* differences with answers at the end of the book. These are expected and both answers will be acceptable. Differences stem from different accuracy of graphs/GDC/software used, and in some cases different decisions on the number of classes or class sizes.

Exercise 10.1

1. Unit. Example of sensible population. Example of sensible sample. Type of variable

(a) No experimental unit. Total number of students. Students in one school. Qualitative.

(b) No unit. All 10th grade students taking that exam. 10th grade students from a particular school or district. Quantitative and discrete.

(c) Usually cm or inches. All newborn children. Newborn children from a particular hospital or a particular days of the week. Quantitative and continuous.

(d) No unit. All children aged less than 14 years old. Children less than 14 years old from a specific location. Qualitative.

(e) Seconds or minutes or any other units of time. All the times people went to work (all the commuters). The commuters within a city or a company. Quantitative (and continuous)

(f) No units. The population of a country. A subset of that population (from a city, from a given aged group, etc.). Qualitative and ordinal (because there is an order in the qualities)

(g) No units. All the students in all the international schools. Only international schools from a given region, or countries. Qualitative.

2. **(a)** **(i)** The population is the 1176 students from grade 10–12.

(ii) Either a random sample or a random stratified sample (based on age and gender for instance) would be appropriate.

(b) **(i)** The population is all the bolts produced.

(ii) A random stratified sample (based on the shift) or a systematic sampling (collecting each kth bolt produced).

3. **(a)** Blood type is categorical (qualitative)

(b) Number of cars is a discrete numerical (quantitative)

(c) The length of a fish (in whatever units) is a continuous numerical (quantitative)

(d) The amount of time spent studying mathematics is a continuous numerical (quantitative)

(e) The volume of liquid in a canned drink is a continuous numerical (quantitative)

(f) The number of languages spoken in a community is a discrete numerical (quantitative)

(g) The 100m race time is a continuous numerical (quantitative)

(h) Colors used on a whiteboard is a categorical (qualitative)

(i) The rating of a trumpet solo as superior, excellent and good is a categorical (qualitative)
In this case, we can add that the data is ordinal categorical as the categories have an order.

4. **(a)** Volume of paint in a can is continuous numerical (quantitative)
(b) Number of children in a family is discrete numerical (quantitative)
(c) Time taken to finish a marathon is continuous numerical (quantitative)
(d) The length of an average children film is continuous numerical (quantitative)
(e) The height of a student is continuous numerical (quantitative)
(f) The average carbon dioxide emissions is continuous numerical (quantitative)
(g) The religion(s) practiced in each household is categorical (qualitative)

5. **(a)** Inferential statistics. We are going to make an inference based on a sample of a population (16 years old football players in the UK).
(b) Descriptive statistics. We will check all free throw of that specific player for that 10 games season.

6. **(a)** To find the average mass of all 12 years in Spain, we will need to collect a sample. So inferential statistics.
(b) To find the average amount of money spent on entertainment for the next 3 months, we will record all such expenses. So descriptive statistics.
(c) To find the number of heads after flipping a coin 100 times, we will record the number of head. So descriptive statistics.
(d) To determine if the number of woman wearing seat belts is greater than the number of man wearing seat belts, we will collect a sample of the whole population. So inferential statistics.
(e) To determine if the IB Mathematics score for boys is greater than for the girls, we can check the data from the IB (IB statistical report). So descriptive statistics.

7. **(a)** This is a non-random sampling techniques as it is based on his knowledge and expertise.
(b) As presented, this would be a judgment sample.
(c) Biased in the sample is introduced with this sampling techniques and poor generalization will result.
(d) This sample is random. Actually, it is pseudo-random as it has been selected through a software who generates pseudo-random numbers.
(e) This is a simple random sampling technique.
(f) We might still have some differences in profiles between the sample and the whole class.

8. This is a stratified sampling technique as the population has been divided in 2 strata based on gender.

9. A stratified sample could be used. The strata are the different faculties. It could even stratified further by considering gender, origin, social classes, etc.

10. A random quota sample has been used. Same number of men and women has been chosen. Within each gender, individuals are chosen randomly.

Exercise 10.2

1. **(a)** To find the x and y value, add the number of students to the cumulative frequency.
$x = 137 + 31 = 168$, $y = 194 + 6 = 200$

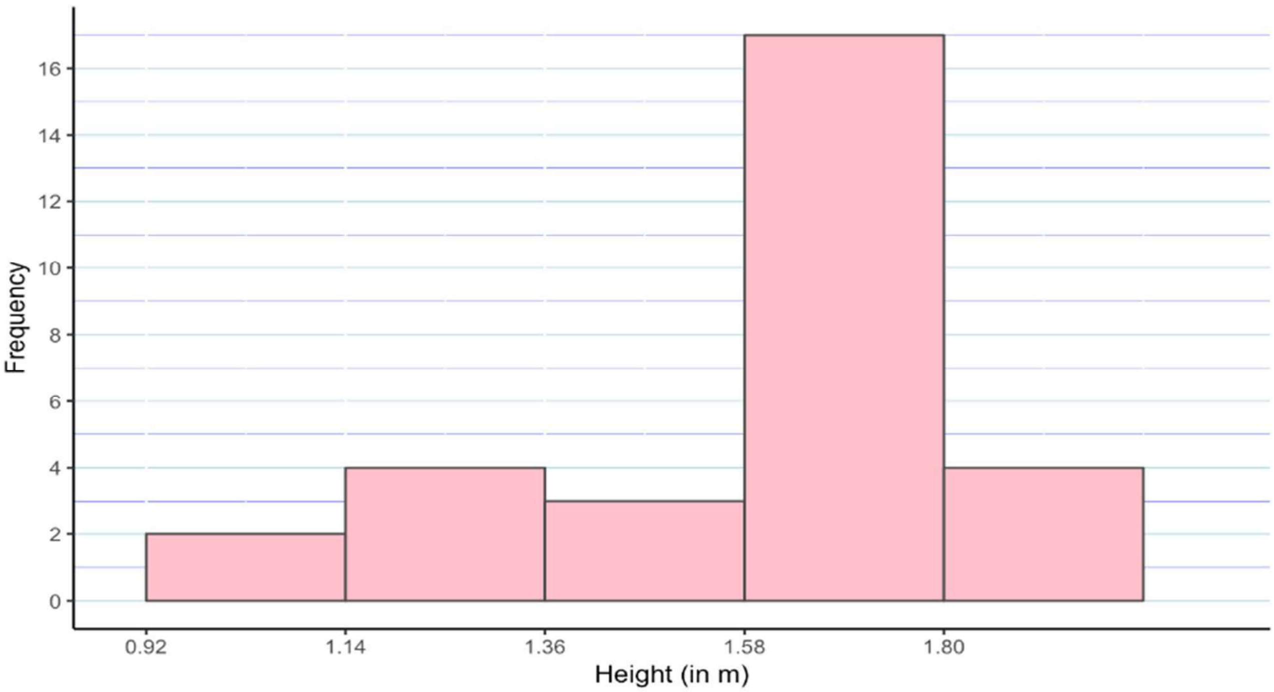
(b) Students younger than 21 are either 17, 18, 19 or 20. Check cumulative frequency for 20 years old. Cumulative frequency when age is 20 = 168.

(c) 25th percentile is the first quartile. So $\frac{1}{4}$ of 200 is 50.
The 50th student is 18 years old.

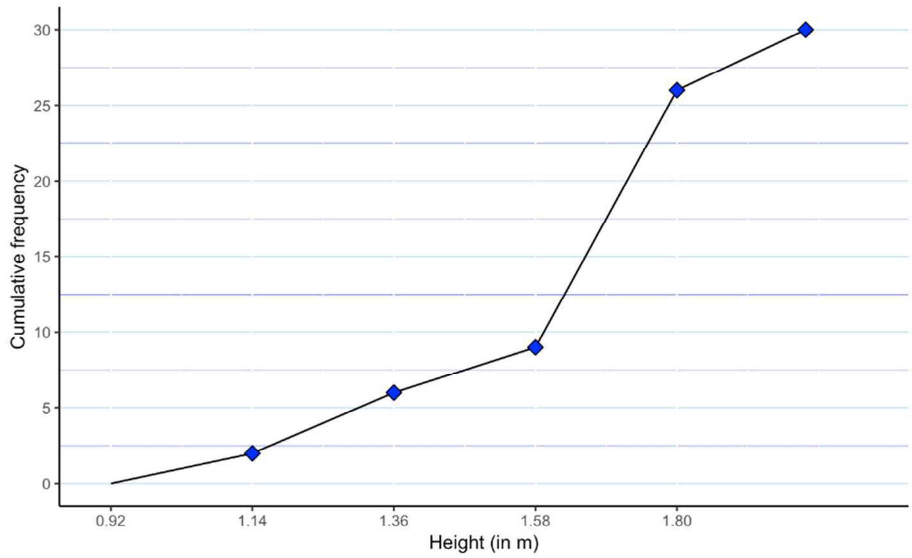
2. **(a)** As mentioned in 10.2, a good rule of thumb is to choose between 5 and 15 classes. With 30 data points, 5 class is an appropriate choice. Height in metres is continuous. Min is 0.92 m, max is 2 m. The width of each class interval can be $\frac{2-0.92}{5} = 0.216$ which is rounded to 0.22.

Height (in m)	Frequency
$h < 1.14$	2
$1.14 \le h < 1.36$	4
$1.36 \le h < 1.58$	3
$1.58 \le h < 1.80$	17
$h \ge 1.80$	4

(b)

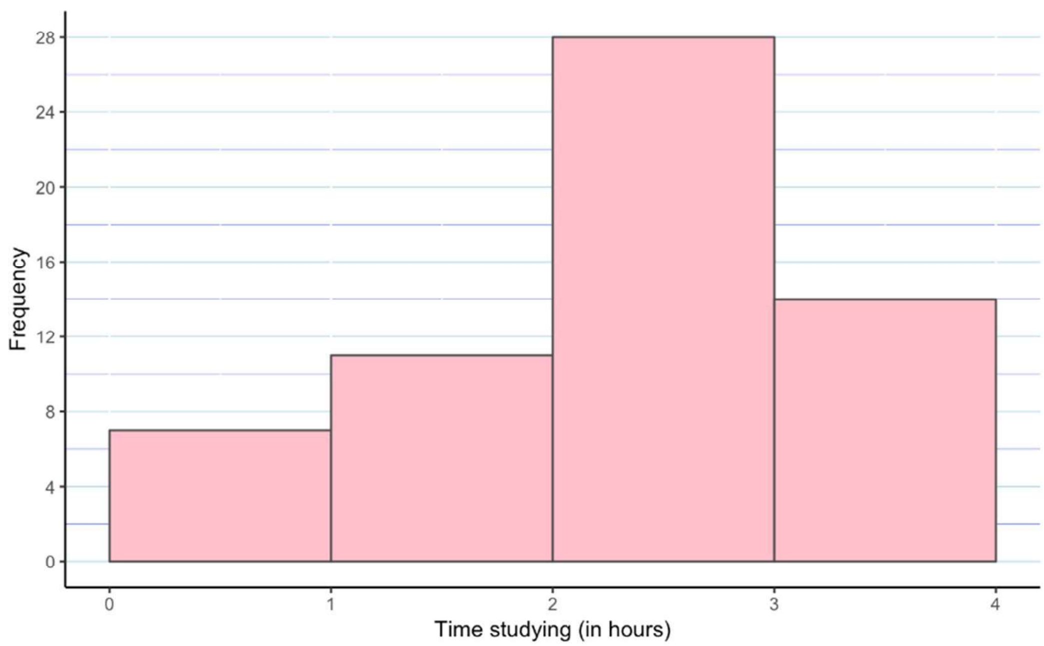


(c)

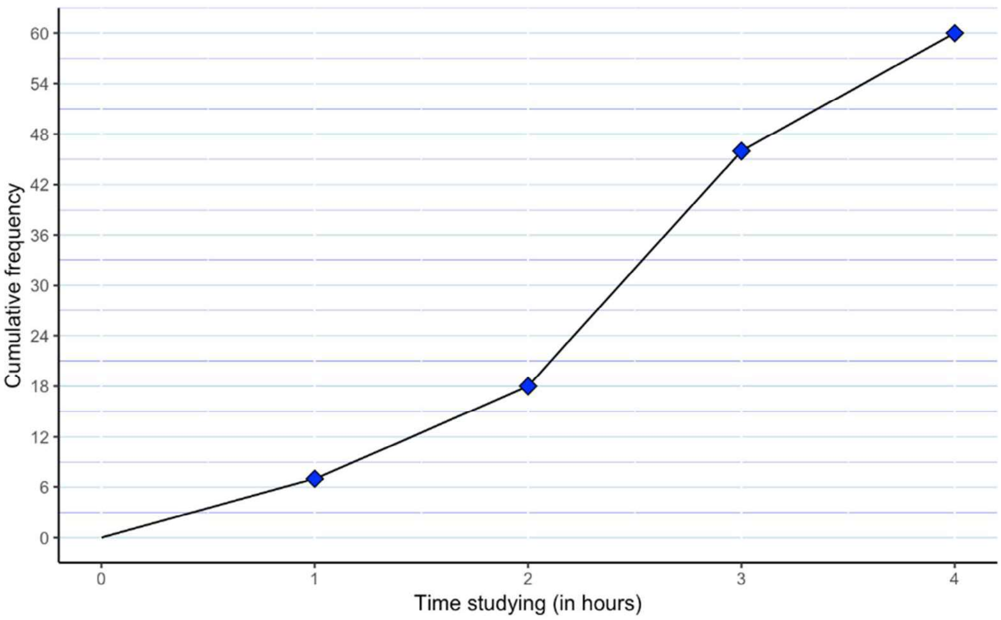


(d) Following our 5 classes, one notice that most students are between 1.58 and 1.80 m. The 2 classes, 1.14 to 1.36 m and 1.8 to 2.0 m have the same number of students (4). We also noticed that the data is skewed to the left, that is quite a few students have heights less than the modal class.

3. **(a)** To find the frequency p and r, we consider the cumulative frequency at that row and subtract from it the cumulative frequency of the previous row.
To find the cumulative frequency q, we take the frequency at that row and add it to the cumulative frequency of the previous row.
$p = 18 - 7 = 11$, $q = 18 + 28 = 46$, $r = 60 - 46 = 14$



(b)



(c) There are 60 students that have been surveyed. Most students (28) spend between 2 and 3 hours studying per evening. Few students (7) are spending less than 1 hours studying each evening.

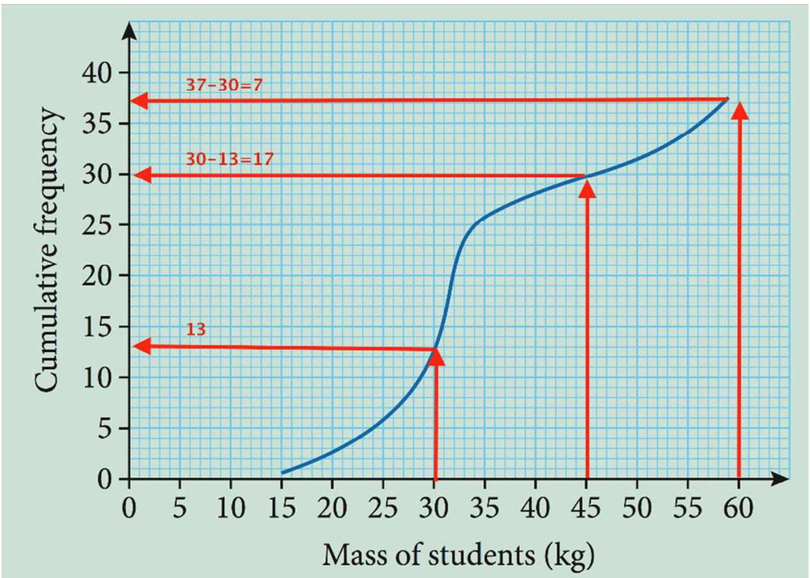
4. **(a)** To find the number of students that were included in the sample, add all the frequency for each class. $3+5+12+18+10+4+3=55$ students.

(b) 4 students achieved a score of 6. $\frac{4}{55}=0.0727$ (3 s.f.). 7.27% of students achieved a score of 6.

(c) Score below 3: 3 students have a score of 1 and 5 have a score of 2.
So 8 students have a score below 3. $\frac{8}{55}=0.145$ or 14.5%

(d) In the bar chart, 3 students received a score of 7. $\frac{3}{55}=0.0545$ (3 s.f.). In a population of 800, we could infer that $800 \cdot 0.0545 = 43.6$, or 44 students would get a score of 7.

5. **(a)**

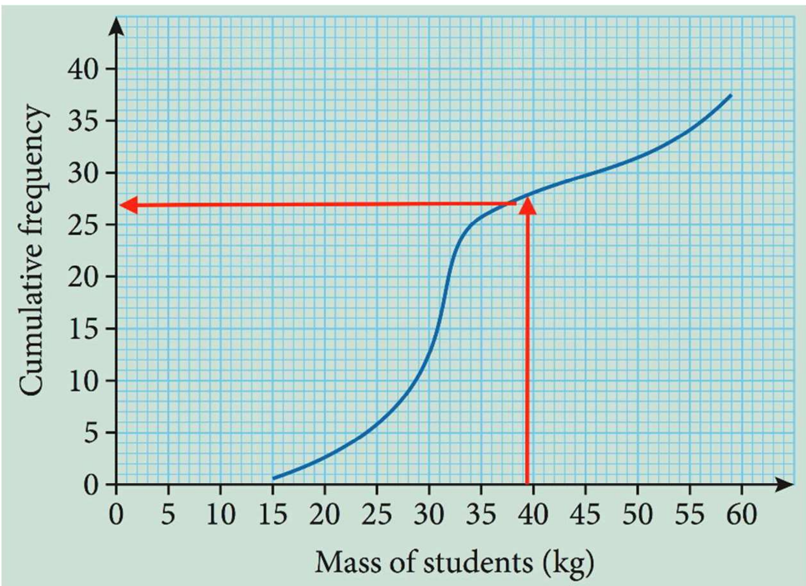


Mass of students, M (kg)	Frequency
$15 \le M < 30$	13
$30 \le M < 45$	17
$45 \le M < 60$	7

(b) 37

(c) The biggest frequency is 17. This is where $30 \le M < 45$

(d) 27. We need to check below 40 and see what the frequency is.

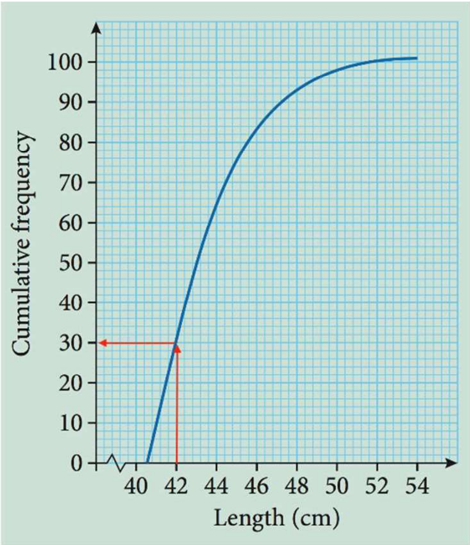


(e) From **(a)** we know that there are 37 students. 25% percentile is the first quartile. $\frac{37}{4} = 9.25$. If we check at 9 on the cumulative frequency, we see that the maximum weight is 28 kg.

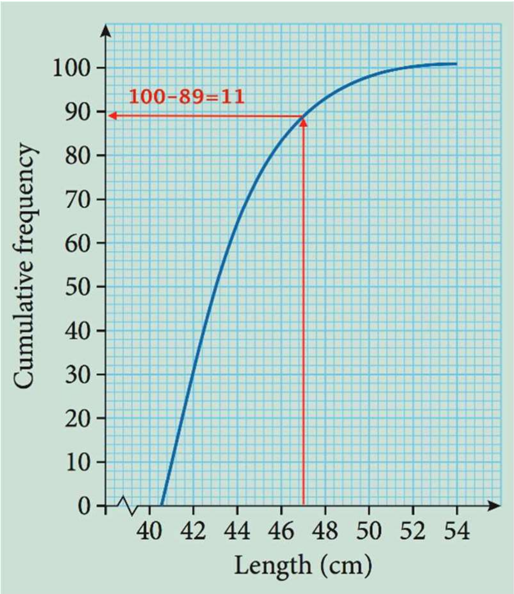
6.

(a) Based on the graph, we see that pike measured between 40.4 cm to 54 cm. Hence a range of 13.6 cm.

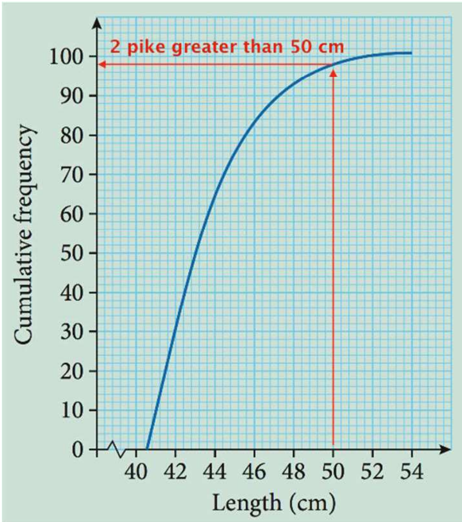
(b)



(c) 11 pike measure more than 47 cm.



(d) 2 pike greater than 50 cm. Hence 2% .

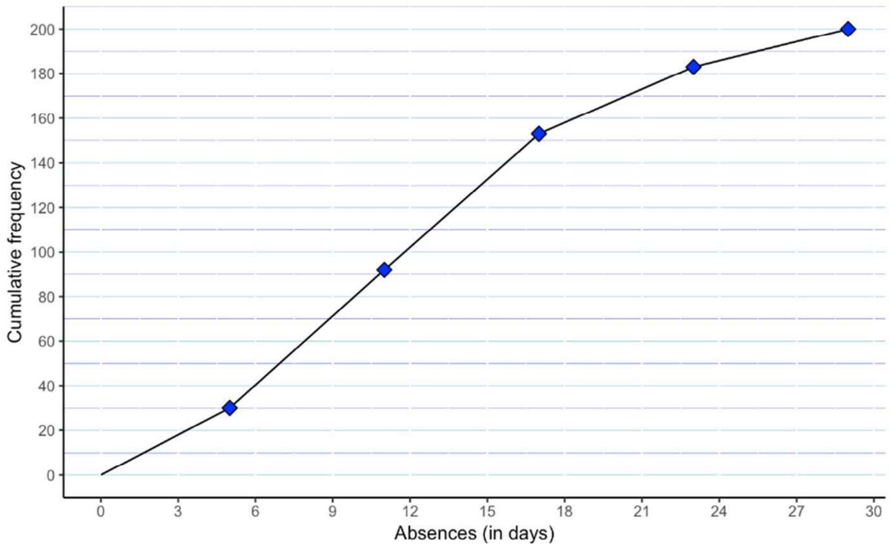


(e) In the second survey, there are fewer pike (89) comparing to the first survey (100). The range of pike in the second survey is also greater, going from 40 to 55.2 cm. In the second survey half of the pike are greater than 51 cm when in the first survey half of the pike are greater than 43 cm. In the second sample fewer pike were shorter than 42 cm (4), more pike were longer than 47 cm (70) and a larger percentage are big (40.4%).

7. (a) You get the cumulative frequency by adding the frequency to the cumulative frequency of the previous row.

Absences (in days)	Frequency	Cumulative Frequency
$0 \le x \le 5$	30	30
$6 \le x \le 11$	62	92
$12 \le x \le 17$	61	153
$18 \le x \le 23$	30	183
$24 \le x \le 29$	17	200

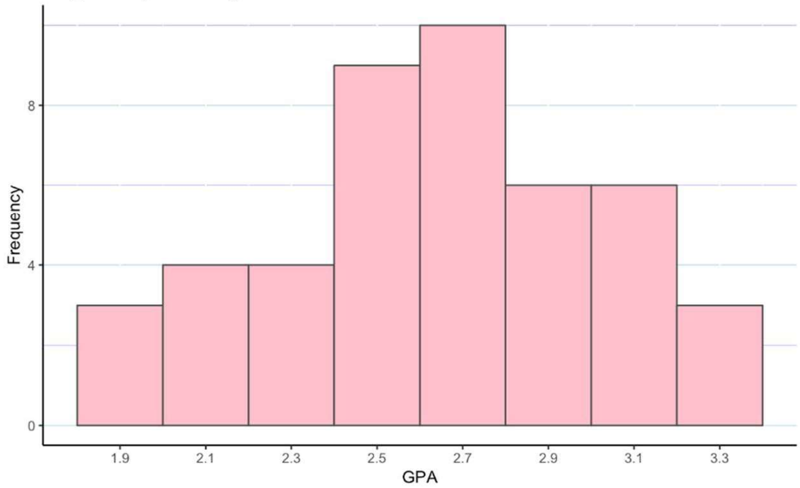
(b)



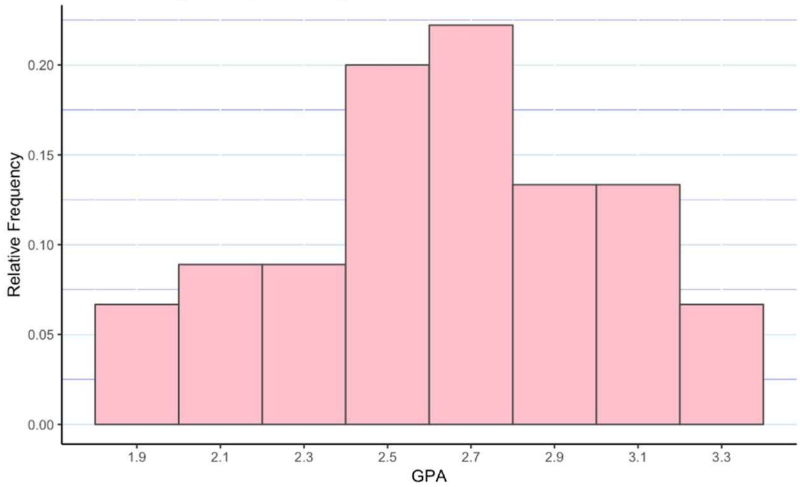
8. We first make a table with our values and bin. We chose a bin of 0.2 width. To calculate the relative frequency, we divide each frequency by the total number of students (45).

GPA	Frequency	Cumulative Frequency	Relative Frequency
$GPA < 2$	3	3	0.0667
$2 \le GPA < 2.2$	4	7	0.0889
$2.2 \le GPA < 2.4$	4	11	0.0889
$2.4 \le GPA < 2.6$	9	20	0.2
$2.6 \le GPA < 2.8$	10	30	0.222
$2.8 \le GPA < 3$	6	36	0.133
$3 \le GPA < 3.2$	6	42	0.133
$GPA >= 3.2$	3	45	0.0667

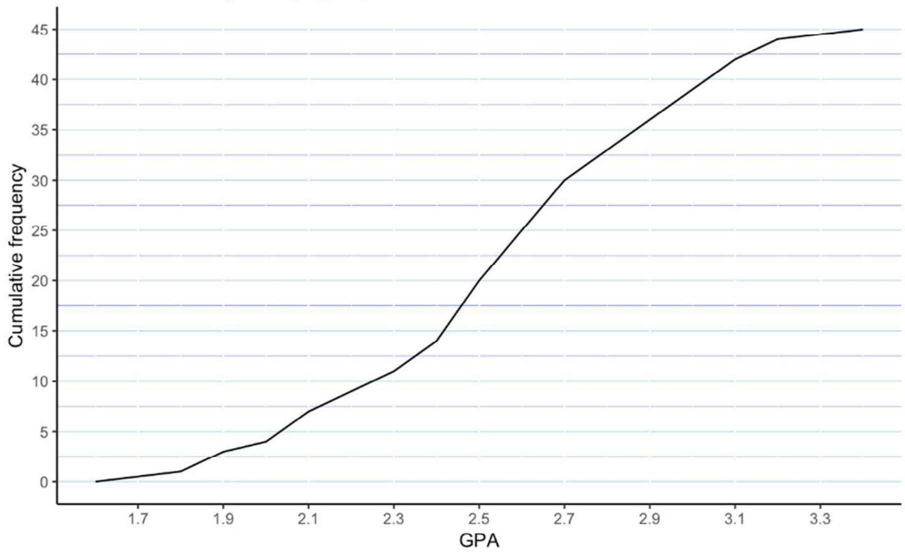
A frequency histogram



A relative frequency histogram

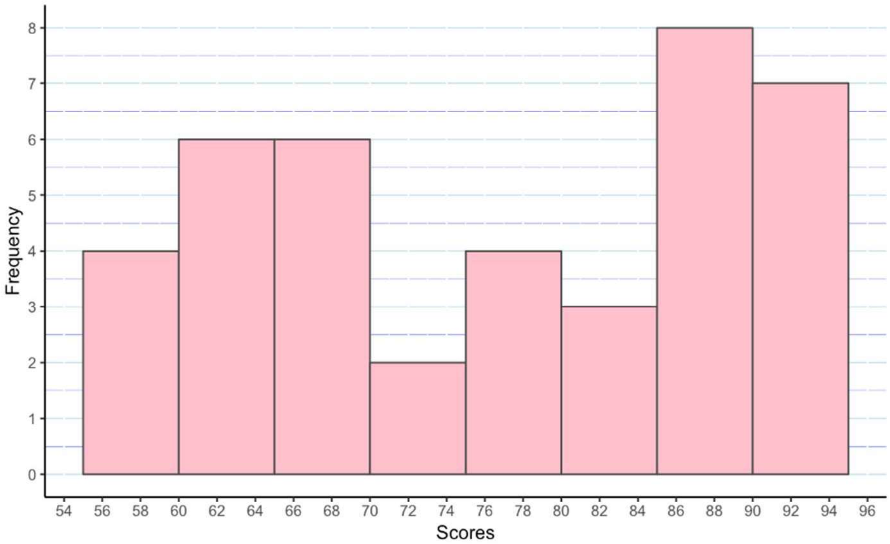


A cumulative frequency graph



Most students have a GPA between 2.4 and 2.7. 15 students have a GPA of 2.8 or above and 11 students have a GPA of 2.3 or below. The modal class is $2.6 \le GPA < 2.8$.

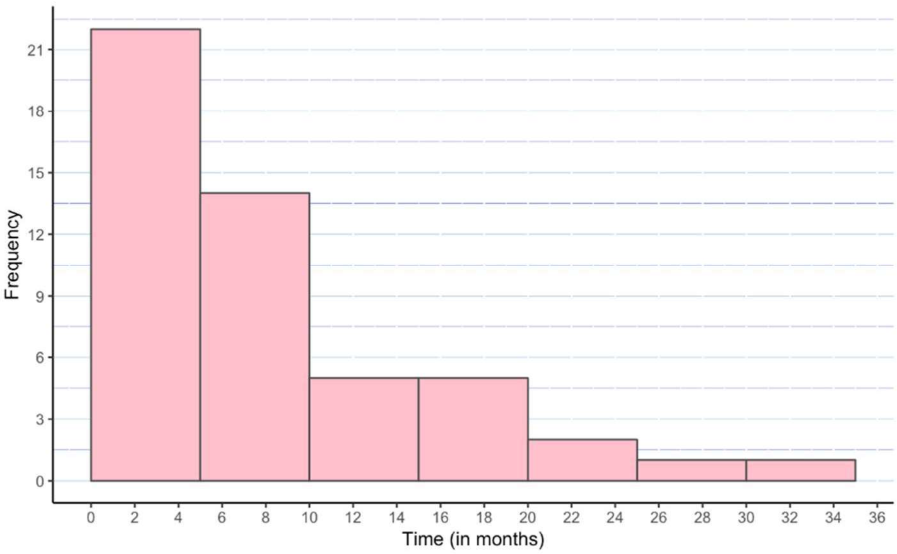
9. To visualise the data, we have chosen a frequency histogram with a class width of 5.



We see that our data can be clustered into 3 main groups. Students with scores of 55 to 69, students having score between 70 and 84 and students with scores of 85 or above.

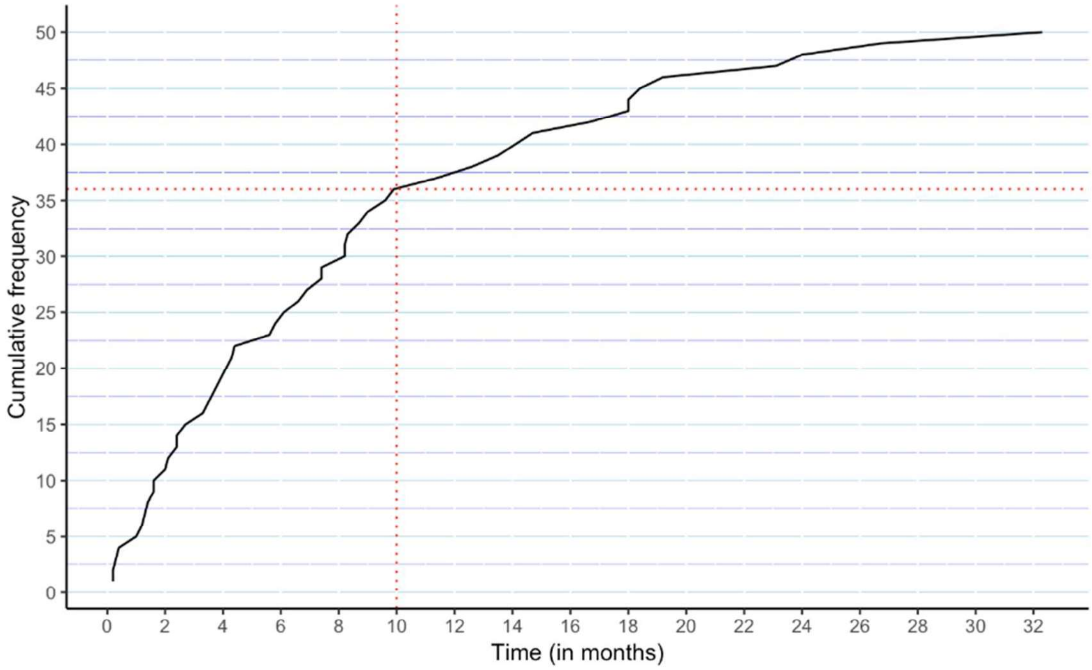
10. (Slightly different from book answers because of our choice of class width of 5 months while the book chose 2 months.)

(a) We notice that our lowest value is 0.2 and highest value is 32.3.
We have chosen a class width of 5 months.



(b) The symmetry of the graph will of course depend on the choice made for the class width. In our case, the data does not display any symmetry. Rather, the data is very positively skewed.

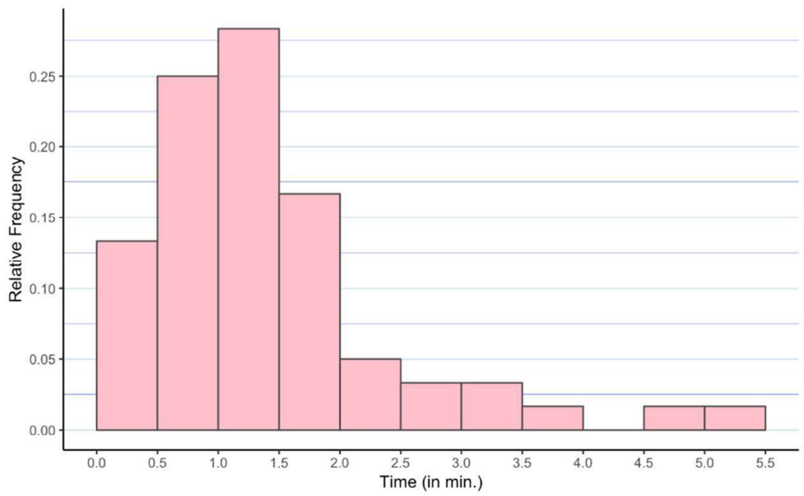
(c) We notice that 36 drivers have a repeated violation within 10 months. In this case, $\frac{36}{50} = 72\%$



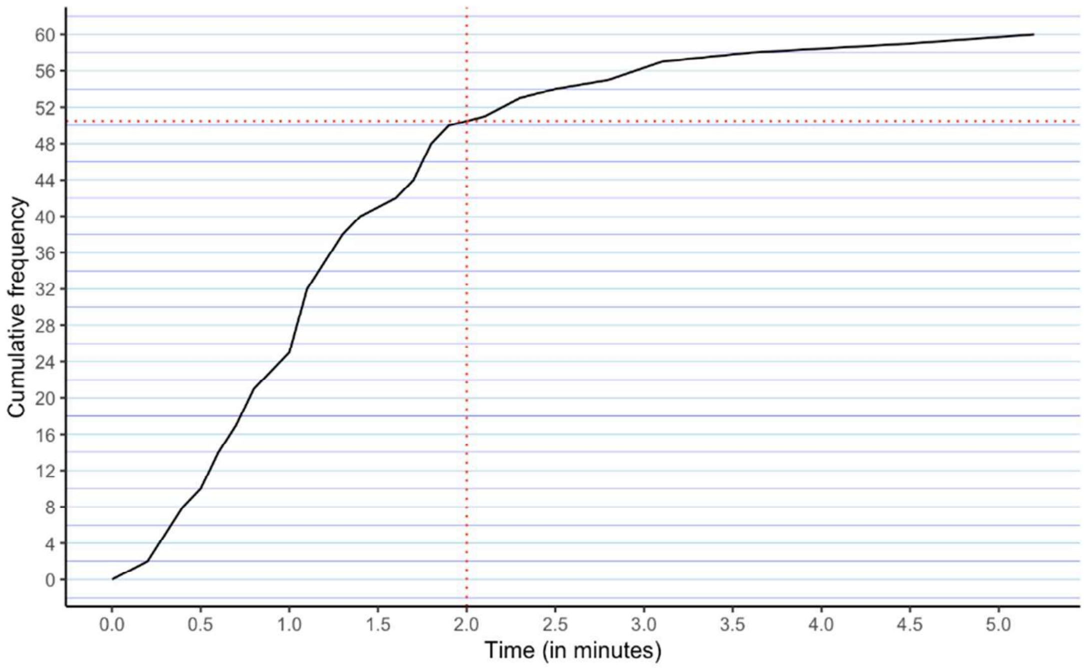
11. **(a)** To construct a relative frequency histogram, we first need to determine our class width, our frequency then our relative frequency. To get the relative frequency, we divide each frequency by the total number of customer surveyed (60 in this case).

Time to wait to be served (in minutes)	Frequency	Relative Frequency.
$t < 0.5$	8	0.133
$0.5 \le t < 1$	15	0.25
$1 \le t < 1.5$	17	0.283
$1.5 \le t < 2$	10	0.167
$2 \le t < 2.5$	3	0.05
$2.5 \le t < 3$	2	0.0333
$3 \le t < 3.5$	2	0.0333
$3.5 \le t < 4$	1	0.0167
$4 \le t < 4.5$	0	0
$4.5 \le t < 5$	1	0.0167
$t \ge 5$	1	0.0167

A relative frequency histogram.

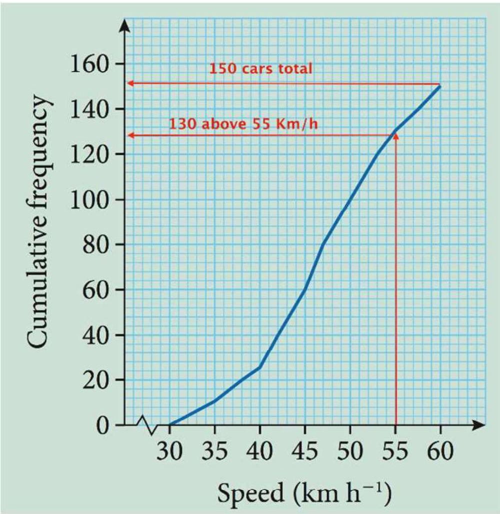


(b) Using a cumulative frequency graph, we notice that around 10 customers (60 – 50) have to wait 2 minutes or more.

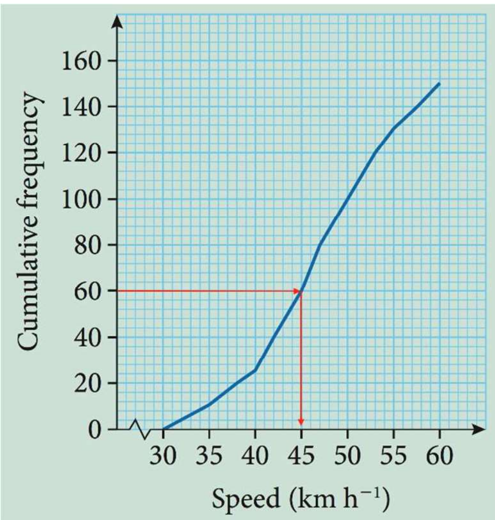


12. **(a)** Based on the cumulative frequency graph, the minimum speed of car passing through that intersection is 30 km h^{-1}.

(b) Based on the graph, we see that there are 150 cars passed through that intersection. 20 of these cars (150−130) had speed above 55 km h^{-1} which is $\frac{20}{150} = 0.133$ or 13.3% of the cars.



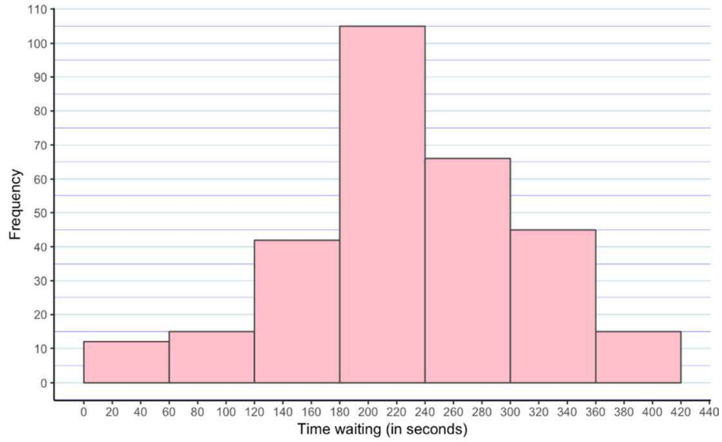
(c) 40% of 150 cars is $\frac{40}{100} * 150 = 60$. So we need to look at what is the speed at the 60th car.
Based on the graph, we see that the speed is 45 km h^{-1}. $k = 45$.



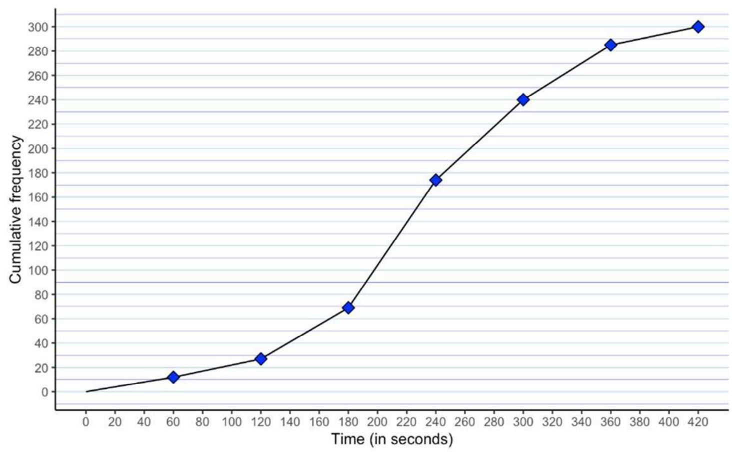
(d) The 60th percentile is $\frac{60}{100} * 150 = 90$. Using the same method as above, we see that the 90th car is travelling around 48 km h^{-1}.

13.

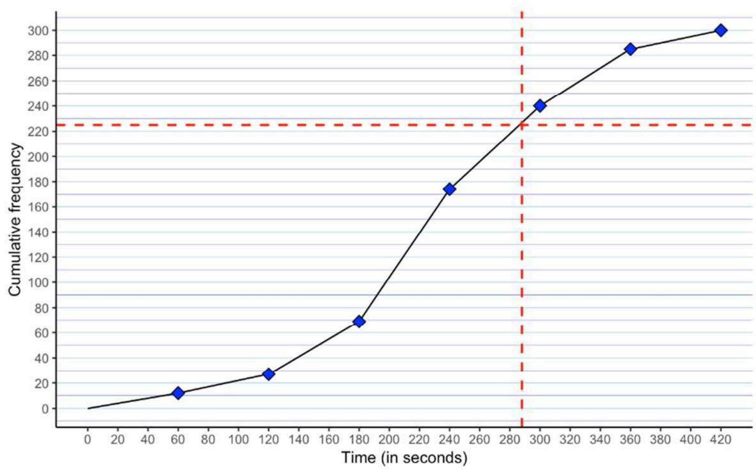
(a)



(b)



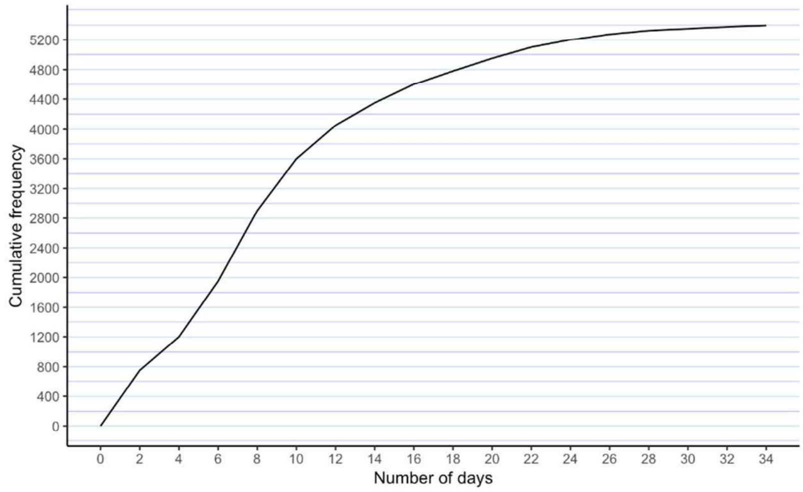
(c) The waiting time that is exceeded by 25% of the 300 customers, we are looking at the 75th percentile or the 225th customer. We go on the cumulative frequency graph using the same method as above and see that the waiting time would be around 288 seconds.



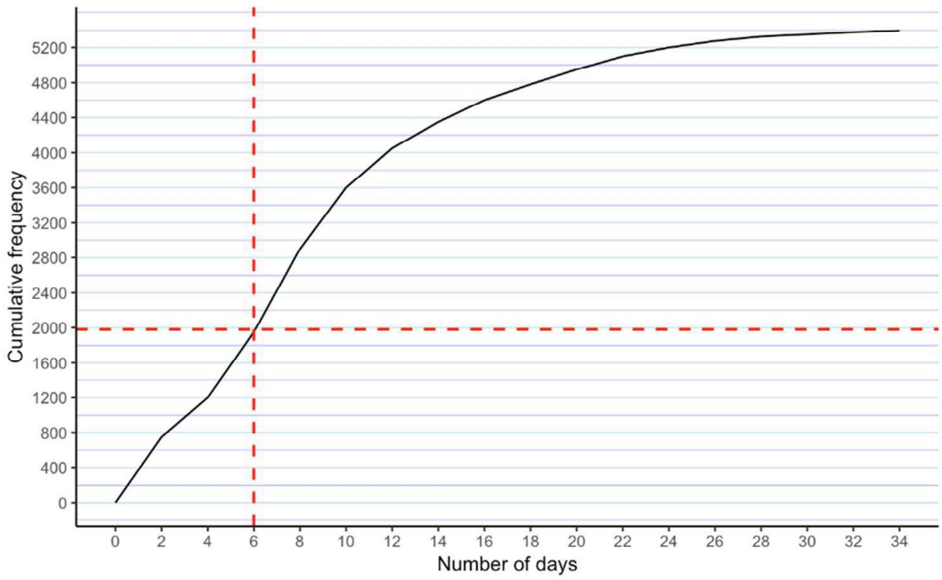
(d) The 75% percentile is the same as the waiting time exceeded by 25% of the customers. So around 288 seconds.

14. **(a)** We see that the data are right skewed with a mode at around 7 days. A few people have long stayed over 20 days. The vast majority of patients stays in hospital less than 7–8 days.

(b)



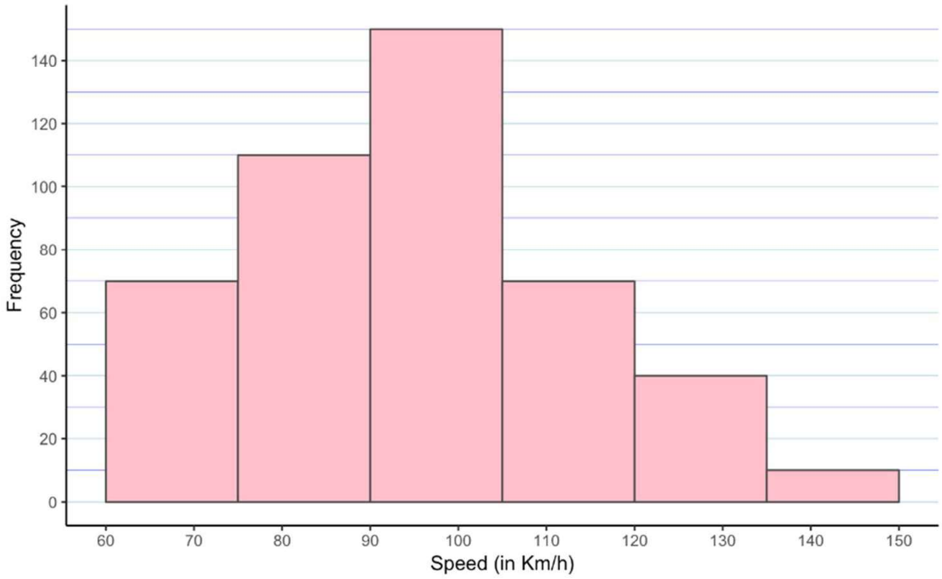
(c) Based on the above graph, we see that a little fewer than 2000 patients on the around 5400 patients stayed less than 6 days. $\dfrac{2000}{5400} = 0.37 = 37\%$ of patients. (You can expect a small discrepancy depending of your cumulative frequency graph.)



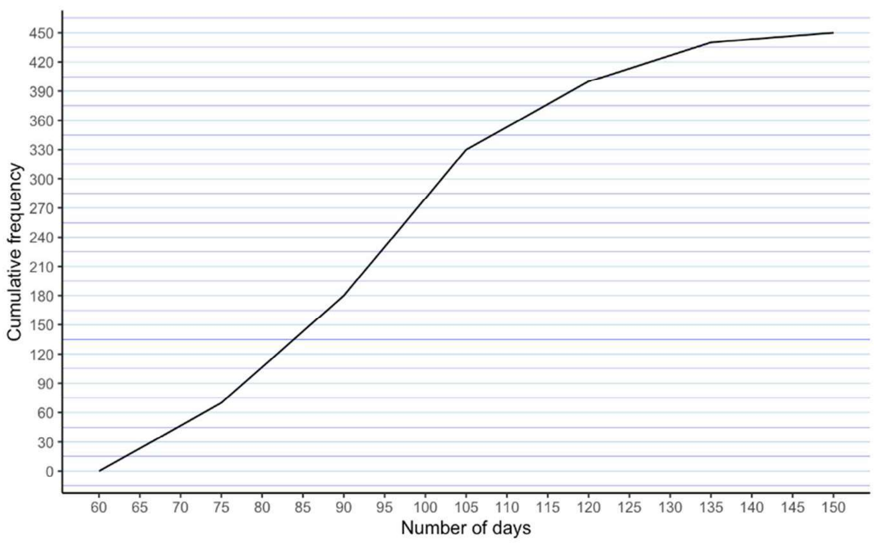
15. **(a)**

Speed (in km h^{-1})	Frequency	Cumulative Frequency
$60 \le s < 75$	70	70
$75 \le s < 90$	110	180
$90 \le s < 105$	150	330
$105 \le s < 120$	70	400
$120 \le s < 135$	40	440
$s \ge 135$	10	450

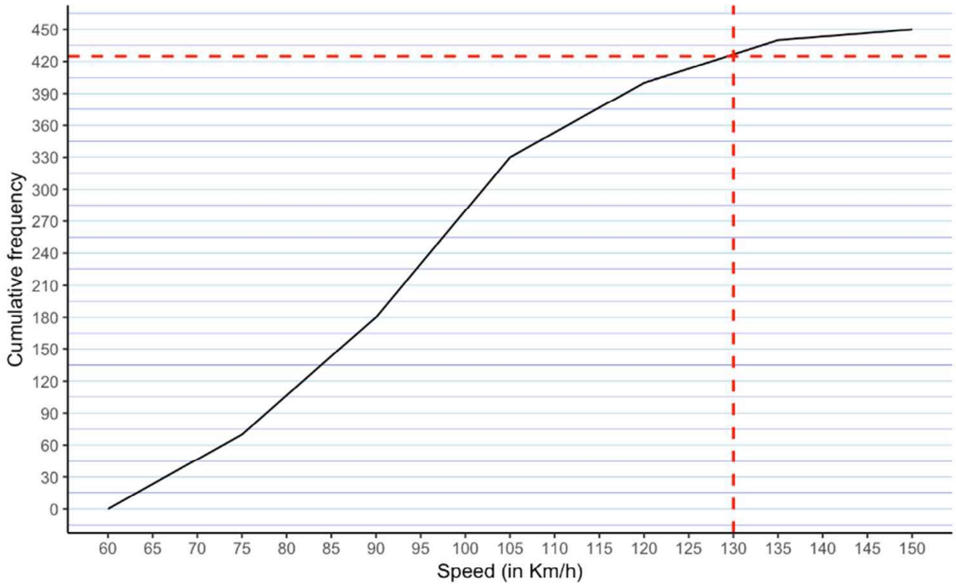
(b) For the histogram, we use the end-values of each speed class.



(c)



(d) Based on the graph, we see that around 425 people drives less than 130 km h^{-1}. Hence, 25 people are driving more than 130 km h^{-1} which is $\frac{25}{450} = 0.556\,(3 \text{ s.f.})$ or 5.56%.



Exercise 10.3

1. **(a)** $n = 4$. 8 appears 4 times in the data.

(b) To find the mean number of visitor, we have a couple of ways to do this. We can either add all the numbers and divide by the number of data points. All answers are in thousands.

$$\frac{9+7+8+11+9+6+10+8+12+6+8+13+7+9+10+9+10+11+12+8+7+13+10+7+7}{25} = 9.08$$

Or we can use the frequencies already given in the table (more efficient)

□

$$\frac{6\cdot 2+7\cdot 5+8\cdot 4+9\cdot 4+10\cdot 4+11\cdot 2+12\cdot 2+13\cdot 2}{25} = 9.08$$

To find the standard deviation, using the raw data and the mean we just found:

$$\sqrt{\frac{(9-9.08)^2+(7-9.08)^2+(8-9.08)^2+\cdots+(7-9.08)^2+(7-9.08)^2}{25}} = 2.04$$

Or using the frequencies table, we can calculate the standard deviation this way:

$$\sqrt{\frac{(6-9.08)^2\cdot 2+(7-9.08)^2\cdot 5+(8-9.08)^2\cdot 4+\cdots+(12-9.08)^2\cdot 2+(13-9.08)^2\cdot 2}{25}} = 2.04$$

(c) 6 weeks have more than 10 (thousands) visitors. $\frac{6}{25} = 0.24$ or 24%.

(d) 7 is the modal number of visitor. So 7 000.

2. **(a)** Using GDC, mean = 11 or manually $\frac{1+1+8+3+1+1+7+26+51}{9}=11$

Median=3; we need to order the data first, 1, 1, 1, 1, 3, 7, 8, 26, 51.
Mode = 1

(b) $(1-11)^2\cdot 4+(3-11)^2+(7-11)^2+(8-11)^2+(26-11)^2+(51-11)^2=2314$

(c) range: $51-1=50$

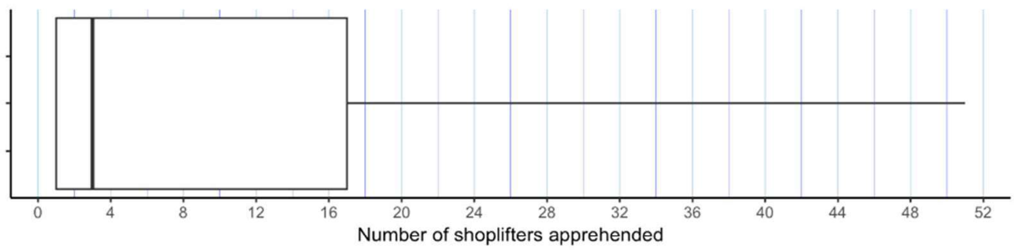
Variance: $\frac{(1-11)^2\cdot 4+(3-11)^2+(7-11)^2+(8-11)^2+(26-11)^2+(51-11)^2}{9}=257$ (3 s.f.)

Standard Deviation:

$$\sqrt{\frac{(1-11)^2\cdot 4+(3-11)^2+(7-11)^2+(8-11)^2+(26-11)^2+(51-11)^2}{9}}=16.0 \text{ (3 s.f.)}$$

(d) For the boxplot, we first calculate the five-point summary:

min = 1, Q_1 = 1, Q_2 = 3, Q_3 = 17 $\left(\text{since } \frac{26+8}{2}\right)$, max = 51



There is no whisker on the lower side as the minimum value (1) and the first quartile (Q_1 = 1) are the same.

(e) $\text{IQR}=Q_3-Q_1=17-1=16$

Any value greater than Q3 + 1.5 IQR is an outlier.
Upper fence: $17+1.5\cdot 16=41$. So, 51 is an outlier.

3. **(a)** Mean (using GDC) = 55.8 (3 s.f.).

Manually: $\frac{60+30+70+10+140+30+80+70+20+60+90+10}{12}=55.8$

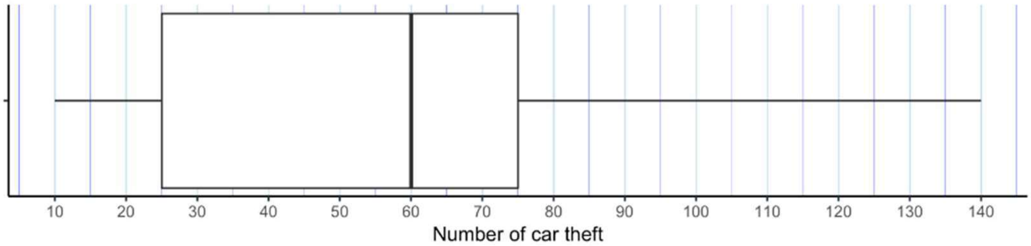
Median: 60. Order the data, take the middle value (between 6th and 7th value)
10, 10, 20, 30, 30, 60, 60, 70, 70, 80, 90, 140
Mode: 10, 30, 60, 70 (each numbers appear twice). This set of data is multi-modal. It has several modes.

(b) Range: $140-10=130$

Standard Deviation: (using GDC) 36.6

(c) To draw a boxplot, we need the five points summary.

Min = 10, Q_1 = 25, Q_2 = 60, Q_3 = 75, Max = 140



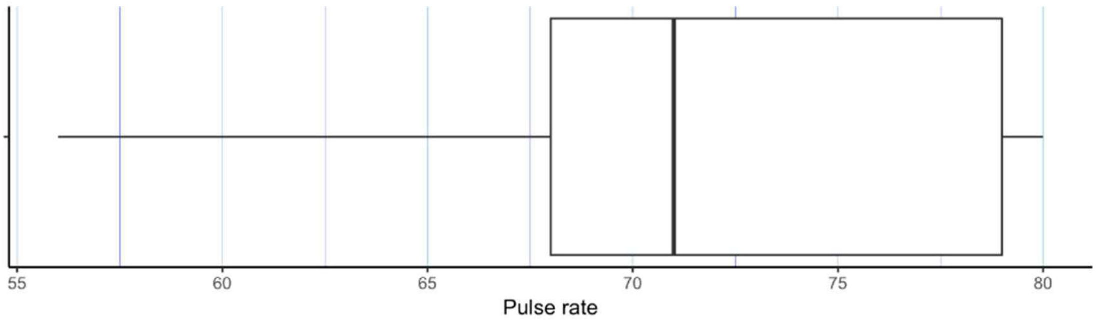
(d) To find outliers, we will check the fences.
Upper fence: $Q_3 + 1.5 \cdot \text{IQR} = 75 + 1.5 \cdot (75 - 25) = 150$

Lower fence: $Q_1 - 1.5 \cdot \text{IQR} = 25 - 1.5 \cdot (75 - 25) = -25$

There are no values less than 25 or more than 150, so we have no outliers.

4. **(a)** (Using GDC)
Mean: 71.5, Standard deviation: 7.04

(b) To calculate the 5 points summary, we first order the data
56, 60, 67, 68, 68, 68, 71, 71, 72, 76, 76, 79, 80, 80, 80
Min = 56, Q_1 = 68, Q_2 = 71, Q_3 = 79, Max = 80

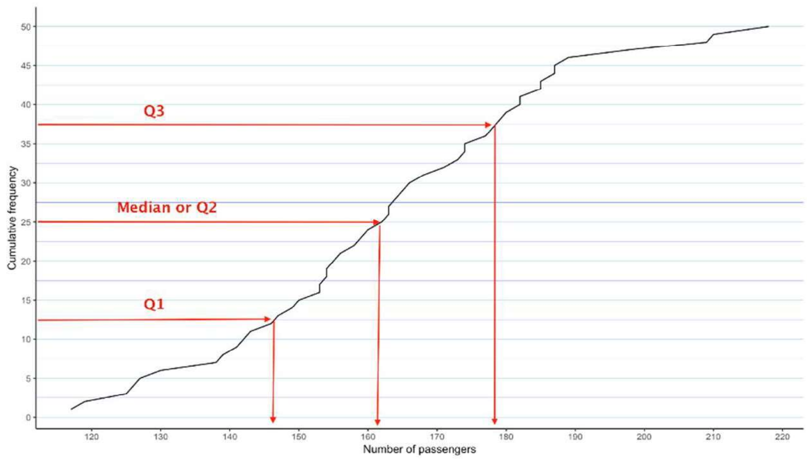


(c) To calculate outliers, we find the fences.
Upper fence: $Q_3 + 1.5 \cdot \text{IQR} = 79 + 1.5 \cdot (79 - 68) = 95.5$

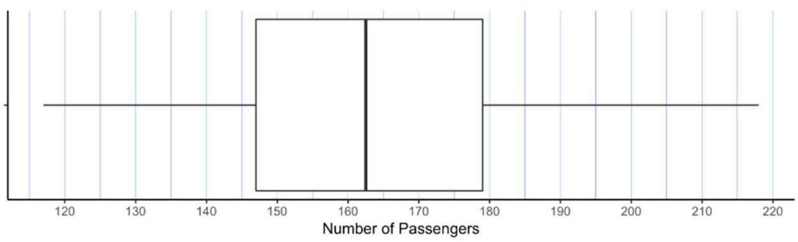
Lower fence: $Q_1 - 1.5 \cdot \text{IQR} = 68 - 1.5 \cdot (79 - 68) = 51.5$

There are no values less than 51.5 or more than 95.5, so there are no outliers.

5. **(a)** (Using GDC)
Mean: 163 (3 s.f.). Standard Deviation: 23.1



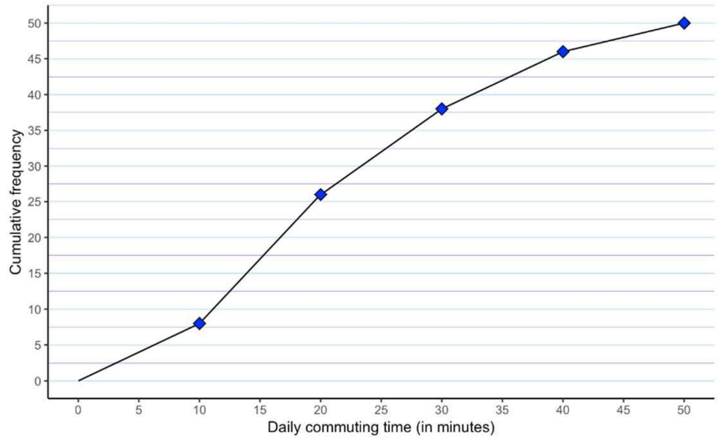
(b) Based on the cumulative frequency graph, we estimate the median at 161, the first quartile at 148, the third quartile at 178.
To draw a boxplot, we order the data first.
Min = 117, Q_1 = 147, Q_2 = 162.5, Q_3 = 179, Max = 218



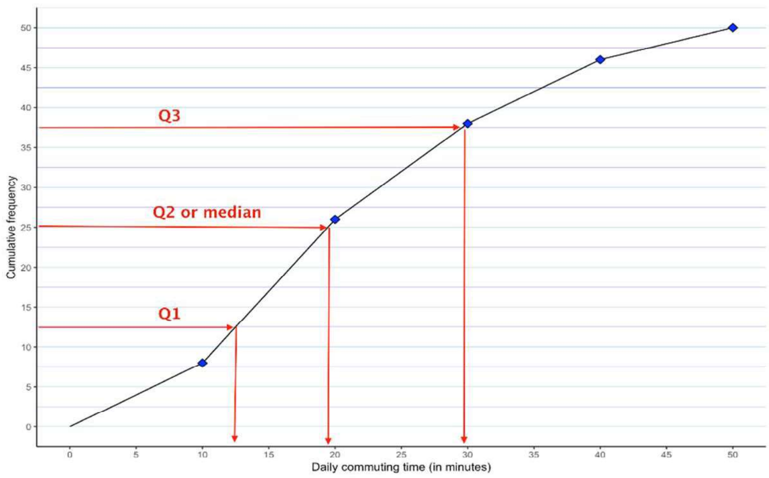
(c) (Using GDC).
$\text{IQR} = Q_3 - Q_1 = 179 - 147 = 32$
To calculate the outliers, we find the fences.
Upper fence: $Q_3 + 1.5 \cdot \text{IQR} = 179 + 1.5 \cdot (179 - 147) = 227$
Lower fence: $Q_1 - 1.5 \cdot \text{IQR} = 147 - 1.5 \cdot (179 - 147) = 99$
There are no values less than 99 or more than 227, so there are no outliers.

6. **(a)** The cumulative frequency table

Daily commuting time (in minutes)	Number of employees	Cumulative Number of employees
$0 \le t < 10$	8	8
$10 \le t < 20$	18	26
$20 \le t < 30$	12	38
$30 \le t < 40$	8	46
$40 \le t < 50$	4	50



(b) Based on the graph, we estimate $Q_1 \approx 12$, median $(Q_2) \approx 19$, $Q_3 \approx 29$



(c) (Using GDC)
Remember to enter the middle value for each daily commuting time class.
Mean = 21.4, standard deviation = 11.6 (3 s.f.)

7. **(a)** The average telephone bill is calculated using the middle value of each amount class.

Amount of telephone bills (in €) – middle value	Number of families
55	9
85	11
115	16
145	10
175	4

$$\text{Average bill} = \frac{55\cdot 9+85\cdot 11+115\cdot 16+145\cdot 10+175\cdot 4}{50} = €108$$

(b) Standard deviation (using GDC) = 35.6

(c) The mean of just the phone calls will be €24 less expensive, the standard deviation will stay the same (this make sense as we are not changing the spread of the data by subtracting an amount).
Mean of phone calls: $108.4 - 24 = 84.4$
Standard deviation: 35.6

8. **(a)** Using GDC
Mean: \$106,500. Median: \$76,000. Standard deviation: \$93,200
IQR: $Q_3 - Q_1 = \$97,000 - \$60,000 = \$37,000$

(b) To detect outliers,
Lower fence: $Q_1 - 1.5 \cdot IQR = 60 - 1.5 \cdot 37 = 4.5$
Upper fence: $Q_3 + 1.5 \cdot \text{IQR} = 97 + 1.5 \cdot 37 = 152.5$.
392 is an outlier as it is bigger than our upper fence.
Recalculate statistics without outlier (all values in thousand \$).
Mean: \$75,900. Median: \$74,000. Standard deviation: \$16,600.
IQR: $\$90,500 - \$59,500 = \$31,000$
The standard deviation has changed by a greater amount. This makes sense as the standard deviation is squaring the values from the mean.

(c) The set without the outlier provide a better summary of the data.

9. **(a)** There are 9 students in the class.
Solve for x. $\dfrac{5+\ 5+\ 4+\ 6+\ 3+\ 7+7+3+x}{9} = 5$
$40 + x = 45$. Hence $x = 5$

(b) Median = 5.
Standard deviation = 1.51 (using GDC)

(c) The section has then 10 students. For the new average to be 6, solve for x.
$$\frac{5+\ 5+\ 4+\ 6+\ 3+\ 7+7+3+5+x}{10} = 6$$
$45 + x = 60$. Hence $x = 15$

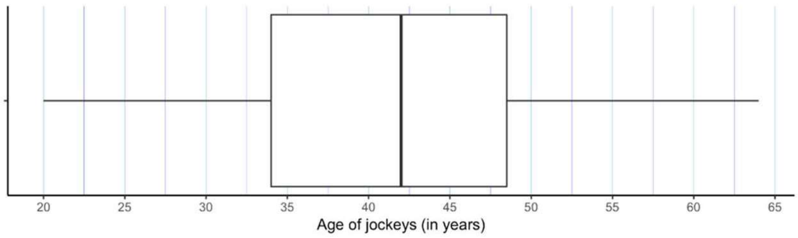
(d) There are now 22 people. The 10 from the first section and the 12 from the second.
$$\frac{50 + 12 * 4.5}{22} = \frac{104}{22} = 4.72 \ (3 \text{ s.f.})$$

10. **(a)** As a detailed summary of the data, we could look for the mean and standard deviation as well as for the 5 points summary. (Using GDC)
(Using GDC). Mean: 31.4 minutes. Standard deviation: 9.35 minutes.
(Manually, all answers in minutes).
Min = 20, Q_1 = 26, Median = 28.5, Q_3 = 32, Max = 56, IQR = 6.

(b) Almost 75% of the customers spent less than the mean time. It is a few customers who are spending greater amount of time. This can also be seen as the mean is quite greater than the median, there are values that skew the data set to the right.

11. **(a)** (using GDC)
Min = 20, $Q_1 = 34$, Median = 42, $Q_3 = 48.5$, Max = 64, IQR = 14.5.

(b)

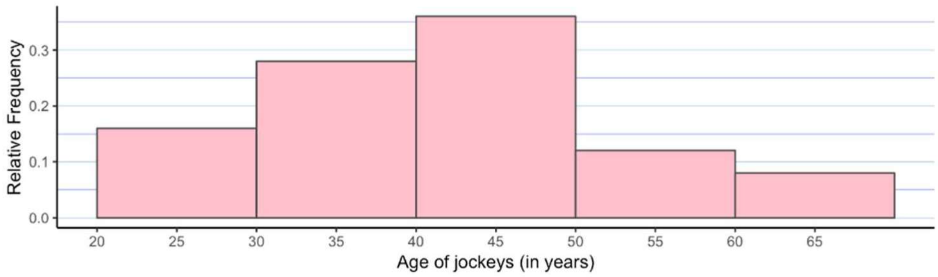


The data are symmetrical around the median. No skewness observed.

(c)

Class	freq.	rel freq
$20 \le x < 30$	4	4
$30 \le x < 40$	7	11
$40 \le x < 50$	9	20
$50 \le x < 60$	3	23
$60 \le x < 70$	2	25

(d)



The most frequent interval is 40–50 and the majority of the jockeys are under 50.

Chapter 10 practice questions

1. **(a)** To find the mean temperature, we need to solve the following equation for T

$$\frac{21+12+15+12+24+T+30+24}{8}=21$$

$138+T=8\cdot 21$ or $T=168-138=30\,°\text{C}$

(b) Mode = 12, 24 and 30°C (as they each appear twice). This is a multi-modal distribution.

(c) Median = 22.5°C

2. **(a)** (using GDC)
(i) Mean = 13.7 **(ii)** standard deviation = 2.52

(b) median = 13.1

3. **2, b, 3, a, 6, 9, 10, 12**

(a) Median (middle value) is 5. So $\frac{a+6}{2}=5$. Hence, $a=4$

To find the mean, solve for *b*, replacing *a* by 4.

$$\frac{2+b+3+4+6+9+10+12}{8}=6$$

$46+b=48$ or $b=2$

4. **(a)** There are 50 children, hence the sum of the frequency should be 50 as well.
Solving for *x*. 4+3+8 +x+ 4+ 1=50 Hence $x=30$

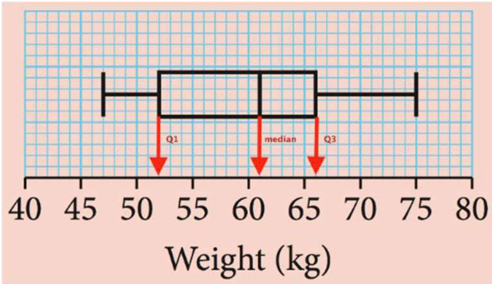
(b) (using GDC)
(i) mean = 2.6
(ii) median = 3
(iii) standard deviation = 1.06

(c) 2 should be added to each number of filling, so the same should happen to the mean and median. This does not create a wider spread; hence the standard deviation stays the same.
(i) mean = 4.6
(ii) median = 5
(iii) standard deviation = 1.06

5. (Using GDC)
(a) mean = 1.75

(b) median = 2

6.



(a) Median or $Q_2 = 61$ kg

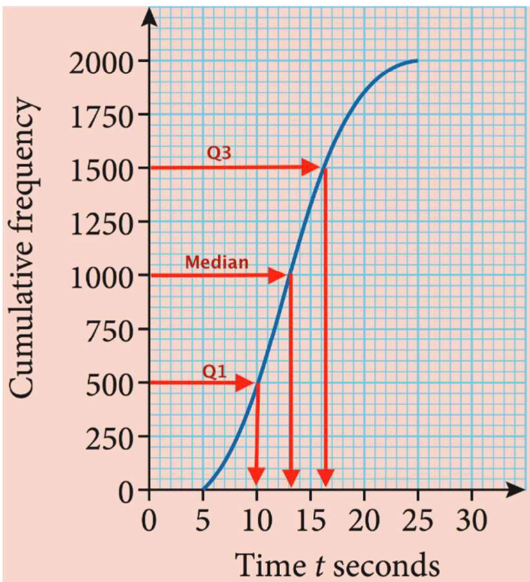
(b) IQR $= Q_3 - Q_1 = 66 - 52 = 14$ kg

(c) Between 51 and 66 kg, this is between Q_2 and Q_3 or 25%.
25% of 80 = 20 males.

(d) The lightest 40 males are the ones below the median or Q_1.. To calculate the mean mass of adults based on the box-and-whisker plot, we assume the people are equally spread within their respective quartiles. 20 people are equally spread between 47 and 52 kg (within Min and Q_1) and 20 people are equally spread between 52 and 61 kg (within Q_1 and Q_1). Hence

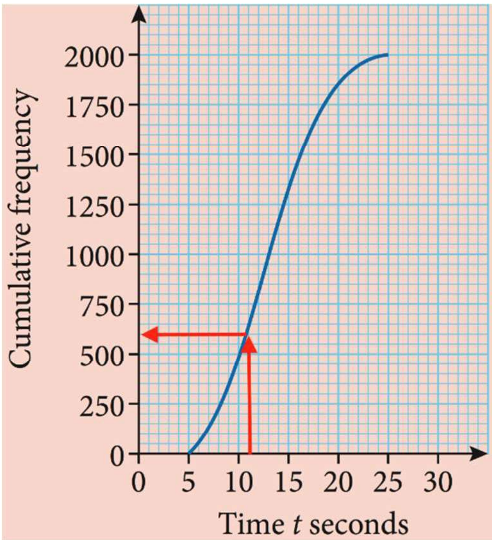
$$\text{mean} = \frac{20 \cdot \frac{47+52}{2} + 20 \cdot \frac{52+61}{2}}{40} = 53\,\text{kg}.$$

7. **(a)** With 2000 men, the median will be at the 1000th person, the Q_1 will be at the 500th and Q_3 will be that the 1500th person. See diagram below:



(i) Median time = 13 s
(ii) Upper quartile (Q3) = 16 s. and lower quartile (Q1) = 10 s.
(iii) IQR = 6 s.

(b) We can see that 600 men take up to 11 seconds to carry that task. So 1400 men (2000 – 600) takes more than 11 seconds. On the cumulative frequency axis, each square is equivalent to 50 men.



(c) 55% of 2000 men = 1100th men. On the cumulative frequency graph, we see that it corresponds to 14 s. Hence, 55% of men are taking less than 14 s. to complete the task.

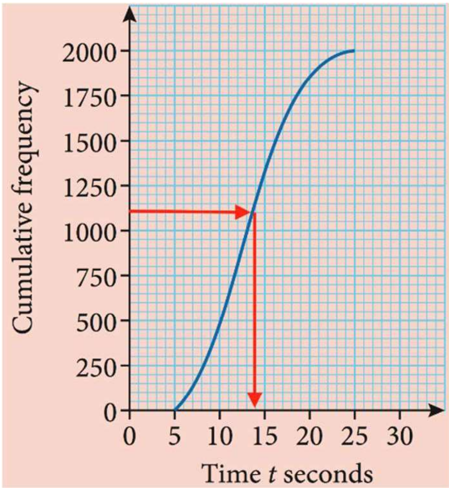
(d) a = 500. From 15 to 20 sec., we go from 1350 to 1850 men or a difference of 500 men.
b = 150. From 20 to 25 sec., we go from 1850 to 2000 men or a difference of 150 men.

(e) To find the mean, we will use the mid-interval value as we are assuming that number of men are equally spread within each class interval.

$$\mu = \frac{7.5 \cdot 500 + 12.5 \cdot 850 + 17.5 \cdot 500 + 22.5 \cdot 150}{2000} = 13.25\text{ s.}$$

$\sigma = 4.41$ s. (using GDC)

(f) We estimated the mean to be 13.25 s. and 1 standard deviation to be 4.41, so the most amount of time to get a bonus is $13.25 - 4.41 = 8.84$ s. Hence, Pedro's time of 9.5 s. does not warrant a bonus.



8. **(a)** The modal group is 170 to 180 cm.

(b) Using a GDC with the mid-value of each class interval for the values and the frequencies.

$$\mu = \frac{145 \cdot 2 + 155 \cdot 28 + 165 \cdot 63 + 175 \cdot 74 + 185 \cdot 20 + 195 \cdot 11 + 205 \cdot 2}{200} = 171.5 \text{ cm.}$$

$\sigma = 11.1$ cm.

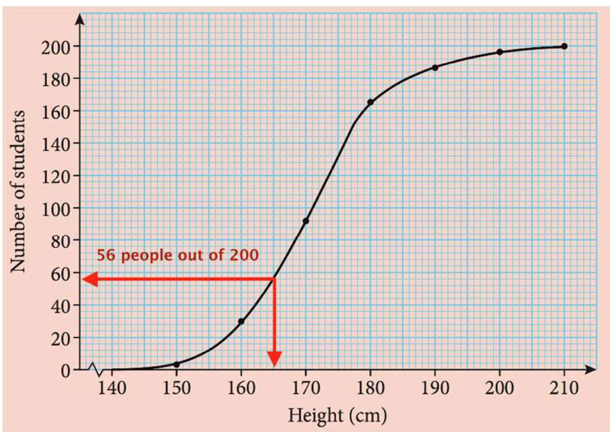
(c) With 200 students, the median height is the one of the 100th student. Based on the graph, we notice that it is 171 cm.

(d) To find the IQR, we need both Q_3 (given as 177.3 cm) and Q_1. To find Q_1, we check at the $\frac{1}{4}$ of 200 or the 50th student's height which is 164 cm. Hence,

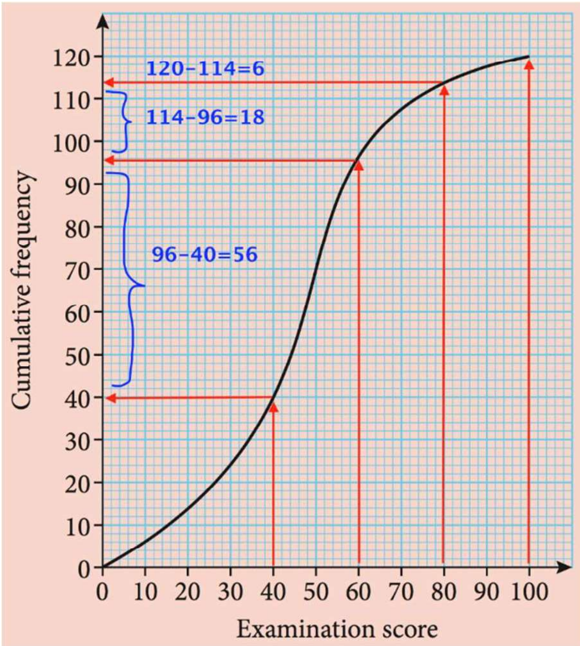
$\text{IQR} = Q_3 - Q_1 = 177.3 - 164 = 13.3$ cm.

(e) Based on the graph, we notice that 56 out of 200 students measure less than 165 cm.

$\frac{56}{200} = 0.28$ or 28%.



9. **(a)** Based on the graph.



Examination score (in %)	$40 \le x < 60$	$60 \le x < 80$	$80 \le x < 100$
Frequency	56	18	6

(b) To find the mid-interval value: $\frac{40+60}{2} = 50$

(c) To find an estimate of the mean, we multiply the mid-interval values by its frequency.

$$\frac{10 \cdot 14 + 30 \cdot 26 + 50 \cdot 56 + 70 \cdot 18 + 90 \cdot 6}{120} = 46$$

□

Exercise 11.1

1. The total number of outcomes of the three throws is $2^3 = 8$.

The outcomes corresponding to at most one head are (*TTT, HTT, THT, TTH*),

so the probability is $\frac{4}{8} = \frac{1}{2}$

2. We can use a table of outcomes for this problem:

		Dice 1					
		1	2	3	4	5	6
Dice 2	1	0	1	2	3	4	5
	2	1	0	1	2	3	4
	3	2	1	0	1	2	3
	4	3	2	1	0	1	2
	5	4	3	2	1	0	1
	6	5	4	3	2	1	0

(a) **(i)** $\frac{10}{36} = \frac{5}{18}$

(ii) $\frac{6}{36} = \frac{1}{6}$

(b) $\frac{5}{18} \cdot 500 \approx 139$

3. **(a)** The multiples of 5 from 1 to 20 are four, so $\frac{4}{20} = \frac{1}{5}$

(b) The prime numbers in the bag are: 2, 3, 5, 7, 11, 13, 17, 19,

so the probability is $\frac{8}{20} = \frac{2}{5}$

(c) The factors of 60 are: 1, 2, 3, 4, 5, 6, 10, 12, 15, 20,

so the probability is $\frac{10}{20} = \frac{1}{2}$

4. Using a table of outcomes:

		Dice 1					
		1	2	3	4	5	6
Dice 2	1	1	2	3	4	5	6
	2	2	4	6	8	10	12
	3	3	6	9	12	15	18
	4	4	8	12	16	20	24
	5	5	10	15	20	25	30
	6	6	12	18	24	30	36

(a) **(i)** $\frac{6}{36} = \frac{1}{6}$

(ii) This corresponds to the main diagonal in the table, so $\frac{6}{36} = \frac{1}{6}$

(iii) The triangular numbers are 1, 3, 6, 10, 15, 21, 28, 36.

There are 12 of these in the table

so the probability is $\frac{12}{36} = \frac{1}{3}$

(iv) $\frac{4}{36} = \frac{1}{9}$

(v) There is only one case where the product is 36,

so, $\text{P(product} < 36) = 1 - \frac{1}{36} = \frac{35}{36}$

(b) $\frac{1}{6} \cdot 200 \cong 33$

5. **(a)** If we call k the probability of rolling a number from 1 to 5, the probability of rolling a 6 is $3k$.

Since the total probability is 1, we have

$k + k + k + k + k + 3k = 1 \Rightarrow 8k = 1 \Rightarrow k = \frac{1}{8}$

The probability of rolling a 6 is then $\frac{3}{8}$

(b) Each of the number 1, 3, 5 have a probability of $\frac{1}{8}$ so the total is $\frac{3}{8}$

6. $P(A') = 1 - P(A) = 1 - 0.27 = 0.73$

7. **(a)** There are 13 clubs so $\frac{13}{52} = \frac{1}{4}$

(b) $\frac{3 \times 4}{52} = \frac{3}{13}$

(c) $\frac{52-1}{52} = \frac{51}{52}$

8. The total number of students is $4+12+8+3+2+1=30$

(a) $\frac{4+12+8}{30} = \frac{24}{30} = \frac{4}{5}$

(b) $\frac{8+3}{30} = \frac{11}{30}$

(c) All the students in the survey studied for less than 6 hours, so the probability is 1.

9. **(a)** **(i)** $\frac{1221}{4318}$

(ii) $\frac{611+262+87}{4318} = \frac{960}{4318} = \frac{480}{2159}$

(b) **(i)** $\frac{1221+1439}{4318} \cdot 27.2 = 16.8$ millions

(ii) $\frac{698+611}{4318} \cdot 27.2 = 8.25$ millions

10. **(a)** **(i)** $\frac{3}{60} = \frac{1}{20}$

(ii) $\frac{18+15+14}{60} = \frac{47}{60}$

(b) $\frac{14}{60} \cdot 50 \cong 12$

11. **(a)** For drug A the relative frequency of improvement is $\frac{75}{75+12} = 0.862$

for drug B is $\frac{62}{62+7} = 0.899$

(b) Drug B as the improvement is higher

(c) $0.862 \cdot 200 \cong 172$

12. **(a)** $\frac{3633}{7322} = 0.496$

(b) $\frac{611}{7322} = 0.0834$

(c) $\frac{636}{7322} = 0.0869$

13. The total number of people is 440251

(a) $\frac{39943 + 62782}{440251} = 0.233$

(b) $\frac{261634}{440251} = 0.594$

14. **(a)**

Region	**Relative frequency**
Bangkok	$\frac{7474}{10801} = 0.692$
Central	$\frac{12840}{24227} = 0.530$
North	$\frac{5958}{14391} = 0.414$
Northeast	$\frac{8367}{23243} = 0.360$
South	$\frac{5194}{11218} = 0.463$
Whole kingdom	$\frac{39833}{83880} = 0.475$

(b) Possible answers include: better infrastructure, easier access to internet, more people who can afford it.

(c) $\frac{39833}{83880} \cdot 62.8 = 29.8$ millions

15. **(a)** $0.002 \cdot 10000 + 0.0015 \cdot 3000 + 0.001 \cdot 2500 + 0.0004 \cdot 100000 + 0.003 \cdot 4000 =$ \$79

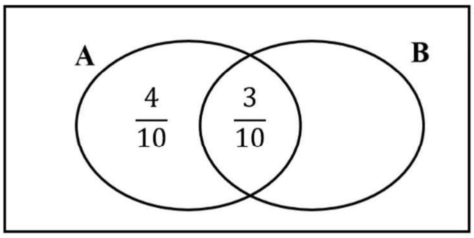
(b) $79 \cdot 1.6 =$ \$126.4

Exercise 11.2

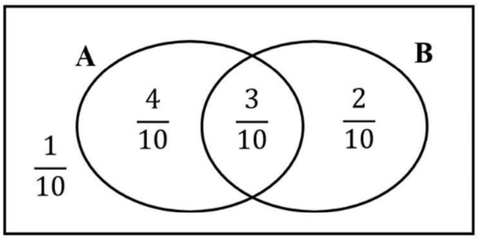
1. Rearranging $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ to make P *(B)* the subject, we get $P(B) = P(A \cup B) - P(A) + P(A \cap B)$. Thus

$$P(B) = \frac{4}{5} - \frac{3}{4} + \frac{3}{10} = \frac{16 - 15 + 6}{20} = \frac{7}{20} = 0.35$$

2. For this problem it's better to use a Venn diagram. With the information given we already know two of the probabilities in the diagram.



We also know that $P\left((A \cup B)'\right) = 1 - P(A \cup B) = \frac{1}{10}$ and then from the fact that the sum of the probabilities of the four regions is 1, we can complete the diagram.



From the diagram:

(a) $P(B) = \frac{3}{10} + \frac{2}{10} = 0.5$

(b) $\frac{2}{10} = 0.2$

(c) $\frac{7}{10} + \frac{1}{10} = 0.8$

(d) $\frac{1}{10} = 0.1$

(e) $\frac{2}{10} + \frac{1}{10} + \frac{4}{10} = 0.7$

3. Rearranging $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ one gets

$\mathrm{P}(A \cap B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cup B)$

In this case $\mathrm{P}(A \cap B) = \frac{1}{3} + \frac{2}{9} - \frac{4}{9} = \frac{1}{9}$

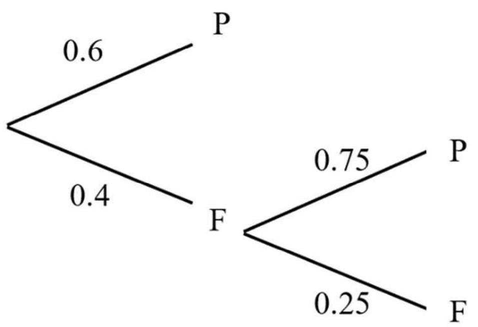
Since $\mathrm{P}(A \cap B) \neq 0$ the events are not mutually exclusive.

Then $\mathrm{P}(A) \cdot \mathrm{P}(B) = \frac{1}{3} \cdot \frac{2}{9} = \frac{2}{27} \neq \mathrm{P}(A \cap B)$ and therefore they are not independent.

4. Since the two events are independent, we have $\mathrm{P}(B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} = \frac{\frac{3}{10}}{\frac{3}{7}} = \frac{7}{10}$

Thus $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = \frac{3}{7} + \frac{7}{10} - \frac{3}{10} = \frac{58}{70} = \frac{29}{35}$

5. Using a tree diagram for pass (P) or fail (F)



The probability that the driver does not need to retrain is $0.6 + 0.4 \cdot 0.75 = 0.9$

6. **(a)** $1 - 0.08 = 0.92$

(b) **(i)** $\mathrm{P}(\text{both O-negative}) = 0.08 \cdot 0.08 = 0.0064$

(ii) $\mathrm{P}(\text{at least 1 O-negative}) = 1 - \mathrm{P}(\text{none O-negative}) = 1 - 0.92^2 = 0.1536$

(iii) $\mathrm{P}(\text{only 1 O-negative}) = 0.08 \cdot 0.92 + 0.92 \cdot 0.08 = 0.1472$

(c) $\mathrm{P}(\text{at least 1 O-negative}) = 1 - \mathrm{P}(\text{none O-negative}) = 1 - 0.92^2 = 0.487$

7. **(a)** There are 10 choices for each of the four digits, so $10^4 = 10000$

(b) In this case we have 9 choices for the first and 10 for the other three so the probability is $\mathrm{P} = \frac{9 \cdot 10^3}{10^4} = 0.9$

(c) $\mathrm{P}(\text{at least 1 zero}) = 1 - \mathrm{P}(\text{no zeros}) = 1 - \frac{9 \cdot 9 \cdot 9 \cdot 9}{10000} = 0.3439$

8. $\text{P}(\text{exactly 1 red}) = \text{P}(\text{R,B}) + \text{P}(B, R) = \frac{6}{8}\cdot\frac{2}{8} + \frac{2}{8}\cdot\frac{6}{8} = \frac{3}{8}$

9. **(a)** $\text{P(at least one 6)} = 1 - \text{P(no 6s)} = 1 - \frac{5}{6}\cdot\frac{5}{6} = \frac{11}{36}$

(b) We can list the combinations that give a sum above 10.

They are three: $(6,5),(5,6),(6,6)$

Since there are $6\cdot 6 = 36$ combinations in total

$\text{P}(\text{sum is at most 10}) = \frac{36-3}{36} = \frac{33}{36} = \frac{11}{12}$

(c) The combinations with at least one of the dice showing a 4 are 11.

These already contain two combinations with a sum of 10: $(6,4)$ and $(4,6)$.

The only missing one is $(5,5)$.

Therefore the probability is $\frac{12}{36} = \frac{1}{3}$

10. **(a)** **(i)** $\frac{700}{800+700} = \frac{700}{1500} = \frac{7}{15}$

(ii) $\frac{220}{1500} = \frac{11}{75}$

(iii) $\frac{220+180}{1500} = \frac{400}{1500} = \frac{4}{15}$

(iv) $\frac{220+180+200+130+190}{1500} = \frac{920}{1500} = \frac{46}{75}$

(b) $\frac{\text{number of male grade 12}}{\text{number of grade 12}} = \frac{220}{400} = \frac{11}{20}$

(c) No they are not as the ratio male:female is different for different years, sometimes significantly, as you can see by comparing grade 9 and grade 10.

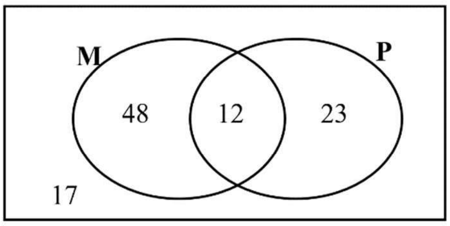
11. **(a)** **(i)** $0.41 + 0.15 = 0.56$

(ii) 0.15

(b) $\frac{0.15}{0.41+0.15} = \frac{0.15}{0.56} = 0.268$

(c) They are not independent, as the proportion of students using glasses among those who need them is much higher than among those who don't need them, as one would expect.

12. Using the information to fill a Venn diagram, starting from the intersection, we get



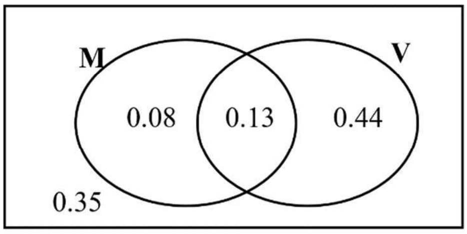
Where the 17 comes from $100-(48+12+23)=17$

(a) $\mathrm{P}\left(M'\cap P'\right)=\dfrac{17}{100}$

(b) $\mathrm{P}\left(M\cap P\right)=\dfrac{12}{100}$; $\mathrm{P}\left(M\right)\cdot\mathrm{P}\left(P\right)=\dfrac{60}{100}\cdot\dfrac{35}{100}=\dfrac{21}{100}\neq\mathrm{P}\left(M\cap P\right)$

Therefore the events are not independent

13. Displaying the information in a Venn diagram,



(a) $\mathrm{P}\left(M\cup V\right)=0.08+0.13+0.44=0.65$

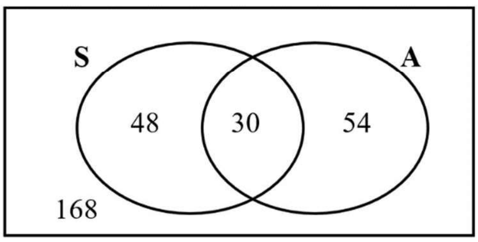
(b) $\mathrm{P}\left(M'\cap V'\right)=0.35$

(c) $\mathrm{P}\left(\text{exactly one card}\right)=0.08+0.44=0.52$

14. From the data we can calculate:

$n(S'\cap A')=300-132=168$ and
$n(S\cap A)=n(S)+n(A)-n(S\cup A)=78+84-132=30$

and use these to complete a Venn diagram:



(a) $\mathrm{P}(S'\cap A')=\dfrac{168}{300}=\dfrac{14}{25}$

(b) $\mathrm{P}(S\cap A)=\dfrac{30}{300}=\dfrac{1}{10}$

15. **(a)** $\text{P}(\text{three 2s}) = \left(\frac{1}{6}\right)^3 = \frac{1}{216}$

(b) $\text{P}(\text{at least one 2}) = 1 - \text{P}(\text{no 2s}) = 1 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$

(c) $\text{P}(\text{exactly one 2}) = \text{P}(2, \text{not 2, not 2}) + \text{P}(\text{not 2}, 2, \text{not 2}) + \text{P}(\text{not 2, not 2}, 2)$

$$= \frac{1}{6}\cdot\frac{5}{6}\cdot\frac{5}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} + \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6} = \frac{75}{216} = \frac{25}{72}$$

16. H = hit, M = miss

(a) $\text{P}(H, M) = 0.7 \cdot 0.3 = 0.21$

(b) $\text{P}(\text{exactly once}) = \text{P}(H, M, M) + \text{P}(M, H, M) + \text{P}(M, M, H)$

$$= 0.7 \cdot 0.3 \cdot 0.3 + 0.3 \cdot 0.7 \cdot 0.3 + 0.3 \cdot 0.3 \cdot 0.7 = 0.441$$

(c) $\text{P}(\text{at least once}) = 1 - \text{P}(\text{none hitting the centre}) = 1 - 0.7 \cdot 0.7 \cdot 0.7 = 0.657$

17. There are in total $12 \cdot 12 = 144$ possible outcomes.

(a) There are 23 combinations where at least one of the dice scores 12, so the probability is $\text{P}(\text{at lease one 12}) = \frac{23}{144}$

(b) The combinations that give a sum of 12 are

$(1,11), (2,10), (3,9), (4,8), (5,7), (6,6), (7,5), (8,4), (9,3), (10,2)$ and $(11,1)$

and so $\text{P}(\text{sum} = 12) = \frac{11}{144}$

(c) There is one way to get a sum of 24 (12 and 12), two ways of getting a sum of 23, three ways for a sum of 22, four ways for 21 and five ways for 20.

$\text{P}(\text{sum is at least 20}) = \frac{1+2+3+4+5}{144} = \frac{15}{144} = \frac{5}{48}$

(d) **(i)** The combinations that are in $A \cap B$ are

$(1,10), (2,10), (3,10), (4,10), (5,10), (10,5), (10,4), (10,3), (10,2)$ and $(10,1)$

Hence, $\text{P}(A \cap B) = \frac{10}{144} = \frac{5}{72}$

(ii) To count the element in B we can consider that the number of way to score a sum of *2* is 1, the number of ways to score a 3 is 2, … the number of ways to score a 13 is 12, the number of ways to score a 14 is 11 and the number of ways to score a 15 is 10, so $n(B) = 1+2+...+12+11+10 = 99$. From these, we are missing the pairs of scores where at lease on dice contain a 10 but the total is above 15.

These are 13:

$(6,10),(7,10),(8,10),(9,10),(10,10),(11,10),(12,10),$
$(10,12),(10,11),(10,9),(10,8),(10,7)$ and $(10,6)$

Therefore $\mathrm{P}(A\cup B) = \frac{n(A\cup B)}{144} = \frac{99+13}{144} = \frac{112}{144} = \frac{7}{9}$

(iii) $\mathrm{P}\left((A\cap B)'\right) = 1-\mathrm{P}(A\cap B) = 1-\frac{5}{72} = \frac{67}{72}$

(iv) $\mathrm{P}(A'\cap B') = \mathrm{P}\left((A\cup B)'\right) = 1-\mathrm{P}(A\cup B) = 1-\frac{7}{9} = \frac{2}{9}$

18. The combinations of scores that give a sum of 6 are 5: $(1,5),(2,4),(3,4),(4,2),(5,1)$.

At each roll then the probability of winning is $\frac{5}{36}$

(a) $\mathrm{P}(\text{K wins on 2nd roll}) = \mathrm{P}(\text{K loses})\cdot\mathrm{P}(\text{G loses})\cdot\mathrm{P}(\text{K wins})$

$$= \frac{31}{36}\cdot\frac{31}{36}\cdot\frac{5}{36} = \frac{4805}{46656}$$

(b) $\mathrm{P}(\text{G wins on 2nd roll}) = \mathrm{P}(\text{K loses})\cdot\mathrm{P}(\text{G loses})\cdot\mathrm{P}(\text{K loses})\cdot\mathrm{P}(\text{G wins})$

$$= \frac{31}{36}\cdot\frac{31}{36}\cdot\frac{31}{36}\cdot\frac{5}{36} = \frac{148955}{1679616}$$

(c) $\mathrm{P}(\text{K wins}) = \frac{5}{36}+\left(\frac{31}{36}\right)^2\cdot\frac{5}{36}+\left(\frac{31}{36}\right)^4\cdot\frac{5}{36}+\left(\frac{31}{36}\right)^6\cdot\frac{5}{36}+...$

$$= \frac{5}{36}\left(1+\left(\frac{31}{36}\right)^2+\left(\left(\frac{31}{36}\right)^2\right)^2+\left(\left(\frac{31}{36}\right)^2\right)^3+\left(\left(\frac{31}{36}\right)^2\right)^4+...\right)$$

This corresponds to the sum to infinity of a geometric sequence with

$u_1 = \frac{5}{36}$ and $r = \left(\frac{31}{36}\right)^2$. Therefore $P(\text{K wins}) = \frac{\frac{5}{36}}{1-\left(\frac{31}{36}\right)^2} = \frac{36}{67}$

19. Total number of people in the survey = 51 + 16 + 9 + 9 = 85

(a) **(i)** $\mathrm{P}(\text{both bedroom}) = 0.51^2 = 0.2601$

(ii) $\mathrm{P}(\text{at least one}) = 1 - \mathrm{P}(\text{none}) = 1 - 0.49^2 = 0.7599$

(b) $0.09 \cdot 66.2 = 5.958$ millions

20. **(a)**

Driving Centre	Pass rate
Singapore Safety Driving Centre	$\frac{1947}{3678} = 0.529$
Bukit Batok Driving Centre	$\frac{3846}{6418} = 0.599$
Comfort Driving Centre	$\frac{3931}{7345} = 0.535$

From the table we can see that the centre with the highest pass rate is Bukit Batokn Driving Centre.

(b) **(i)** $\mathrm{P}(\text{both passing}) = 0.529^2 = 0.280$

(ii) $\mathrm{P}(\text{none passing}) = \frac{3678 - 1947}{3678} = \frac{1731}{3678}$

$$\mathrm{P}(\text{at least one passing}) = 1 - \mathrm{P}(\text{none passing}) = 1 - \left(\frac{1731}{3678}\right)^2 = 0.779$$

(c) **(i)** $\mathrm{P}(\text{both passing}) = \frac{3846}{6418} \cdot \frac{3931}{7345} = 0.321$

(ii) $$\mathrm{P}(\text{exactly one passing}) = \frac{3846}{6418} \cdot \frac{7345 - 3931}{7345} + \frac{6418 - 3846}{6418} \cdot \frac{3931}{7345}$$
$$= 0.493$$

(d) Overall pass rate = $\frac{1947 + 3846 + 3931}{3678 + 6418 + 7345} = \frac{9742}{17441}$

Expected number of pass = $\frac{9742}{17441} \cdot 18200 = 10147$

21. **(a)** **(i)** Since black is dominant only *bb* will have brown fur so the probability is $\frac{1}{4}$

(ii) $\frac{2}{4}=\frac{1}{2}$

(b) $\frac{3}{4}\cdot 20=15$

(c) The possible genes of their kittens now are BB, BB, Bb, Bb

(i) no bb so the probability is 0

(ii) $\frac{2}{4}=\frac{1}{2}$

(d) The possible genes for fur thickness of the kittens are TT, Tt, Tt, tt.

(i) Since fur and thickness are independent

$\mathrm{P}(\text{thick brown})=\mathrm{P}(\text{thick})\cdot\mathrm{P}(\text{brown})=\frac{3}{4}\cdot\frac{1}{4}=\frac{3}{16}$

(ii) $\mathrm{P}(\text{thin black})=\mathrm{P}(\text{thin})\cdot\mathrm{P}(\text{black})=\frac{1}{4}\cdot\frac{3}{4}=\frac{3}{16}$

(iii) $\mathrm{P}(\text{thick black})=\mathrm{P}(\text{thick})\cdot\mathrm{P}(\text{black})=\frac{3}{4}\cdot\frac{3}{4}=\frac{9}{16}$

(e) Expected number with thin brown $=\mathrm{P}(\text{thin brown})\cdot 32=\frac{1}{4}\cdot\frac{1}{4}\cdot 32=2$

(f) If 62 rabbits have black fur, then $80-62=18$ have brown fur. If 55 have thick fur, then $80-55=25$ have thin fur. The rabbits with thick brown fur are $18-4=14$. The rabbits with thin black fur are $25-4=21$. Finally, the number of rabbits with thick black fur are $62-21=41$.

With these numbers:

(i) $\mathrm{P}(\text{thick black})=\frac{41}{80}$

(ii) $\mathrm{P}(\text{thin black})=\frac{21}{80}$

Exercise 11.3

1. Total number of males = $42+23+57=122$

Total number of females = $33+46+61=140$

Total number of visitors = 122 +140 = 262

(a) $\frac{46}{262}=\frac{23}{131}$

(b) $\frac{46}{46+23}=\frac{46}{69}=\frac{2}{3}$

(c) $\frac{46}{140}=\frac{23}{70}$

(d) $\frac{42+33}{262}=\frac{75}{262}$

(e) $\frac{42}{122}=\frac{21}{61}$

2. The total number of tests is 96

(a) $\frac{12+15}{96}=\frac{27}{96}=\frac{9}{32}$

(b) $\frac{2}{29+15+2}=\frac{2}{46}=\frac{1}{23}$

(c) $\frac{34}{34+29}=\frac{34}{63}$

(d) $\frac{34+12}{34+12+4}\cdot 100=\frac{46}{50}\cdot 100=92$

3. **(a)** 29% of $105=30.5$ millions

(b) $\frac{18+40+29}{300}=\frac{87}{300}=\frac{29}{100}$

(c) 0.27

(d) $\frac{70}{70+36+37}=\frac{70}{143}$

(e) $\frac{40+7+17}{300-(70+36+37)}=\frac{64}{157}$

4. **(a)** $\dfrac{2}{218+2}=\dfrac{2}{220}=\dfrac{1}{110}$

(b) $\dfrac{192}{408+192}=\dfrac{192}{600}=\dfrac{8}{25}$

(c) Since the proportion of incorrect responses in the Peer Pressure Group is much greater than the in the control group, the experiment indicates that group pressure has an impact.

5. The number of ways of scoring a sum of 10 is 3: $(6,4),(5,5),(4,6)$.

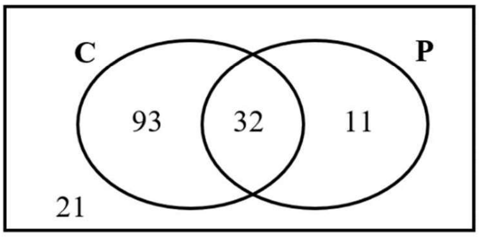
Therefore the probability is $\dfrac{1}{3}$

6. In order to complete a Venn diagram we need to find $n(C\cap P)$.

Since 21 ordered neither coffee nor pastry we have $n(C\cup P)=157-21=136$.

Then, $n(C\cap P)=n(C)+n(P)-n(C\cup P)=125+43-136=32$

The corresponding Venn diagram is



(a) $\mathrm{P}(C\cap P)=\dfrac{32}{157}$

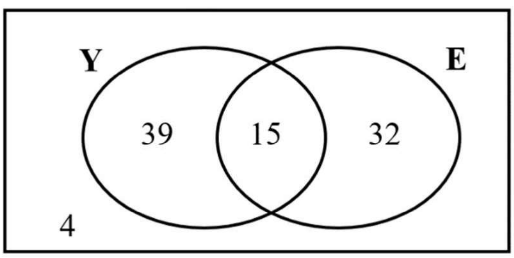
(b) $\mathrm{P}(P\mid C)=\dfrac{n(P\cap C)}{n(C)}=\dfrac{32}{125}$

(c) $\mathrm{P}(C\mid P')=\dfrac{n(C\cap P')}{n(P')}=\dfrac{93}{93+21}=\dfrac{93}{114}=\dfrac{31}{38}$

7. $n(Y\cup E)=n(Y)+n(E)-n(Y\cap E)=54+47-15=86$

$n\left((Y\cup E)'\right)=90-n((Y\cap E)=90-86=4$

The Venn diagram then is:



(a) $\mathrm{P}(E \mid Y)=\frac{n(E\cap Y)}{n(Y)}=\frac{15}{54}=\frac{5}{18}$

(b) $\mathrm{P}(Y \mid E')=\frac{n(Y\cap E')}{n(E')}=\frac{39}{39+15}=\frac{39}{43}$

(c) $\mathrm{P}(\text{at least one Y})=1-\mathrm{P}(\text{no Y})=1-\frac{36}{90}\cdot\frac{35}{89}\cdot\frac{34}{88}=\frac{1839}{1958}$

$$\mathrm{P}(\text{all Y} \mid \text{at least one Y})=\frac{\mathrm{P}(\text{all Y } \cap \text{ at least one Y})}{\mathrm{P}(\text{at least one Y})}=\frac{\frac{54}{90}\cdot\frac{53}{89}\cdot\frac{52}{88}}{\frac{1839}{1958}}=\frac{689}{3065}$$

8. **(a)** **(i)** 0.18

(ii) 0.9

(b) $(0.74\cdot 0.1+0.18\cdot 0.18+0.08\cdot 0.35)\cdot 12000=1613$

(c) $\mathrm{P}(\text{a customer has an accident this year})=(0.74\cdot 0.1+0.18\cdot 0.18+0.08\cdot 0.35)$
$=0.1344$

$\frac{0.08\cdot 0.35}{0.1344}=0.208$

9. **(a)** **(i)** The two events are independent so the probability is still $\frac{8}{20}=\frac{2}{5}$

(ii) $\mathrm{P}(\text{at least 1 red})=1-\mathrm{P}(\text{no reds})=1-\frac{12}{20}\cdot\frac{12}{20}=\frac{16}{25}$

(iii) $\frac{\frac{2}{5}}{\frac{16}{25}}=\frac{5}{8}$

(b) **(i)** $\frac{7}{19}$

Since there is no replacement the red ball left for Jessica are 7 out of 19

(ii) $\mathrm{P}(\text{at least 1 red})=1-\mathrm{P}(\text{no reds})=1-\frac{12}{20}\cdot\frac{11}{19}=\frac{62}{95}$

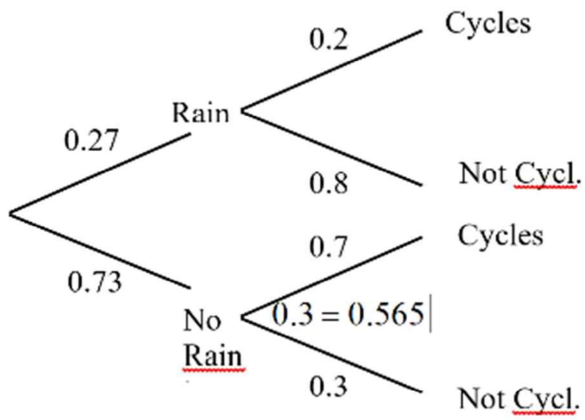
(iii) $\frac{\frac{8}{20}}{\frac{62}{95}}=\frac{19}{31}$

10. **(a)** $0.3\cdot 0.2+0.5\cdot 0.1+0.2\cdot 0.5=0.21$

(b) $\dfrac{\text{P}(\text{A}\cap\text{ he won a prize})}{\text{P}(\text{he won a prize})}=\dfrac{0.3\cdot 0.2}{0.21}=0.286$

11. Using a tree diagram,

(a)



$0.27\cdot 0.2+0.73\cdot 0.3=0.565$

(b) $\dfrac{0.27\cdot 0.2}{0.565}=0.0956$

12. **(a)** **(i)** $\text{P}(\text{positive for HIV})=\dfrac{1.1}{326}\cdot 0.99+\dfrac{324.9}{326}\cdot 0.01=0.0133$

(ii) $\text{P}(\text{negative for HIV})=1-\text{P}(\text{positive for HIV})=1-0.0133=0.987$

(iii) $\text{P}(\text{infected}\mid\text{negative test})=\dfrac{\text{P}(\text{infected AND negative test})}{\text{P}(\text{negative test})}$

$$=\frac{\frac{1.1}{326}\cdot 0.01}{0.987}$$

$$=3.42\times 10^{-5}$$

(iv) $\text{P}(\text{not infected}\mid\text{positive test})=\dfrac{\text{P}(\text{not infected AND positive test})}{\text{P}(\text{positive test})}$

$$=\frac{\frac{324.9}{326}\cdot 0.01}{0.0133}$$

$$=0.749$$

(b) The probability of a false negative is very low: it is unlikely that an infected person is not detected by the test.

(c) The probability of a false positive is very high; actually, $\frac{3}{4}$ of the people testing positive are not infected. This implies that centres delivering the test need to find strategies to improve the accuracy, like repeating the test more than once.

13. This problem is similar to **Q12.**

(a) **(i)** $\text{P}(\text{positive test}) = 0.07\cdot 0.98 + 0.93\cdot 0.02 = 0.0872$

(ii) $\text{P}(\text{negative test}) = 1-\text{P}(\text{positive test}) = 1-0.0872 = 0.9128$

(iii) $\text{P}(\text{false negative}) = \dfrac{0.07\cdot 0.02}{0.9128} = 0.00153$

(iv) $\text{P}(\text{false positive}) = \dfrac{0.93\cdot 0.02}{0.0872} = 0.213$

(b) The probability of a drunk driver undetected by the test is low (0.15%), meaning the test is good from this point of view.

(c) When a driver test positive, there is about 20% chance that their BAC is below 0.5. This is an issue as it means that with this test a significant proportion of punished drivers are innocent.

14. **(a)** $\text{P}(A\cap B) = \text{P}(A)\text{P}(B) = 0.6\cdot 0.7 = 0.42$

$\text{P}(A\cup B) = 0.6+0.7-0.42 = 0.88$

(b) $\text{P}(A\cap B) = \text{P}(B)\cdot\text{P}(A\,|\,B) = 0.7\cdot 0.8 = 0.56$

$\text{P}(A\cup B) = 0.6+0.7-0.56 = 0.74$

15. $\text{P}(B') = 1-\text{P}(B) = 1-0.55 = 0.45$.

$\text{P}(A\cap B) = \text{P}(B)\text{P}(A\,|\,B) = 0.55\cdot 0.7 = 0.385$

$\text{P}(A\cap B') = \text{P}(A)-\text{P}(A\cap B) = 0.4-0.385 = 0.015$

$\text{P}(A\,|\,B') = \dfrac{\text{P}(A\cap B')}{\text{P}(B')} = \dfrac{0.015}{0.45} = 0.0333$

16. **(a)** $\text{P}(A') = 1-\text{P}(A) = 1-0.3 = 0.7$

$\text{P}(B) = \text{P}(A)\cdot\text{P}(B\,|\,A)+\text{P}(A')\cdot\text{P}(B\,|\,A') = 0.3\cdot 0.5+0.7\cdot 0.7 = 0.64$

(b) $\text{P}(A\,|\,B) = \dfrac{\text{P}(A\cap B)}{\text{P}(B)} = \dfrac{0.15}{0.64} = 0.234$

17. **(a)** $\text{P}(A\cap B)=\text{P}(A)P(B);\quad \text{P}(B)=\text{P}(A\cup B)-\text{P}(A)+\text{P}(A\cap B)$

$\text{P}(B)=\text{P}(A\cup B)-\text{P}(A)+\text{P}(A)\text{P}(B)=$

$\text{P}(B)=0.8-0.6+0.6\text{P}(B)\Rightarrow 0.4\text{P}(B)=0.2\Rightarrow \text{P}(B)=\dfrac{0.2}{0.4}=0.5$

(b) $\text{P}(A\cap B)=\text{P}(A)\cdot\text{P}(B\,|\,A)=0.6\cdot 0.25=0.15$

$\text{P}(B)=\text{P}(A\cup B)-\text{P}(A)+\text{P}(A\cap B)=0.8-0.6+0.15=0.35$

(c) $\text{P}(A'\cap B')=\text{P}\left((A\cup B)'\right)=1-0.8=0.2$

$\text{P}(A'\,|\,B')=1-\text{P}(A\,|\,B')=0.8$

$\text{P}(B')=\dfrac{\text{P}(A'\cap B')}{\text{P}(A'\,|\,B')}=\dfrac{0.2}{0.8}=0.25$

$\text{P}(B)=1-\text{P}(B')=1-0.25=0.75$

Chapter 11 practice questions

1. **(a)** $\text{P}(A)\text{P}(B)=\text{P}(A\cap B)\Rightarrow k(k+0.5)=0.14$

$k^2+\dfrac{1}{2}k=\dfrac{7}{50}\Rightarrow 50k^2+25k-7$

$(5k-1)(10k+7)=0\Rightarrow k=0.2$

(b) $\text{P}(A\cup B)=\text{P}(A)+\text{P}(B)-\text{P}(A\cap B)=0.2+0.7-0.14=0.76$

(c) $\text{P}(A'\cap B')=1-\text{P}(A\cup B)=1-0.76=0.24$

$\text{P}(A'\,|\,B')=\dfrac{\text{P}(A'\cap B')}{\text{P}(B')}=\dfrac{0.24}{0.3}=0.8$

2. **(a)** $\text{P}(B)=\dfrac{\text{P}(A\cap B)}{\text{P}(A\,|\,B)}=\dfrac{0.24}{0.4}=0.6$

(b) $\text{P}(A\cap B)\neq 0$

(c) $\text{P}(B\,|\,A)=\text{P}(B)$ so the events are independent

(d) $\text{P}(B\cap A')=\text{P}(B)-\text{P}(B\cap A)=0.6-0.24=0.36$

3. **(a)** $\text{P}(A \cap B) = \text{P}(B) \cdot \text{P}(A \mid B) = \frac{3}{8} \cdot \frac{1}{4} = \frac{3}{32}$

(b) $\text{P}(\text{only one happening}) = \text{P}(A \cup B) - \text{P}(A \cap B)$

$$= \text{P}(A) + \text{P}(B) - 2\text{P}(A \cap B) = \frac{9}{16} + \frac{3}{8} - 2 \cdot \frac{3}{32} = \frac{24}{32} = \frac{3}{4}$$

(c) $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B) = \frac{9}{16} + \frac{3}{8} - \frac{3}{32} = \frac{27}{32}$

$\text{P}(A' \cap B') = 1 - \text{P}(A \cup B) = 1 - \frac{27}{32} = \frac{5}{32}$

4. **(a)** $\text{P}(A \cap B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cup B) = \frac{3}{11} + \frac{4}{11} - \frac{6}{11} = \frac{1}{11}$

(b) $\text{P}(A \cap B) = \text{P}(A)\text{P}(B) = \frac{3}{11} \cdot \frac{4}{11} = \frac{12}{121}$

5. $\text{P}(A) = \text{P}(A \cap B) + \text{P}(A \cap B') = 0.3 + 0.3 = 0.6$

$\text{P}(A \cap B) = \text{P}(A)\text{P}(B) \Rightarrow \text{P}(B) = \frac{\text{P}(A \cap B)}{\text{P}(A)} = \frac{0.3}{0.6} = 0.5$

$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B) = 0.6 + 0.5 - 0.3 = 0.8$

6. **(a)** $\text{P}(\text{at least one } 6) = 1 - \text{P}(\text{no } 6\text{s}) = 1 - \left(\frac{5}{6}\right)^2 = \frac{11}{36}$

(b) $\dfrac{\frac{1}{36}}{\frac{11}{36}} = \frac{1}{11}$

7. **(a)** From the information, we can find that the total number of male is $200 - 90 = 110$. The total Number of employed people is $200 - 60 = 140$
20 males are unemployed, so $60 - 20 = 40$ are the unemployed females.
These numbers finally give 90 employed males and 50 females.

	Males	Females	Totals
Unemployed	20	40	60
Employed	90	50	140
Totals	110	90	200

(b) **(i)** $\frac{40}{200} = \frac{1}{5}$

(ii) $\frac{90}{140} = \frac{9}{14}$

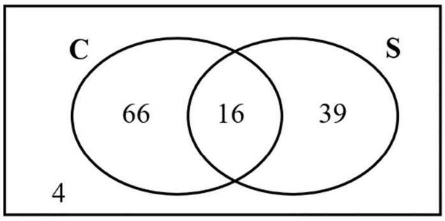
8. **(a)** Since the systems are independent the probability is $0.002 \cdot 0.001 = 0.00002$

(b) $\text{P}(\text{at least one not failing}) = 1 - \text{P}(\text{both fail}) = 1 - 0.00002 = 0.99998$

(c) $\text{P}(\text{at least one failing}) = 1 - \text{P}(\text{none failing}) = 1 - 0.998 \cdot 0.99 = 0.01198$

$\text{P}(\text{main failing} \mid \text{at least one failing}) = \frac{0.002}{0.01198} = 0.167$

9. The Venn diagram for the problem is

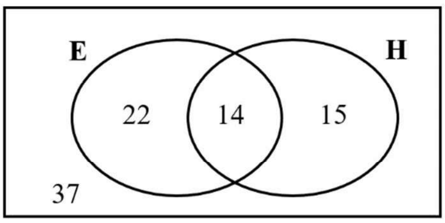


(a) **(i)** $\text{P}(C \cup S) = \frac{66+16+39}{125} = \frac{121}{125}$

(ii) $\frac{66+39}{125} = \frac{105}{125} = \frac{21}{25}$

(b) $\frac{16}{66+16} = \frac{16}{82} = \frac{8}{41}$

10. The corresponding Venn diagram is



(a) **(i)** $\frac{14}{88} = \frac{7}{44}$

(ii) $\frac{22}{22+14} = \frac{22}{36} = \frac{11}{18}$

(b) **(i)** $\frac{52}{88} \cdot \frac{51}{87} \cdot \frac{50}{86} = \frac{5525}{27434}$

(ii) $\text{P}(\text{at least one E}) = 1 - \text{P}(\text{no E}) = 1 - \frac{5525}{27434} = \frac{21909}{27434}$

11. Total number of components in the batch = 370

(a) **(i)** $\frac{6+120}{370}=\frac{126}{370}=\frac{63}{185}$

(ii) $\frac{4}{370}=\frac{2}{185}$

(iii) $\frac{120+80+150+6}{370}=\frac{356}{370}=\frac{178}{185}$

(iv) $\frac{6}{6+4+10}=\frac{3}{10}$

(b) The fraction of defective components in the three machines are respectively $\frac{1}{21},\frac{1}{21},\frac{1}{16}$.

Thus, the first two machines show the same quality, whilst machine III is less accurate.

12. **(a)** $100-(24.7+20.9+9.4+7.8+7.5)=29.7\%$

(b) **(i)** $0.209^2=0.0437$

(ii) $\text{P}(\text{at least 1 Samsung})=1-\text{P}(\text{no Samsung})=1-0.753^2=0.433$

(iii) $0.078\cdot 0.922+0.922\cdot 0.078=0.144$

(c) $\text{P}(\text{at least 1 Lenovo})=1-\text{P}(\text{no Lenovo})=1-0.906^3=0.256$

(d) $0.075\cdot 300=22.5$ millions

13. **(a)** $\frac{120000}{7600000}=\frac{3}{190}$

(b) **(i)** $\frac{4}{7.6}\cdot\frac{4}{7.6}=0.277$

(ii) $\text{P}(\text{at least one PNG})=1-\text{P}(\text{none PNG})$

$$=1-\left(\frac{7600000-30000}{7600000}\right)^2$$
$$=0.00788$$

(iii) $\left(\frac{4}{7.6}\right)^2+\left(\frac{0.12}{7.6}\right)^2+\left(\frac{0.114}{7.6}\right)^2+\left(\frac{0.03}{7.6}\right)^2=0.277$

(iv) $\frac{\left(\frac{0.12}{7.6}\right)^2}{0.277}=0.000898$

(c) $\text{P}(\text{at least one speaks Eng}) = 1 - \text{P}(\text{none}) = 1 - \left(\frac{7600000 - 114000}{7600000}\right)^5 = 0.0728$

14. **(a)** $0.01 \cdot 16000 + 0.055 \cdot 7000 + 0.09 \cdot 3500 = \$860$

(b) $860 \cdot 1.5 = \$1290$

(c) $\frac{0.01}{0.01 + 0.055 + 0.09} = 0.0645$

15. **(a)** **(i)** $0.0006 \cdot 0.995 + 0.9994 \cdot 0.01 = 0.0106$

(ii) $\frac{0.9994 \cdot 0.01}{0.0106} = 0.944$

(iii) $0.0006 \cdot 0.005 + 0.9994 \cdot 0.99 = 0.989$

(iv) $\frac{0.0006 \cdot 0.005}{0.989} = 3.03 \times 10^{-6}$

(b) **(i)** $0.0006 \cdot 0.995^2 + 0.9994 \cdot 0.01^2 = 0.000694$

(ii) $\frac{0.9994 \cdot 0.01^2}{0.000694} = 0.144$

16. **(a)** $0.003 \cdot 0.005 = 1.5 \times 10^{-5}$

(b) $0.003 \cdot 0.995 + 0.997 \cdot 0.005 = 0.00797$

(c) $\text{P}(\text{at least one}) = \text{P}(\text{both}) + \text{P}(\text{exactly one}) = 1.5 \times 10^{-5} + 0.00797 = 0.007985$

$\text{P}(\text{Bea} \mid \text{at least one}) = \frac{0.003}{0.007985} = 0.376$

17. **(a)** $\text{P}(\text{failing at least a test}) = 1 - \text{P}(\text{passing both}) = 1 - 0.02^2 = 0.9996$

(b) $0.02^2 = 0.0004$

(c) $0.02^2 = 0.0004$

18. Total number of people = 213

(a) **(i)** $\frac{58}{213}$

(ii) $\frac{35 + 18}{213} = \frac{53}{213}$

(b) **(i)** $\left(\frac{33}{213}\right)^2 = 0.0240$

(ii) $\text{P}(\text{at least 1 comedy}) = 1 - \text{P}(\text{none comedy}) = 1 - \left(\frac{213 - 33}{213}\right)^2 = 0.286$

(c) **(i)** $\left(\frac{43}{213}\right)^{10} = 1.12\times10^{-7}$

(ii) $P(\text{at least 1 action}) = 1 - P(\text{none action}) = 1 - \left(\frac{213-43}{213}\right)^{10} = 0.895$

19. $P(\text{disease} \mid \text{positive test}) = \frac{0.0001\cdot 0.99}{0.0001\cdot 0.99 + 0.9999\cdot 0.05} = 0.00198$

20. **(a)** $\frac{4}{6} = \frac{2}{3}$

(b) $P(\text{Jill wins on 1st}) = P(\text{Jack does not win})\cdot P(\text{Jill wins}) = \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$

(c) $P(\text{Jack wins}) = \frac{2}{3} + \left(\frac{1}{3}\right)^2\cdot\frac{2}{3} + \left(\frac{1}{3}\right)^4\cdot\frac{2}{3} + \left(\frac{1}{3}\right)^6\cdot\frac{2}{3} + \ldots$

$$= \frac{2}{3}\left(1 + \left(\frac{1}{3}\right)^2 + \left(\left(\frac{1}{3}\right)^2\right)^2 + \left(\left(\frac{1}{3}\right)^2\right)^3 + \left(\left(\frac{1}{3}\right)^2\right)^4 + \ldots\right)$$

This corresponds to the sum to infinity of a geometric sequence with

$u_1 = \frac{2}{3}$ and $r = \left(\frac{1}{3}\right)^2$

Therefore $P(\text{Jack wins}) = \frac{\frac{2}{3}}{1-\left(\frac{1}{3}\right)^2} = \frac{3}{4}$

21. **(a)** **(i)** $\frac{15}{22}$

(ii) $P(\text{at least 1 black}) = 1 - P(\text{no blacks}) = 1 - \left(\frac{15}{22}\right)^2 = \frac{259}{484}$

(iii) $\frac{\frac{15}{22}}{1-\left(\frac{7}{22}\right)^2} = \frac{22}{29}$

(b) In this case there is no replacement, so the two events are not independent and the probabilities for Shihui are not the same as for Jasmine.

(i) $\frac{15}{21}$

(ii) $1-\frac{15}{22}\cdot\frac{14}{21}=\frac{6}{11}$

(iii) $\dfrac{\frac{15}{22}}{1-\frac{7}{22}\cdot\frac{6}{21}}=\frac{3}{4}$

22. **(a)**

Test centre	Pass rate
Aberdeen	$\frac{3027+3249}{5892+5805}=0.537$
Bangor	$\frac{1412+1461}{3112+3017}=0.469$
Cheltenham	$\frac{1751+1829}{3332+3130}=0.554$
Doncaster	$\frac{3682+3938}{8506+8072}=0.460$

Cheltenham has the highest overall pass rate.

(b) $\frac{3938}{3249+1461+1829+3938}=\frac{3938}{10477}$

(c) **(i)** $\left(\frac{3249}{5805}\right)^2=0.313$

(ii) $\dfrac{\left(\frac{3249}{5805}\right)^2}{1-\left(\frac{5805-3249}{5805}\right)^2}=\frac{361}{929}$

(d) **(i)** $\frac{1461}{3017}\cdot\frac{1751}{3332}=0.254$

(ii) $\frac{1461}{3017}\cdot\frac{3332-1751}{3332}+\frac{3017-1461}{3017}\cdot\frac{1751}{3332}=0.501$

(e) $\frac{3027}{5892}\cdot 6000=3082$

23. **(a)** **(i)** $\frac{2}{4}=\frac{1}{2}$

(ii) $\frac{2}{4}=\frac{1}{2}$

(b) $\frac{1}{2}\cdot 10=5$

(c) The possible genes of their chicks now are PP, Pp, Pp, pp

(i) $\frac{3}{4}$

(ii) $\frac{2}{4}=\frac{1}{2}$

(d) The possible genes for feather structure of the chicks are HH, Hh, Hh, hh.

(i) $\frac{1}{4}$

(ii) Since the genes for pea comb and feather structure are independent

$$\begin{aligned}\text{P}(\text{pea comb and normal feather}) &= \text{P}(\text{pea comb})\cdot\text{P}(\text{normal feather})\\ &=\frac{1}{2}\cdot\frac{3}{4}=\frac{3}{8}\end{aligned}$$

(iii) $$\begin{aligned}\text{P}(\text{pea comb and woolly}) &= \text{P}(\text{pea comb})\cdot\text{P}(\text{woolly})\\ &=\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}\end{aligned}$$

(e) $\frac{1}{2}\cdot\frac{3}{4}\cdot 16=6$

(f) The number of chickens with a pea comb and a woolly appearance is $87-55=32$. The number of chickens with no pea comb and a woolly appearance is $38-32=6$. The number of chickens with no pea comb and normal feather structure is $120-87-6=27$.

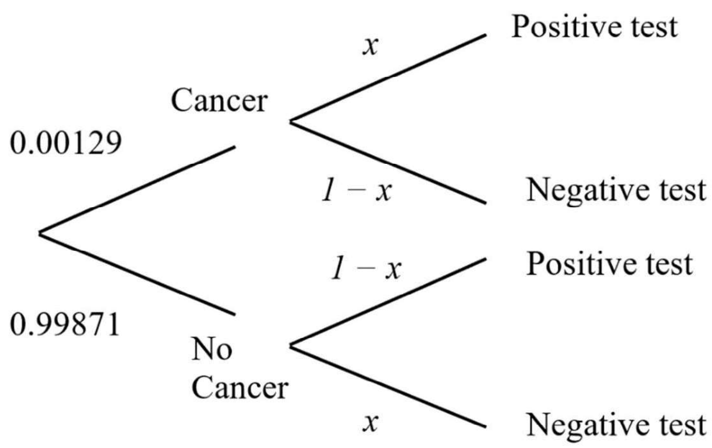
With these numbers:

(i) $\text{P}(\text{pea comb and woolly})=\frac{32}{120}=\frac{4}{15}$

(ii) $\text{P}(\text{no pea comb and normal feather structure})=\frac{27}{120}=\frac{9}{40}$

(iii) $\text{P}(\text{no pea comb} \mid \text{normal feather structure})=\frac{27}{120-38}=\frac{27}{82}$

24. **(a)** If the accuracy of the PSA test is x the tree diagram for the problem is:



Since the probability of a false positive is 65% we have

$$\frac{0.99871\cdot(1-x)}{0.00129\cdot x+0.99871\cdot(1-x)}=0.65$$

$$\frac{0.99871-0.99871x}{0.00129x+0.99871-0.99871x}=0.65$$

$$\frac{0.99871-0.99871x}{00.99871-0.99742x}=0.65\Rightarrow 0.99871-0.99871x=0.65(0.99871-0.99742x)$$

$$0.3495485=0.350387x\Rightarrow x=\frac{0.3495485}{0.350387}=0.998$$

Therefore the accuracy of the test is 99.8%

(b) $\frac{129}{100\,000}\cdot 0.002\cdot 800\,000=2.064\cong 2$

Exercise 12.1

1. **(a)** 4; 11; notice that a loop contributes 2 to the degree of a vertex {5, 6, 5, 6}

(b) 4; 6; {3, 3, 3, 3}

(c) 5; 5; {2, 1, 3, 2, 2}

2. **(a)** No. Consider people to be vertices of a graph. consider 4 of these vertices, then there is an edge from each of them to the other 3. When you add the fifth vertex, then it will add one edge to each of the other vertices, increasing the edges to 4.

(b, c) In fact, with 5 people, each chatting with everyone else, makes the graph a complete graph, K_5. Each vertex of the 5 vertices is connected to the other 4. So, Yes. As mentioned in the discussion, this is a K_5, whose graph you are familiar with (page 455).

3. Recall that a **simple graph** $G(V, E)$ is a graph that contains no loops or parallel edges. To be connected, there should be no isolated vertices. [The simplest case is when one vertex is connected to all other $n-1$ vertices with $n-1$ edges. You cannot remove any edge because it will render one vertex isolated.] Also, in a connected graph, no vertex has a degree of 0. First, if there are no vertices of degree 1, then every vertex has a degree at least 2. This implies that $2|E| = \sum \deg(v_i) \geq 2n.$

This in turn means that $|E| \geq n.$ However, if at least 2 vertices have a degree of 1 (remember that number of vertices with odd degree must be even), then our argument can be rewritten as $2|E| = \sum \deg(v_i) \geq 2n-2 \Rightarrow |E| \geq n-1.$

4. In a complete graph, every vertex is connected to all other vertices. For one vertex, there are $n-1$ edges connecting it to the other $n-1$ vertices. However, there are n vertices, and thus we have $n(n-1)$ such cases. Furthermore, every edge is contributing twice to the counting of edges since the edge is counted leaving vertex A and incident to vertex B, but it is counted again when leaving B and incident to A. Thus the number of edges is $\frac{n(n-1)}{2}$.

5. **(a)** $K_{3,4}$ has 3 vertices in on part and 4 in the other, thus 7 vertices altogether.

Each of the 3 vertices in one part are connected to the 4 vertices in the other part, and thus there are $3 \times 4 = 12$ edges.

(b) Similarly, $K_{13,17}$ has 13 + 17 = 30 vertices and 13 × 17 = 221 edges.

(c) Also, $K_{m,n}$ has $m + n$ vertices and mn edges

6. We have to solve the system of equations

$mn = 128; m + n = 24$

This can be solved in several ways. Either looking at the factors of 128 and choosing those that add up to 24, or setting up a quadratic equation in m or n. Therefore, they are 8 and 16.

7. **(a)** If we have v vertices, then the number of degrees is $3|v|$.

Recall that $\sum \deg .v = 2e \Rightarrow 3|v| = 24 \Rightarrow |v| = 8$.

(b) Yes: if $e = 14$, then $\sum \deg .v = 2e = 28$.

Thus, we can have 14 vertices each with degree 2, or 7 vertices each with degree 4.

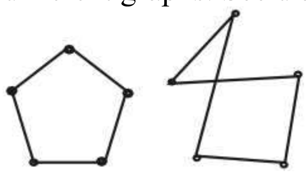
(c) A proof is not required. This is an investigation type of question.

As we know, $\sum \deg .v = 2e \Rightarrow rv = 2p$, with r is the vertex degree of the regular graph and v is the number of vertices. That means $r = \frac{2p}{v}$.

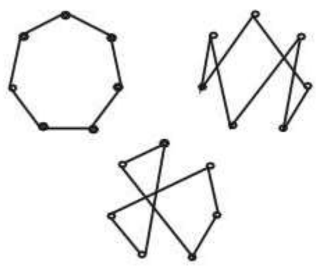
Since p is a prime number, the only viable situation is when $v = p$ and $r = 2$.

Let us consider cases:

If $p = 2$, then the possible number of vertices would be 1, or 2, in both cases, the graph will not be simple because it will either have a loop or two parallel edges. If $v = 3$, then we cannot have a regular graph.
If $p = 3$, we have one such graph, K_3.
If $p = 5$, the only viable situation is when $v = 5$. In that case, we have two different graphs. See diagram



If $p = 7$, a viable situation is when $v = 7$. We have 3 different graphs



Other graphs would be equivalent to the ones above.

The number of graphs is then $\left\lfloor \frac{p}{2} \right\rfloor$. In case you need to look at more cases, your next step is to check the case with $p = 11$.

(d) In exercise 3, we showed that the minimum number of edges in a simple graph is $n - 1$. In number 4, we showed that the largest number of edges a graph can have and still be simple is when it is complete. That is with a number of edges $\frac{n(n-1)}{2}$.

8. If graph with n vertices is connected then the possible degrees will be any number from 1 to $n-1$. Since we have n vertices, by the pigeonhole principle at least 2 vertices must occupy the same number!

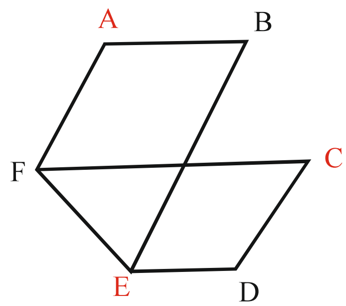
9. A subgraph must contain elements from both components. You cannot have a subgraph from one side only because each vertex is connected to a vertex from the other component and once a vertex is selected you need to select another vertex, and to be connected, there should be an edge, but the edges go to vertices in the opposite part. Thus, you will have vertices from both parts, and it is bipartite.

10. Use different colours for the vertices. No two adjacent vertices can have the same colour:

(a) The "vertical" vertices are one part and the "horizontal ones are another part.

(b) Each vertex is adjacent to all other vertices. You cannot divide the graph into two parts containing non-adjacent vertices each.

(c) See colours in the diagram blow. The red vertices are one part and the black ones are another part.



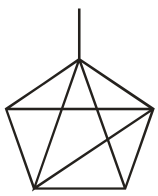
(d) If you take any vertex to be a member of one part, there are two vertices that are adjacent to it. These two vertices cannot be members of the second part because they are also adjacent. This is true for all the vertices of this graph. So, it cannot be bipartite.

11. Recall that $\sum \deg.v = 2+3+\cdots+5 = 24 = 2e \Rightarrow e = 12$.

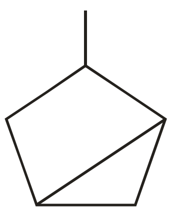
12. **(a)** $\sum \deg.v = 1+3+3+4+4 = 15$; $\sum \deg.v = 2e$,

so, it must be even. No, it is not a simple graph.

(b) Yes,



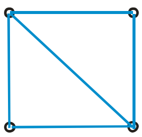
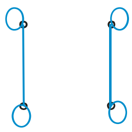
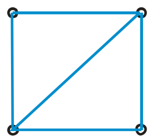
(c) Yes,

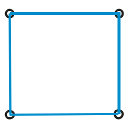
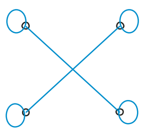
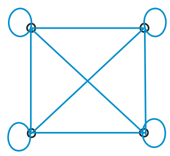


13. **(a)** $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & 3 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

14. **(a)**  **(b)**  **(c)** 

(d)  **(e)**  **(f)** 

Graphs in **a** and **c** are the same as well as **b** and **e**.

15. $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

Adjacency matrices can be rearranged and are similar with degrees: 2,2,2,3,3,3,3

16. A and C are the same with degree sequence: 1,1,2,3,3. We can set up the vertices to show their correspondence: ABCDE ↔ NOMLK. B & D are the same with degree sequence: 2,2,2,3,3. Vertex correspondence: FGHIJ ↔ PRSTQ.

Exercise 12.2

1. Vertices have even degrees.

(a) 123174263456751

(b) 1234543251

2. **(a)** 1234214241

(b) 12345241

(c) vertices 2 and 5 have degree 5 each.

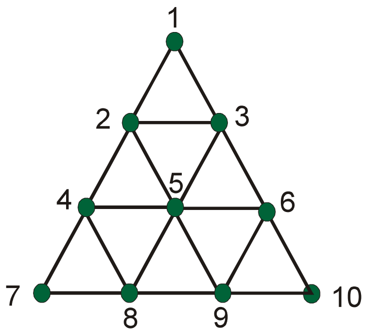
3. **(a)** When n is odd because then each vertex has a degree of $n - 1$, which will be even.

(b) When m and n are both even because vertices have either m or n degrees.

4. **1(a)** Hamiltonian: 12345671; **1(b)** Hamiltonian: 123451

2(a) Hamiltonian: 12341; **2(b)** Hamiltonian path: 12345; 2**(c)** neither.

5. **(a)** For T_1, consider triangle 123 in the diagram: (1231).



For T_2, consider triangle 146 in the diagram: (6532542136)

For T_3, consider triangle 1710 in the diagram:

(10, 9, 6, 5, 9, 8, 5, 4, 8, 7, 4, 2, 5, 3, 2,1, 3, 6, 10)

(b) For T_1, consider triangle 123 in the diagram: (123).

For T_2, consider triangle 146 in the diagram: (6542136)

For T_3, consider triangle 1710 in the diagram: (10, 9, 8, 7, 4, 5, 2, 1, 3, 6, 10)

(c) An Eulerian circuit is always possible ($n \geq 3$), because the degree of every vertex is even. Hamiltonian cycle is also possible using the same plan as above: Visit all vertices except one side, and then go back along that side.

6. Length 1 = 0.Length 2 = 2: *abe*, *ace*. Length 3 = 4:
abde, *abce*, *acbe*, *acfe*. and length 4 = 18: *abcbe*, *acbce*, *acfce*, *acabe*, …

7. Consider the adjacency matrix for K_5 and powers of 4, 5, 6 respectively.

Use GDC/software. We will demonstrate the first one only:

(a)
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}^4 = \begin{pmatrix} 52 & 51 & 51 & 51 & 51 \\ 51 & 52 & 51 & 51 & 51 \\ 51 & 51 & 52 & 51 & 51 \\ 51 & 51 & 51 & 52 & 51 \\ 51 & 51 & 51 & 51 & 52 \end{pmatrix}$$

51 between vertices not on the main diagonal, 52 for vertices on the diagonal.

(b) 205 between vertices not on the main diagonal, 204 for vertices on the diagonal.

(c) 819 between vertices not on the main diagonal, 820 for vertices on the diagonal.

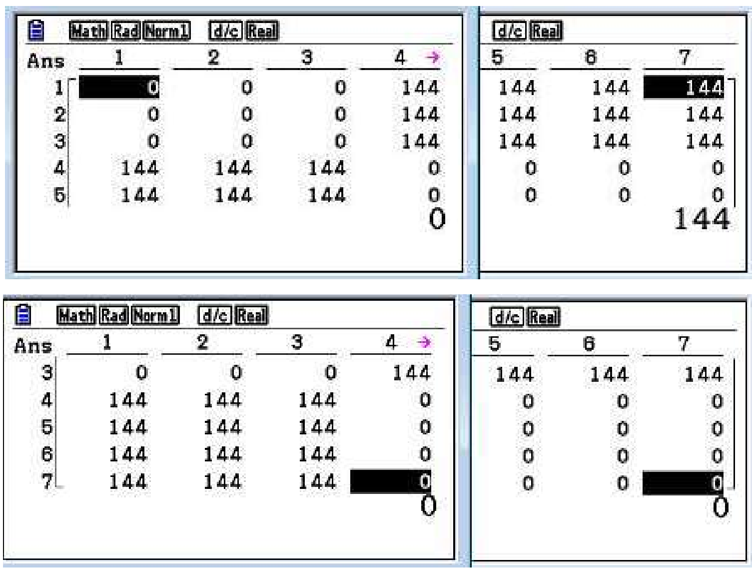
8. Consider the adjacency matrix for $K_{3,4}$ and powers of 4, 5, 6, 7 respectively.

Use GDC/software. We will demonstrate the first two only:

(a)
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}^4 = \begin{pmatrix} 48 & 48 & 48 & 0 & 0 & 0 & 0 \\ 48 & 48 & 48 & 0 & 0 & 0 & 0 \\ 48 & 48 & 48 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 36 & 36 & 36 & 36 \\ 0 & 0 & 0 & 36 & 36 & 36 & 36 \\ 0 & 0 & 0 & 36 & 36 & 36 & 36 \\ 0 & 0 & 0 & 36 & 36 & 36 & 36 \end{pmatrix}$$

48 among vertices of the 3-part, and 36 among the 4-part.

(b) 144 from vertices of 3-part to vertices of 4-part.



(c) 576 among vertices of the 3-part, and 432 among the 4-part.

(d) 1728 from vertices of 3-part to vertices of 4-part.

9. **(a)** No cycle- If you start at the left, you will need to visit *c* and *d* twice.
Path: *abcdef*.

(b) Cycle: *abcdea*

(c) No Cycle since *f* has degree 1.
Path: *eabcdf*.

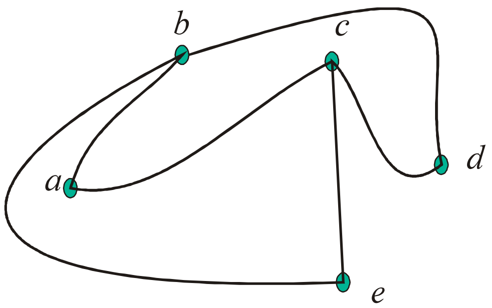
(d) Neither cycle nor path: 3 vertices with degree 1

(e) No Cycle, because in any of them *a* or *d* would have to be visited twice.
Path: *eacdb*.

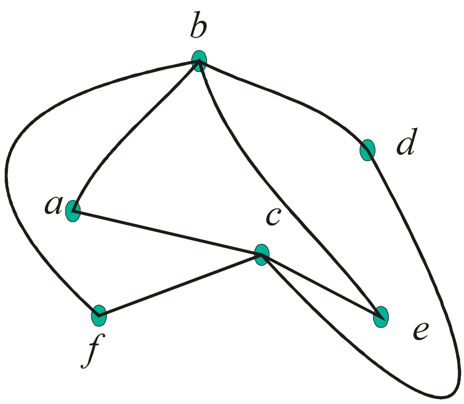
(f) Cycle: *ahgfedcbia*.

Exercise 12.3

1. **(a)** Planar:Redraw, and use Euler's formula: $v - e + f = 5 - 6 + 3 = 2$

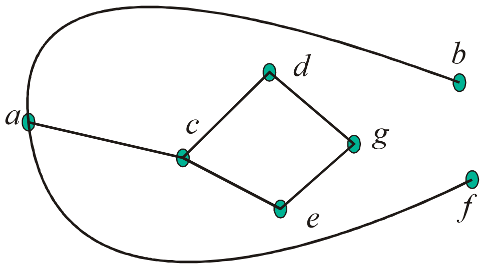


(b) Planar: Redraw, and use Euler's formula: $v - e + f = 6 - 8 + 4 = 2$

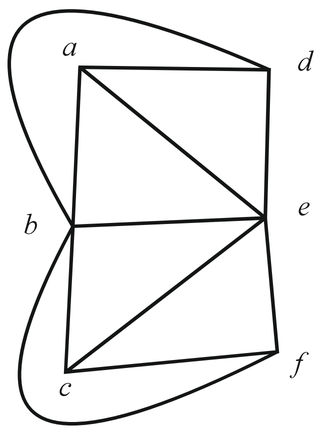


(c) Planar: Redraw, and use Euler's formula:

$v - e + f = 7 - 7 + 2 = 2$



(d) Planar. Redraw, and use Euler's formula:



$v - e + f = 6 - 11 + 7 = 2$

2. We can use Euler's formula for planar graphs:

$v - e + f = 2 \Rightarrow 10 - e + 7 = 2 \Rightarrow e = 15.$

3. In both cases that follow, we use the key fact if G is a connected simple planar graph with e edges and $v > 2$ vertices, then $e \leq 3v - 6$

(a) $e \leq 3v - 6 \Rightarrow e \leq 3 \times 7 - 6 = 15$

(b) $e \leq 3v - 6 \Rightarrow e \leq 3 \times 8 - 6 = 18$

4. Similar to question 3, we use the key fact if G is a connected simple planar graph with e edges and $v > 2$ vertices, then $e \leq 3v - 6$

(a) $e \leq 3v - 6 \Rightarrow 14 \leq 3v - 6 \Rightarrow 3v \geq 20 \Rightarrow v \geq 6.67 \Rightarrow v \geq 7$

(b) $e \leq 3v - 6 \Rightarrow 21 \leq 3v - 6 \Rightarrow 3v \geq 27 \Rightarrow v \geq 9$

5. We use Euler's formula:

$8 - e + f = 2 \Rightarrow f = 2 - 8 + e.$

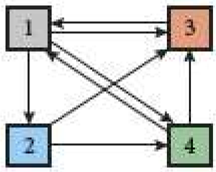
However, if each vertex has a degree of 3, then $\sum \deg.v = 3 \times 8 = 24 = 2e \Rightarrow e = 12.$

Therefore, $f = 2 - 8 + 12 = 6.$

6. **(a)** Redraw and then use Euler's formula as well as the two key facts because it does not have circuits of size 3: $e = 10$, $v = 8$ and $f = 4$, then $8-10+4=2$, or $e \le 3v-6 \Rightarrow 10 \le 3\times 8-6$, or $e \le 2v-4 \Rightarrow 10 \le 2\times 8-4 \Rightarrow$ graph is planar.

(b) Using Euler's formula with $e = 12$, $v = 8$ and $f = 6$, then $8-12+6=2 \Rightarrow$ graph is planar. Also, we can apply the key fact that $e \le 3v-6 \Rightarrow 12 \le 3\times 8-6$.

7. **(a)** label them as 1: inscholastics, 2: ismtf, 3: Pearson, 4: wazir-garry



(b) As described in the section, each column represents the links going from a certain page. Notice that since inscholastics has three links to others, the then the entry corresponding to each is $\frac{1}{3}$.

Similar arguments go for the other columns:

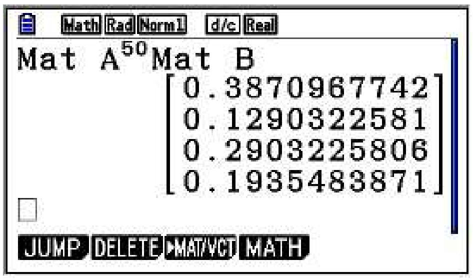
$$\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix}$$

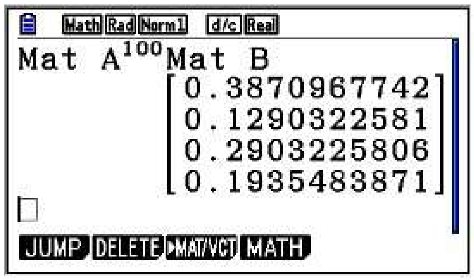
(c) The page rank is the final column corresponding to the long term application of the transition matrix, i.e., T^n as n gets larger and larger.

Considering the surfer starts with any of the pages at random, the initial state is represented by $\begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix}$

After one click, we have $\begin{pmatrix} 0 & 0 & 1 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0.3750 \\ 0.0833 \\ 0.3333 \\ 0.2083 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} \\ \frac{1}{12} \\ \frac{1}{3} \\ \frac{5}{24} \end{pmatrix}$

Using your GDC and having large powers of the transition matrix will yield the page rank vector $\begin{pmatrix} 0.38 \\ 0.12 \\ 0.29 \\ 0.19 \end{pmatrix}$





Exercise 12.4

1. **(a)** 5, 7, 10, 11, 13, 14, 16, 17

(b) 3, 1, 9

(c) 3: 12, 13, 14; 7: No descendants; 15: 16, 17.

(d) 4: 12; 7: No siblings; 9: No siblings.

2. The number of edges in a tree is $n - 1$, when n represents the number of nodes (vertices). Thus, $|e| = |u| - 1 \Rightarrow 17 = |u| - 1 \Rightarrow |u| = 18$.

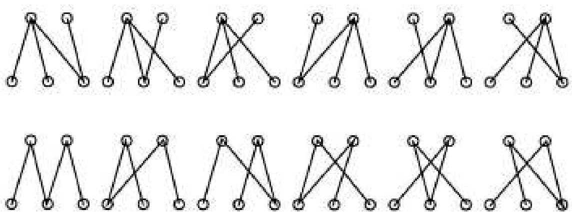
Since $|v| = 2|u| = 36$, and $|f| = |v| - 1 = 36 - 1 = 35$

3. The minimum number of edges in a connected undirected graph is $n - 1$ when n is the number of vertices. Thus if the number of edges is 30, the maximum number of vertices happens when the graph does not have any cycles, that is , when it is a tree. So the maximum number of vertices will be 30 + 1 = 31

4. By definition, in a tree, there is a path between any two vertices, thus there are ${}_nC_2$ different paths.

5. **(a)** There are 12 of such trees.

Every 6 are equivalent (*isomorphic* – a concept not in the syllabus)



(b) One part, V has two vertices, A and B. and the opposite, W has n vertices. In any spanning tree of $K_{2,n}$, if 2 vertices of W are adjacent to both A and B, there would be a cycle, which is not allowed in a tree. On the other hand, if no vertex of W is adjacent to both A and B, then the tree would not be connected. Thus, exactly one of the vertices in W is adjacent to both A and B. Hence, a spanning tree can be created if k vertices in W are connected to A, and we select one vertex to connect to both A and B, then, $k-1$ vertices of the n - 1 will be adjacent only to A. The remaining $n-k$ will be adjacent to B. The number of spanning trees with k vertices joined to A is $n \cdot {}^{n-1}C_{k-1}$

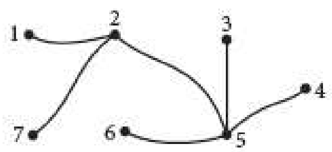
If we join only one vertex to A, there will be n spanning tree. Thus, the total number of spanning tree is $n + n\sum_{k=2}^{n} {}^{n-1}C_{k-1} = n + n\sum_{k=1}^{n-1} {}^{n-1}C_k$

Using the Binomial theorem,

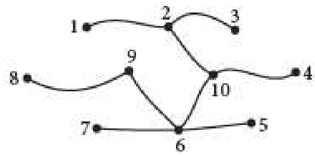
$$\sum_{k=2}^{n} {}^{n-1}C_{k-1} = 2^{n-1} - 1 \Rightarrow n + n\sum_{k=2}^{n} {}^{n-1}C_{k-1} = n + n\left(2^{n-1} - 1\right) = n2^{n-1}$$

6. Answers are not unique.

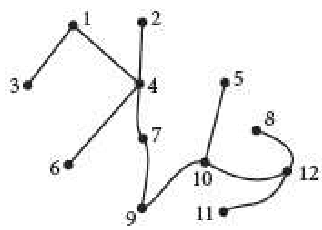
(a) Sample process: remove (1, 4), graph is still connected; similarly, remove (1, 7), Then (7, 5) and (7, 4), and so on. Below is a spanning tree



(b) Remove (1, 9) and (2, 9), keep (1, 2), and so on. Below is a spanning tree



(c) Similarly, here is a spanning tree for this part



7. In Kruskal's algorithm, we select an edge, e_1. If e_1 does not create a cycle, add it to the tree, and set i, the index to 1, and continue till $i = n - 1$.

(a) Sample work:

12, does not create a cycle, add it to tree, $i = 1$, 23, add to tree, $i = 2$, 34, add, $i = 3$, 45, add, $i = 4$, 53, creates a cycle, do not add, 56, add, $i = 5$, 62, do not add, 67, add, $i = 6$, stop. We have a spanning tree: 12 23 34 45 56 67

(b) 12, 23, 34, 45, 56, 67, 78, 89, 9(10)

(c) 12, 24, 45, 58, 8(12), (12)(11), (11)9, 9(10), 47, 76, 63

8. Use any algorithm of your choice. Below are samples of answers.

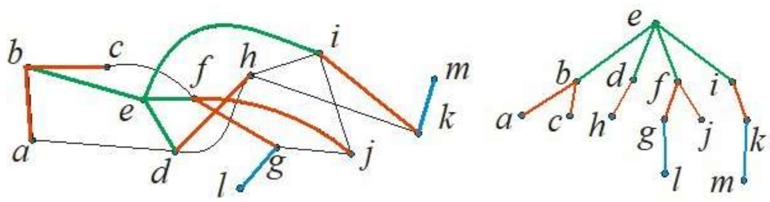
Answers are not unique.

(a) 13, 34, 45, 58, 89, 46, 67, 7 (10), 12

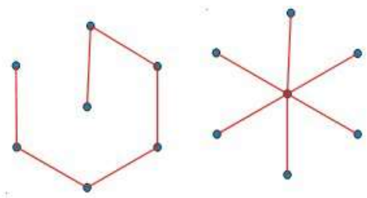
(b) 17, 78, 89, 9(10), (10)(11), (11)6, 65, 54, (10)(14), 9(13), 83, 32

(c) 12, 23, 34, 46, 65, 5(10), (10)9, 98, 87, (10)(11), (11)(12), (12)(13), (13)(14), (14)(15), (10)(16), (16)(17), (17)(19), (19)(20), (20)(18)

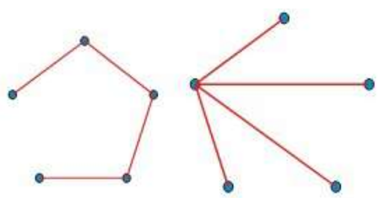
(d) Here is a spanning tree drawn in different ways.



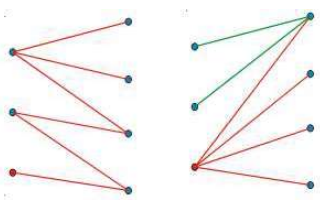
9. **(a)** Here are 2 spanning trees of a W_6.



(b) Two spanning trees for K_5.



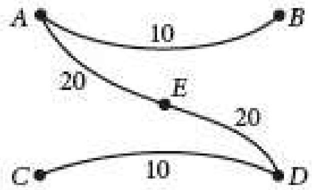
(c) Two spanning trees for $K_{3,4}$



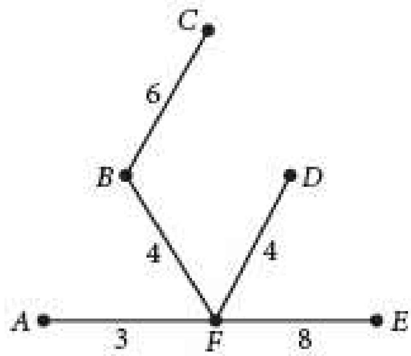
Exercise 12.5

Whether we use Kruskal's algorithm or Prim's the procedure is similar. That is, we select an edge, e, where *e does not create a cycle* and it has the *smallest* possible weight, add it to the tree, and set $i = 1$, and add e to the tree T. In Kruskal's case, we do not pay attention to whether the edge we add is adjacent to one already in the tree. We just stop at $n - 1$. In Prim's, we look for edges that are not chosen yet but adjacent to an existing edge and has the least weight.

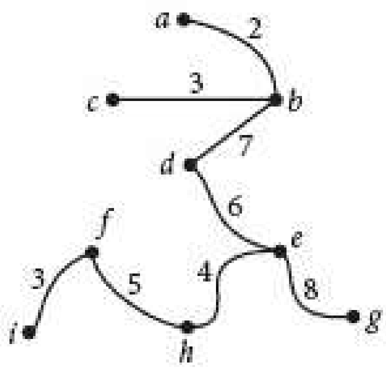
1. **(a)** The smallest are AB and CD, we add one of them, say AB. Next add CD as it does not create a cycle. Next we add AE, and after that ED. By now we are at $I = 5 - 1 = 4$. Stop, and the tree we have is a minimum spanning tree (MST) with weight 60.



(b) Start with AF, then BF, FD, (at this point AB cannot be added because it creates a cycle. BD cannot be added either), now, we add BC and lastly ($i = 5$), we add FE (CD cannot be added – we could have added DE instead) we have a MST with weight of 24.

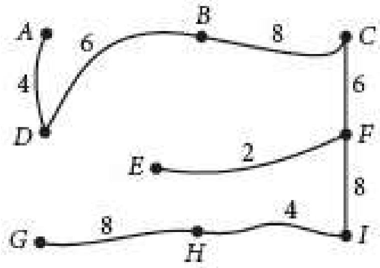


(c) We start with *ab*, the *cb*, and *fi* (even though it is not adjacent to any edge so far). Next is *he*, then *fh*, followed by *db*, and finally *eg*. Tree weight = 38.

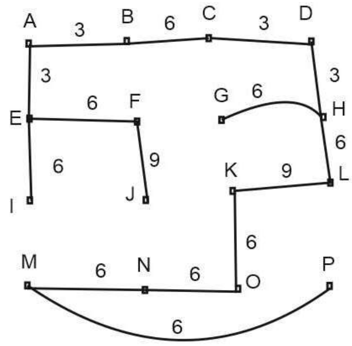


(d) Here is the order. 9 vertices, so we stop at the addition of the 8^{th} edge:

EF, AD, HI, CF, DB, BC, FI, GH. Tree weight = 46.

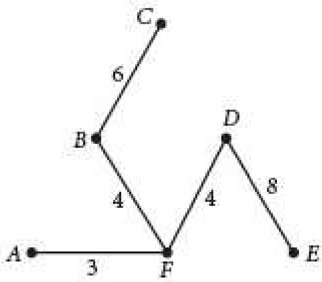


(e) 16 vertices, so, we stop at the 15^{th} edge addition. There are a few other trees with the same weight. Order of addition for this tree: AB, AE, CD, DH, BC, HL, HG, EF, EI, MN, NO, OK, MP, FJ, KL.



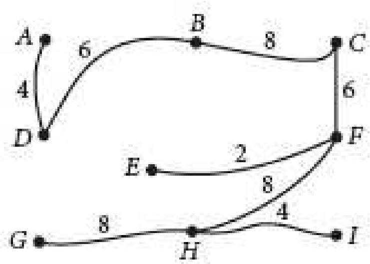
2. **(a)** Possible order of edge addition: AB, AE, AD, CD. The shape of tree is the same as 1**(a)**.

(b) Possible order of edge addition: AF, FB, FD, BC, FE (can also be DE). This scenario has a shape similar to 1**(b)**. If we add DE, here is the shape



(c) *ab*, *bc*, *bd*, *de*, *eh*, *hf*, *fi*, *eg*. Tree shape same as in 1**(c)**.

(d) EF, FC, CB, BD, DA, FH (or FI), HI, HG. Had we added FI instead of FH, tree shape would have been identical to **1(d)**'s tree.



(e) A few trees are possible, one of which is similar to the one in 1**(e)**. Here is a possible order:

AB, AE, BC, CD, DH, EF, HL, EI, HG, FJ, KL, OK, NO, MN, MP.

3. **1(a)** and **2(a)** have the same final tree. However, when building the tree with Kruskal's, AB and CD were added first. In Using Prim's, AB was followed by AE, ED, and then CD.

1(b) and **2(b)**, there is no apparent difference. The different shapes are due to random choices.

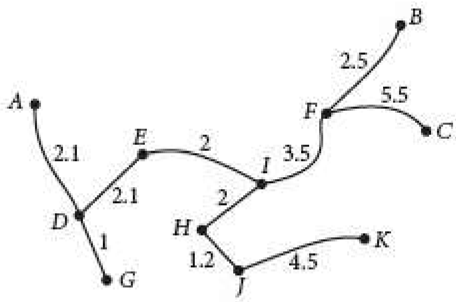
1(c) and **2(c)** have the same final tree too. In Ktuskal's, the order of addition to the tree is the following: *ab*, *bc*, *fi*, *he*, *fh*, *ed*, *bd*, and *eg*. In Prim's: *ab*, *bc*, *bd*, *ed*, *he*, *fh*, , *fi* and *eg*.

1(d) and **2(d)** may have the same tree too. However, Kruskal's, the order of edge addition is: *ef*, *ad*, *hi*, *cf*, *db*, *bc*, *fi*, and *gh*. In Prim"s: *ef*, *fc*, *fh*, *ih*, *cb*, *bd*, *da*, and *gh*.

1**(e)** and 2**(e)** may have the same tree too. However, Kruskal's, the order of edge addition is: *AB*, *AE*, *CD*, *DH*, *BC*, …. Prim's order is: *AB*, *AE*, *BC*, *CD*, *DH*, …

4. We can use either Kruskal's or Prim's for this purpose. Below is a Kruskal's application to the network. We used a spreadsheet to organize work, we first listed the edges with their weights, then sorted them in ascending order according to weight. Then we decided to include an edge as long as no cycle is being created and rejected it if a cycle has been created. 4th column kept track of the number of edges. Since there are 11 cities (vertices), we stop at the 10th addition.

Edge	weight	decision	i	
DG	1	Y	1	
HJ	1.2	Y	2	
EI	2	Y	3	
HI	2	Y	4	
AD	2.1	Y	5	
DE	2.3	Y	6	
AE	2.5	N		
BF	2.5	Y	7	
DH	2.5	N		
GJ	3	N		
FI	3.5	Y	8	
EF	4	N		
JK	4.5	Y	9	
FC	5.5	Y	10	stop



Exercise 12.6

1. We will use a table to show steps in the process

step	1	2	3	4	5
	a	*b*(*a*, 20)	~~*d*(*b*, 70)~~	*d*(*e*, 50)	*f*(*d*, 70)
		~~*c*(*a*, 30)~~	*e*(*b*, 40)		~~*f*(*e*, 80)~~

abedf, 70

2. A table produces the following result

step	1	2	3	4	5	6
	A	~~B(A, 12)~~	D(C, 18)	E(D, 21)	~~F(D, 33)~~	H(G, 48)
		C(A, 9)	~~E(C, 27)~~		G(E, 36)	

ACDEGH, 48

3. The question is put here as a demonstration of the usefulness of the method. It is not an exam question! Making a table, or writing a short program on a GDC, will find the shortest path. For now, a short and adjusted version of the table will suffice

a	abe	abeh	abehl	abehlo		
	6	12	14	26		
	acf	acfi	acfim	acfimp	acfimps	acfimpsu
	12	16	20	24	28	32
	adg	acfj				
	10	20				
		adgk	adgkr	adgkrt	adgkrtu	
		14	18	28	44	

Acfimpsu, 32.

4. From the table in question 1, it is *abed* with weight of 50.

5. For the first one we use the same table as in question 2, we do not cross out F since we are not using it further. So, shortest route is *ACDF*, with a weight of 33.

A new table can be set up to start at B. Shortest route is *BDEGH* with a weight of 45.

6. Starting at A, the nearest neighbour is D. At D, nearest neighbor (nn) is B, then C, and finally go back to A. TSP: *ADBCA* with a weight of 85.

7. There are several scenarios. We consider 2 here: we will use the nearest insertion method. we can start at E or D since they have neighbours that are close: Starting at E, create a cycle EDC. Now, extend the cycle by removing EC and adding vertex B. Thus we have the new cycle, EDCBE, Now remove BC and add vertex A, which completes the tour: *EDCABE* with a weight of 400. Start at D, create cycle DEB. Now, remove DB and add vertex A to make a new cycle, DEBAD. Now remove AD and add vertex C, which completes the route: *DEBACD* with weight of 400.

8. We can use nearest insertion. Create cycle, Vienna-Milan-Frankfurt. Remove Frankfurt Milan and add Prag and thus a new cycle Vienna-Frankfurt-Prag-Milan-Vienna. Lastly, remove Prag-Milan and add Moscow with the tour Vienna-Frankfurt-Prag-Moscow-Milan-Vienna: €1070.

9. We can start with a cycle: NY -V – M. Now, we remove NY-V and add L, so we have a new cycle: NY-L-V-M-NY. Then remove NY-L and add P, So, we now have New York-Paris-London-Vienna-Milan-New York for €1215.

10. We read the table from rows to columns. Let us start with D. The lowest entry is 80 which corresponds to A. So DA is an edge in this tree. Now for the row corresponding to A, 90 is the least entry corresponding to C. Since there is no cycle, we can continue from here. In column for C the least number corresponds to D, which is already in the route and it creates cycle. There is only 130, which corresponds to B. Now the edges are DACB. One vertex is left, E. So, BE is added with 120 weight, and then we go back to D with 130.

	A	B	C	D	E
A		100	90	80	110
B	100		130		120
C	90	130		120	
D	80		120		130
E	110	120		130	

DACBED: 550

11. No need for lengthy steps. Practically, $d(c, 13)$, $g(a, 13)$, and $f(g, 16)$ are clear. The next step is clear too, it is $e(g,19)$ because through *f* or *d* it is obviously longer than 19. So, route is *age*: 19

12. Here is first the route through *f*.

step	**1**	**2**	**3**	**4**	**5**	**6**	**7**
	a	*b(a, 4)*	*g(b, 8)*	*f(g, 10)*	*g(f, 12)*	*k(g, 16)*	*l(k, 20)*
		c(a, 4)	*d(b, 6)*	*f(d, 10)*	*h(f, 16)*	*l(h, 21)*	
			~~*e(c, 7)*~~				

As you see, we have two options. One is to go to *f*, then to *g* (possibility of visiting it twice!), then to *k* and then *l*. Total weight is 20.

The other option if we do not want to visit *g* is to go through *h*. weight is 21.

If we do not need to pass through *f*, then we can show that the route will be *acehl* with weight 13.

13. We start by using nn algorithm. Start at E as required. Nn is G, add G. Nn to G is Y, add Y. Since we are coming back to E, we consider another nn, which is P, add P with edge EP. Nn to Y is S, add S. Here we have a slight problem since we created a cycle. Accepting to visit Y twice, we see that nn to S is Y, so, add Y again. Nn to Y which has not been visited before is F, add F, similarly Add I. Here nn is A, however, choosing A, nn is P and the cycle closes leaving T out. To avoid this, at I, add T, then A, and lastly P. Thus, the route with minimum weight including a visit twice to Y is *EGYSYFITAPE* with weight of 871.

Without visiting any city twice, we end up with *ESYFITAPGE* and weight of 926.

Note: such complex networks would not appear on an exam. These are cases that help demonstrate the method.

14. We need to find an Eulerian cycle for this network. However, two vertices, *f* and *g* have odd degrees. As discussed in the text, we pair these vertices and introduce another route *fg* between them. Now we are ready to find a cycle to visit all edges, once. Here is an Eulerian cycle visiting all edges once except for *fg*: *abcdhghcgbfgfea* with weight of 8300.

15. Again, we need to find an Eulerian cycle for this network. However, four vertices, *e*, *j*, *h* and *g* have odd degrees. As discussed in the text, we pair these vertices and choose the pairing that has minimum weight. The pairing is given in the table below.

Pairing	Shortest path	Distance
ej	*efj*	7
gh	*gh*	5

Now we are ready to find a cycle to visit all edges, once.

Here is an Eulerian cycle visiting all edges once except for *ef*, *fj*, and *hg*: *abcdecjfefibjfgihgha* with weight of 9200.

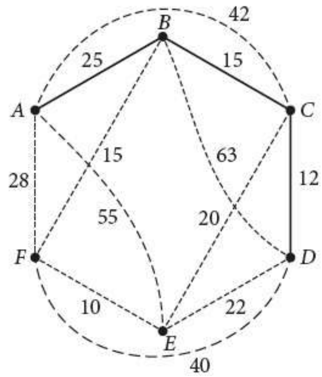
Chapter 12 practice questions

1. We can arrange our work in a table

step	1	2	3	4	5
	B	A(B, 3)	H(A, 18)	G(A, 11)	F(G, 16)
		C(B, 6)		D(B, 10)	~~F(C, 21)~~

BAGF: 16

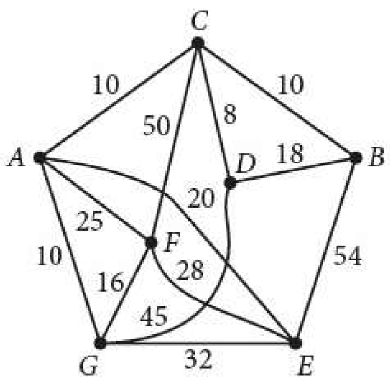
2. **(a)** Here is a graph – not unique shape!



(b) Using the table, first connection is through B for a cost of 25. At B, the row gives several connection with the route to C being the cheapest, cost is 15. From C to D the cost is 12. Thus, the cheapest route is ABCD for a cost of 25 + 15 + 12 = 52

3. With the nearest insertion algorithm, we can start with ACE. Remove edge AE and extend the cycle by adding AH, HG, and GE. The cycle is now ACEGHA. Remove EG and replace it with GF and FE. Now remove EF and replace it with ED and DF. Cycle so far is ACEDFGHA. Remove AH and replace it with HI, and IA. Finally, remove IA and replace with IB and BA. The route we arrived at is now ACEDFGHIBA with a distance of 8.6 km.

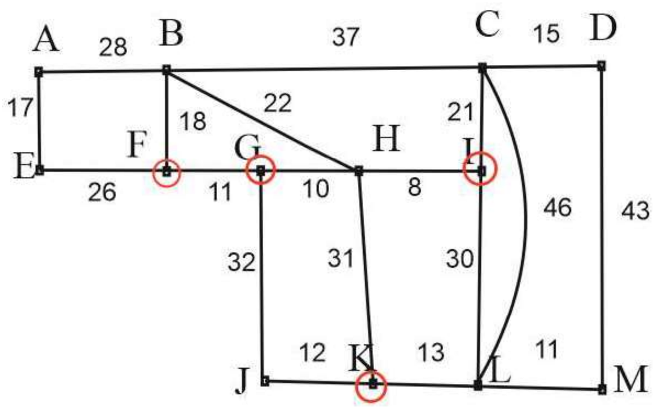
4. **(a)** Here is a graph of the network of cities remembering that the position on the graph does not necessarily represent the geographic position of the town but only its connection to other cities.



(b) A sample is the following using nn algorithm: start at A. A nn is C, and at C a nn is D where B is the nn that has not been visited. At B, the next nn not visited is E, which is very expensive. We will choose to revisit C at 10 000 points instead and A. From A we go to G, then to the nn F then to the last unvisited vertex E before coming back to our starting city A. The round trip is: ACDBCAGFEA at a total cost of 130 000, which she can afford.

Other routes are possible such as ACBDCAEFGA with 130 000 too.

5. Here is a sketch of the plan where we marked vertices with odd degrees clearly



We need to find an Eulerian cycle for this network. However, four vertices, *F, G, I* and *K* have

odd degrees. As discussed in the text, we pair these vertices and choose the pairing that has minimum weight. The pairing is given in the table below.

Pairing	Shortest path	Distance
FG	*FG*	11
IK	*IHK*	39

Here is one sample starting at B: BAEFGFBHGJKHILKHICLMDCB with total time of 481 minutes. He will need to stay 1 extra minute. Other scenarios starting at other points may be possible.

6. **(a)** C_n has n edges.

Additionally, W_n has another n edges connecting the extra vertex with the n vertices.

(b) We use adjacency matrices,

(i) n is odd

$$\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}^3=\begin{pmatrix}0&4&0&4\\4&0&4&0\\0&4&0&4\\4&0&4&0\end{pmatrix},\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}^5=\begin{pmatrix}0&16&0&16\\16&0&16&0\\0&16&0&16\\16&0&16&0\end{pmatrix}$$

Using mathematical induction, this can be proven to be true for all odd numbers

$$\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}^{2k+1}=\begin{pmatrix}0&2^{2k}&0&2^{2k}\\2^{2k}&0&2^{2k}&0\\0&2^{2k}&0&2^{2k}\\2^{2k}&0&2^{2k}&0\end{pmatrix}$$

(ii) n is even:

$$\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}^2=\begin{pmatrix}2&0&2&0\\0&2&0&2\\2&0&2&0\\0&2&0&2\end{pmatrix},\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}^4=\begin{pmatrix}8&0&8&0\\0&8&0&8\\8&0&8&0\\0&8&0&8\end{pmatrix}$$

Using mathematical induction, this can be proven to be true for all even numbers

$$\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}^{2k}=\begin{pmatrix}2^{2k-1}&0&2^{2k-1}&0\\0&2^{2k-1}&0&2^{2k-1}\\2^{2k-1}&0&2^{2k-1}&0\\0&2^{2k-1}&0&2^{2k-1}\end{pmatrix}$$

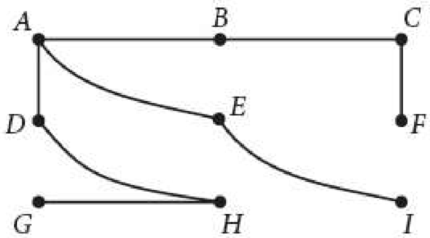
7. Kruskal's: BC, AB, AE, CF, GH, AD, DH, EI;

Below is a table with all decisions for Kruskal's

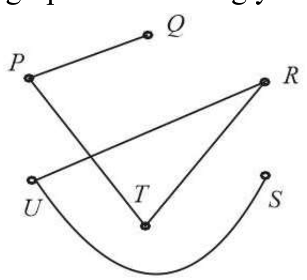
Prim's: BC, AB, AE, CF, AD, DH, GH, EI; 26

In both cases, we end up with the same shape and distance

Edge	weight	decision	i	
BC	2	Y	1	
AB	3	Y	2	
AE	3	Y	3	
GH	3	Y	4	
CF	3	Y	5	
AD	4	y	6	
BE	4	N		
EF	4	N		
DH	4	Y	7	
EI	4	Y	8	STOP



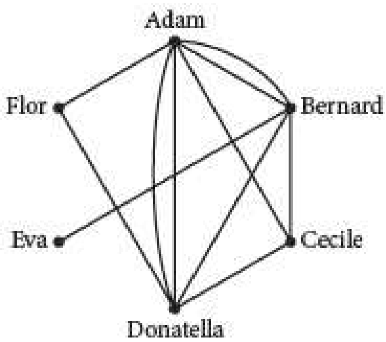
8. We can use Kruskal's algorithm, using a table and sorting lengths in ascending order and making sure no cycle has been created. There are 6 cities, so we stop at the introduction of the fifth stretch of highway. The least distance is PT = 86, then SU = 97, RU = 133, PQ = 200, and lastly TR = 203. Total distance is 719. (making a graph and marking your choices is a good idea so as not to create a cycle)



9. Using Kruskal's algorithm, we make a tree of the following stretches given by length: 1.6, 1.8, 2.1, 2.5, 3.3, 3.6. (A diagram helps avoid creating a cycle.)

Total length of cable needed is 14.9 m at a total cost of 1043 cents (10.43 Dollars or Euros).

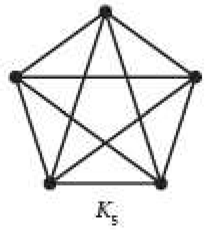
10. **(a)**



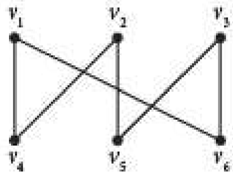
(b) Through Adam.

(c) Bernard, without him, Eva is isolated.

11. **(a)** **(i)** Here is a complete graph k_5:



(ii) Here is a bipartite $k_{3,\,3}$ – not complete. A complete one will need 4 mode edges.



(b) Each vertex has degree 2, so a Hamiltonian cycle is possible.

An example is $v_1v_4v_2v_5v_3v_6v_1$.

12. Here is a degree sequence of all rooms considering the outside as a room, call it O.

Starting clockwise from top-left corner, call them A, B, C, D, E, F, G, H, and I, the innermost room. 2, 4, 2, 3, 2, 4, 2, 3, 4, 2. Requested is an Eulerian circuit, which is not possible because 2 vertices have an odd degree.

13. Remember that in Trim's algorithm, a new edge is added if it is adjacent to an existing one. Tree: *h*, *e*, *d*, *a*, *i*, *g*, weight = 31.

14. **(a)**

	A	B	C	D	E	F
A	0	1	2	2	2	1
B	1	0	1	2	3	2
C	2	1	0	1	2	1
D	2	2	1	0	2	1
E	2	3	2	2	0	1
F	1	2	I	1	1	0

(b) **(i)** City F is the most accessible since its index is 1.5.

City E is the least accessible since its index is 10.

(ii) Cities C and F are the most accessible since their index is 1.5 each.

City E is still the least accessible since its index is 10.

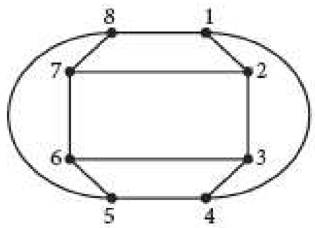
15. **(a)**

U	1	2	3	4	5	6	7	8
1	0	1	0	1	0	0	0	1
2	1	0	1	0	0	0	1	0
3	0	1	0	1	0	1	0	0
4	1	0	1	0	1	0	0	0
5	0	0	0	1	0	1	0	1
6	0	0	1	0	1	0	1	0
7	0	1	0	0	0	1	0	1
8	1	0	0	0	1	0	1	0

(b)

	A	E	B	F	C	G	D	H
A	0	1	0	1	0	0	0	1
E	1	0	1	0	0	0	1	0
B	0	1	0	1	0	1	0	0
F	1	0	1	0	1	0	0	0
C	0	0	0	1	0	1	0	1
G	0	0	1	0	1	0	1	0
D	0	1	0	0	0	1	0	1
H	1	0	0	0	1	0	1	0

(c) V is planar as U is planar.



16. **(a)**

Vertices added to the Tree	Edge added	Weight
3	□	0
5	3, 5	10
6	3, 6	20
7	5, 7	30
10	6, 10	30
1	3, 1	40
2	1, 2	30
11	2, 11	30
9	1, 9	40
4	6, 4	40
8	7, 8	40
		310

(b) Any of two paths: Steps that are marked permanent and are minimal and hence added to the route:

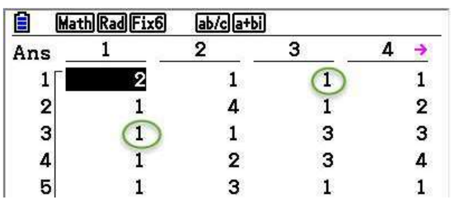
3(1,3), 4(3, 5) [notice here that 4(1, 8) is crossed out], 5(4, 11), 6(5, 18), 8(6, 33), 10(8, 58), 11(10, 80). So, the route is 1–3–4–5–6–8–10–11 and the cost is $8 million.

An alternative route can be 1–3–4–5–6–9–11, also, with weight 80.

17. **(a)** No, vertices C and E have odd degrees.

(b) Here is the adjacency matrix and its square.

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \\ F \end{array}\begin{array}{c} \begin{matrix} A & B & C & D & E & U \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$



When we find the square of this matrix, the entry for (A, C) or (C, A) is 1.

So, there is 1 walk of length 2.

(c) We will need 5 iterations of the algorithm. Resulting tree:

$\{AD, CE, CU, BU, AB\}$ with weight of 28.

Edge	weight	decision	i	
AD	3	Y	1	
CE	3	Y	2	
CU	5	Y	3	
UE	6	NO		CYCLE!
BC	8	Y(NO)	4	
BU	8	NO(Y)		CYCLE!
AB	9	Y	5	STOP

Notice that the choice BC or BU is random. Any of them can be a part of the tree but not both.

18. Kruskal's algorithm:

(a) T: = empty graph for I : =1 to n – 1 begin

e : = any edge in G with smallest weight that does not form a simple circuit when added to T

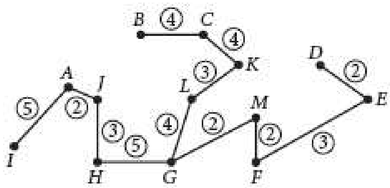
T : = T with e added end {T is a minimum spanning tree of G.}

(b) Arrange the edges in a table similar to question 17.

Notice that there are 13 vertices, and so, we stop adding edges when we reach 12. Also, note that the first 4 edges added has each a weight of 2, then, we moved on to those with weight 3:

AJ, GM, MF, DE, JH, LK, FE, GL, KC, CB, Al, HG.

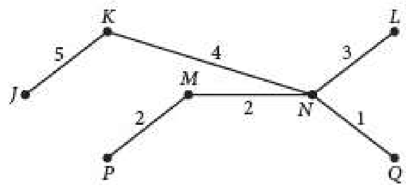
(c) Total weight 39



19. **(a)** B A E B C E F C D F

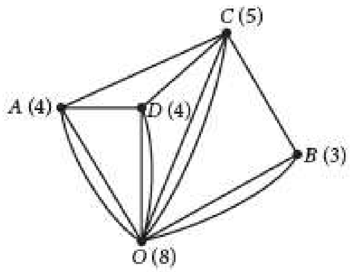
(b) All vertices have even degree.

20. Since we are required to start at J, the first edge we can add is JK, the edge with minimum weight connected to K is KN. The edge with minimum weight connected to present edges is now NQ, followed by NM and MP. Last edge added is NL (notice that even though potentially PQ would have been the next to add after MP, but we did not add it because it creates a cycle).



Weight = 17. (other trees are also possible)

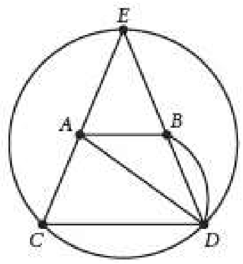
21. **(a)** Here is a sample. The number next to each vertex is the degree of each.



(b) Yes. The graph has exactly two vertices (B and C) with odd degree. It means that there is a path (starting at B or C) that will go once and only once through every door.

(c) Yes. O $\rightarrow$ D $\rightarrow$ A $\rightarrow$ C $\rightarrow$ B $\rightarrow$ O is a Hamiltonian cycle. It means that there is a path (starting anywhere) that will go once and only once through every room before returning to its starting point.

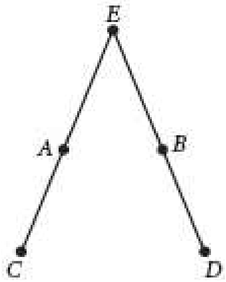
22. **(a)** Here is a sketch of such a graph



(b) Degree of every vertex is even.

(c) Here is one cycle (others are possible): AEBACDBDCEDA

(d) Also, more than one is possible. Here is one.



23. **(a)** MQ, QL, MP, PN, NR.

(b) 11

24. **(a)** G_2 does not have an Eulerian trail since 4 vertices have odd degrees.

(b) BABCECFEFBDEDADC.

25. **(a)** If *H* were planar, then $e \leq 2v - 4$. However, $e = 9$ and $2v - 4 = 8 \Rightarrow e \nleq 2v - 4$.

(b) delete AD.

(c) Subgraphs would have 8 edges which satisfies the condition $e \leq 2v - 4$.

26. **(a)** This is a complete graph K_n with $n = 5$ vertices.

Every vertex is adjacent to all other vertices. To have a Hamiltonian cycle is similar to arranging n objects around a circle.

Thus, there are $(n – 1)! = 4! = 24$ such cycles – this result ignores the fact that some of the cycles are equivalent. Every circuit appears twice – once with one order and the other in reverse. For example, abcda and adcba. Thus the number of "different" cycles is $\frac{(n-1)!}{2} = \frac{4!}{2} = 12$.

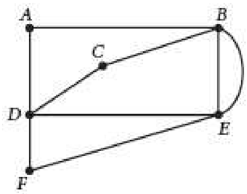
(b) **(i)** BDEC with weight of 20.

(ii) The 2 edges with minimum weight that are removed and can be added are AB and AE, which makes the total 33. So, a lower bound is 33.

(c) DBAEC is a min. spanning tree of 26 weight. An upper bound = 26 × 2 = 52

(d) A minimum tour is ABDCEA with 34. 33 cannot be achieved.

27. **(a)** An example



(b) All vertices are of even order. BEDABCDFEB (not unique).

(c) ABCDEF (not unique).

28. **(a)** **(i)** Eulerian circuit: V1, V2, V3, V4, V2, V6, V5, V4, V6, V1.

(ii) Hamiltonian cycle: V1, V2, V3, V4, V5, V6, V1.

(b) There is no Eulerian circuit since V_2 and V_6 are now of odd degree.

There is a Hamiltonian cycle still, same as above.

(c) **(i)** An Eulerian trail: V_2, V_3, V_4, V_2, V_6, V_5, V_4, V_6, V_1.

(ii) A Hamiltonian path: V_2, V_3, V_4, V_5, V_6, V_1.

29. We adjust Kruskal's algorithm to give us the maximum rather than the least weight. The table below shows our steps

edge	weight	decision	i	
GH	6	Y	1	
EF	5	Y	2	
HI	5	Y	3	
AD	4	Y	4	
CF	4	Y	5	
DG	4	Y	6	
EH	4	Y	7	
AC	3	N		
BC	3	Y	8	stop

The weight of this "maximum" spanning tree is 35.

30. **(a)** One upper bound is the length of any cycle, eg ABCDEA gives 73.

Other methods also apply.

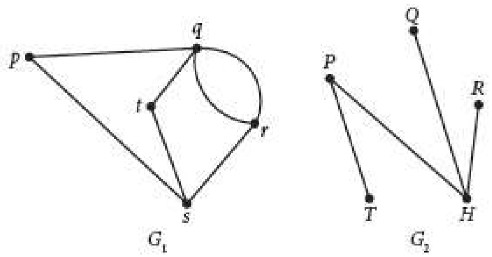
(b) **(i)** AB, AD, BC in that order.

(ii) Weight 33. A lower bound is the length of the minimum spanning tree found plus two of the removed edges with minimum length which are BE = 13 and DE = 14. Thus, a lower bound is 33 + 27 = 60.

31. **(a)** Not planar, since $e = 15 \nleq 3v - 6 = 12$.

(b) BD, DF, FA, FE, EC, in that order. Weight = 12.

32. **(a)** Here is a sample of each



(b) **(i)** G_1 is not simple since it has parallel edges, G_2 is simple.

(ii) Both are connected.

(iii) Both are bipartite: G_1: components are {p, r, t} and {q, s},

G_2: components are {P, R, Q} and {T, S}

(iv) G_1 is not a tree, it has a cycle. G_2 is a tree.

(v) G_1contains an Eulerian trail *rqpsrqts*. G_2 does not have since 4 vertices have odd degrees.

33. **(a)** There are 6 vertices, so, we stop after finding the fifth edge.

The smallest size for edges incident to F is FD, Now F and D are in the tree, the smallest edge incident to either is FC. Now, F, C, and D are in the tree, the smallest edge incident to any of them without creating a cycle is CB. Similarly we add BA and CE.

(b) Since the minimum spanning tree found has a weight of 38, then an upper bound for TSP is $2 \times 38 = 76$.

34. **(a)** **(i)** D, E

(ii) EBD

(iii) Since D and E have odd degrees, and the minimum part between them is EBD, we will consider allowing the use of this path twice.

An example: ABEFGCBDBEGDFCA.

(iv) 36

(b) Example: ABEFDGCA.

35. **(a)** Edges added to tree in order: HP, KQ, QF, FE, PB, ER, PQ, BC, CD; weight = 31.

(b) Lower bound is therefore, 31 + 17 = 48 [17 is the value of the removed edges.]

(c) Total (non-distinct) Hamiltonian circuits in complete graph Kn is $(n-1)!$ $(n > 3)$

This follows from the fact that starting from any vertex we have $n-1$ edges to choose from first vertex, $n-2$ edges to choose from second vertex, $n-3$ to choose from the third and so on. These being independent choices, we get $(n-1)!$ possible number of choices. Having said that, each Hamiltonian circuit has been counted twice (in reverse direction of each other, for example $ABCA$ and $ACBA$. Thus the number of cycles in K_n is $\frac{(n-1)!}{2}$.

(d) Since G is not complete, then the number of cycles to be examined must be less than $\frac{(11-1)!}{2} = 1814400$

36. **(a)** **(i)** Except for C, all other vertices have odd degrees.

(ii) G is bipartite: components are {B, D} and {A, C, E}.

Notice that B and D are not connected, but they are connected to vertices from the other component. Similarly, A, C, and E are not connected to each other.

(iii) $G_A = \begin{pmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \end{pmatrix}$, The entry corresponding to (A, C) in G_A^4 is 36

and so, there are 36 walks of length 4 between the two vertices.

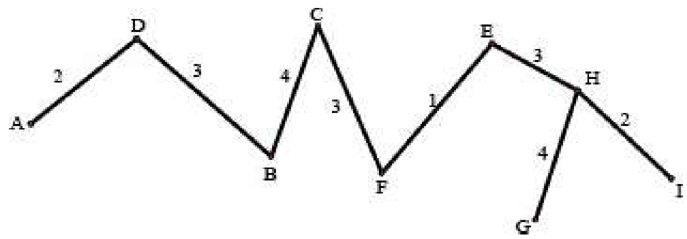
(b) We have two routes:

~~Q(P, 8)~~	Q(U, 7	~~R(Q, 14)~~	S(R, 18)
U(P, 4)	T(Q, 9)	R(T,12)	S(T, 18)
		T(Q, 9)	

Either PUQTS or PUQTRS. Both have length 18.

37. **(a)** Starting at A, the smallest edge is AD.

Connected to A or D, the next smallest edge is DB. This is followed by BC, and CF, FE, EH, HI, HG.



(b) weight = 22

38. **(a)** Every edge creates 2 degrees, one on each end.

Remember that a loop contributes 2 degrees also by convention. Thus with e edges there are $2e$ degrees.

(b) $\sum \deg .v = 2e \Rightarrow$ if we have an odd number of odd vertices and the rest are even, then the total number of degrees will be odd. This cannot happen because it has to be equal to $2e$. Thus, there should be an even number of odd vertices so that when added, their sum is even.

(c) **(i)** Use Euler's formula first: $n + f - e = 2 \Rightarrow n + 4 - e = 2 \Rightarrow e - n = 2$

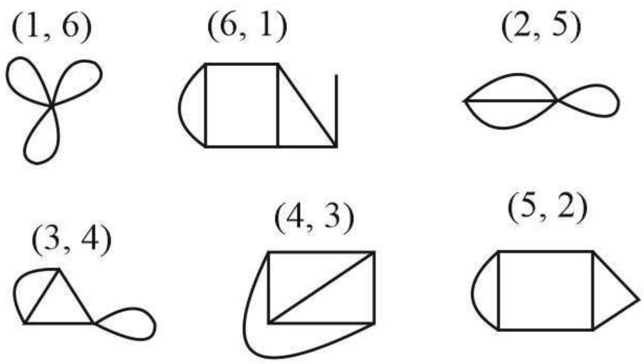
If we have n vertices, $(n - 1)$ with degree 3, and one with degree d, then the number of degrees is $3(n-1)+d$. We know that

$\sum \deg .v = 2e \Rightarrow e = \dfrac{\sum \deg .v}{2} = \dfrac{3(n-1)+d}{2}$. Thus,

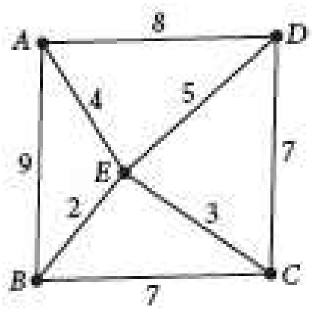
$e - n = 2 \Rightarrow \dfrac{3(n-1)+d}{2} - n = 2 \Rightarrow n + d = 7.$

(n, d) = (1, 6), (2, 5), (3, 4), (5, 2) or (6, 1). [(4, 3) will not be possible as

(ii) Here is a sample



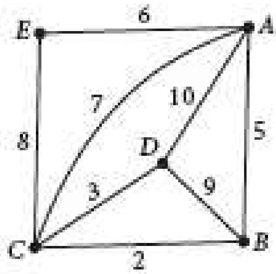
39. **(a)** Here is a sample



(b) To keep it simple and planar, then $e \le 3v - 6 \Rightarrow e \le 9$. We may add one edge.

(c) ABCDEA, weight 32; ABCEDA, weight 32; ABECDA, weight 29; AEBCDA, weight, 28, which is the one with the least weight.

40. **(a)** **(i)** Here is a sample



(ii) It is possible to have an Eulerian trail since only 2 vertices have odd degree.

(iii) A possible walk is ACBDABCDCEA. Length = 55.

(b) Since $\sum \deg.v = 2e$, thus, with one vertex the degree must be an even multiple of e, which cannot happen with an odd degree.

41. **(a)** **(i)** Let the two parts of the bipartite graph be A and B with vertices in A called a_i and in B, b_i. The main diagonal entries in M represent the paths of length 1 of a vertex in one part of the graph to itself, which cannot happen because there are no loops in a bipartite graph. When a number appears on the main diagonal, it represents a path from a vertex to itself. In a bipartite graph, and since no two edges in the same part can be adjacent, the number of edges required to get back to any edge must be 4. For example, leave vertex a_i to a vertex b_j and then back. This makes the length of a path from a vertex to itself at least 2, but always an even number. M^{37} diagonal entries represent paths of length 37 from a vertex back to itself. Since it has to be an even number, then the path of length 37 does not exist from a vertex back to itself and hence the number in the adjacency matrix must be zero.

(ii) Number of paths from v_i to v_j with a maximum length of 3.

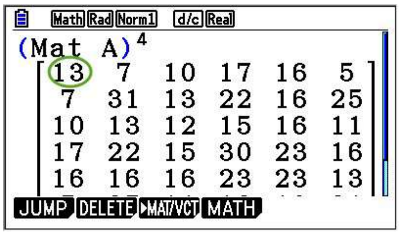
(b) If there is a bipartition (A, B) of the graph, where A, B are independent sets of vertices and the vertices $\{x, y, z\}$ form a triangle. Either A or B must contain at least two vertices from the triangle. However, any pair of vertices in the triangle have an edge between them, contradicting the fact that A is independent. Thus, a bipartite graph cannot contain a triangle.

(c) A bipartite graph is divided into two pieces, of size p and q, where $p + q = n$. Then the maximum number of edges is achieved when each vertex in A is connected to every vertex in B, that is pq. Now, $pq = p(n-p)$ is a parabola with a maximum when mum $p = \frac{n}{2} \Rightarrow \frac{n}{2} \cdot \frac{n}{2} = \frac{n^2}{4}$.

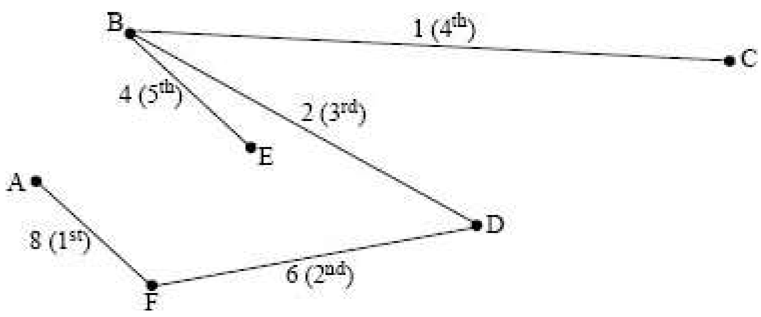
42. **(a)** Not bipartite. It contains more than one triangle.

(b) **(i)** $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$

(ii) This is the entry on the main diagonal in the 4th power of the adjacency matrix corresponding to (A, A), 13.



(c)



Exercise 13.1

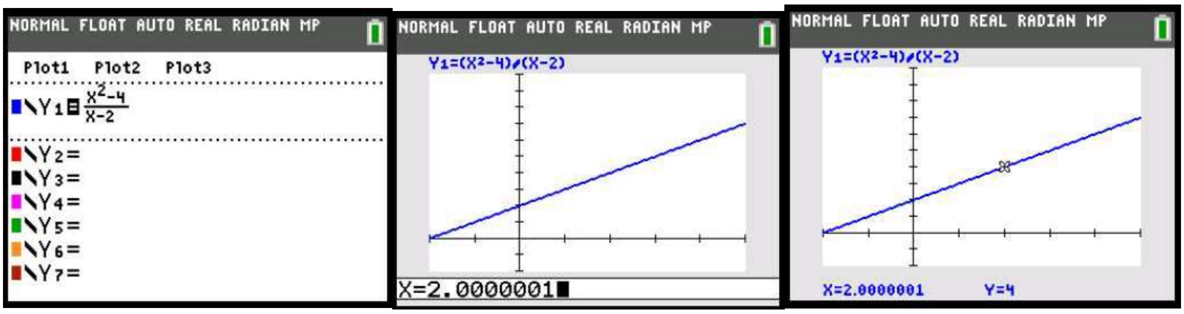
1. If you define as D_n as the distance between Achilles and the tortoise at step n you can see that: $D_0 = 100\,m, D_1 = 10\,m, D_2 = 1\,m, \ldots, D_n = 100 \cdot 10^{-n}$

The distance between the two gets closer to zero, as $\lim_{n\to\infty} D_n = 0$.

Moreover, since the sequence is geometric, the total distance travelled by Achilles will be $S_\infty = \frac{100}{1-0.1} = 111\frac{1}{9}\ m$ which is finite and, therefore will be covered by him in a finite time.

2. You can use a GDC to plot the function and use the graph to see what the limit is.

Pay attention because the function may not be defined for the limit value ($x = 2$ for part **(a)**) and therefore you have to choose a value close to it, but not equal (in the example $x = 2.0000001$).



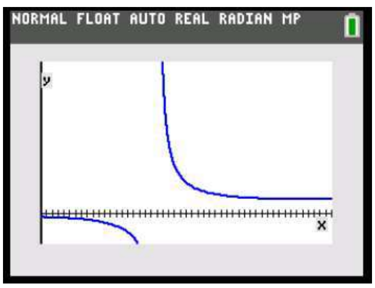
(a) 4

(b) −5

(c) 6

(d) In this case, from the graph it is clear that there is a vertical asymptote for $x = 2$.

When the x approaches 2 from the right ($\lim_{x\to 2^+}$) we can see that the graph tends towards $+\infty$



(e) 3

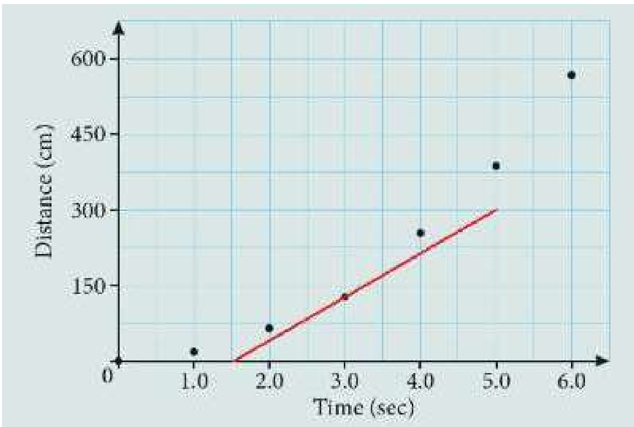
(f) 0.354

3. **(a)** **(i)** $\dfrac{62-0}{2}=31\text{ cm s}^{-1}$

(ii) $\dfrac{247-62}{2}=92.5\text{ cm s}^{-1}$

(iii) $\dfrac{553-247}{2}=153\text{ cm s}^{-1}$

(b) a few methods are available. Two are mentioned in the textbook answer section. Here is another one: Draw the tangent at $x = 3$ by eye and estimate two points in the line, you can estimate the gradient of the line, which corresponds to the instantaneous velocity at $x = 3$. From the graph to the right we can use (1.5,0) and (5.0, 300) which gives a value of 85.7 m s^{-1}. Since it is an approximate procedure similar values are also acceptable.



4. **(a)** **(i)** $\dfrac{30-15}{4.5-0.5}=3.75\text{ km h}^{-1}$

(ii) See graph below. The gradient of the tangent is between 12 and 13

(iii) No, as there is no point with gradient equal to 0.

(b) **(i)** $\dfrac{19-11}{2.5-1.5}=8\text{ km h}^{-1}$

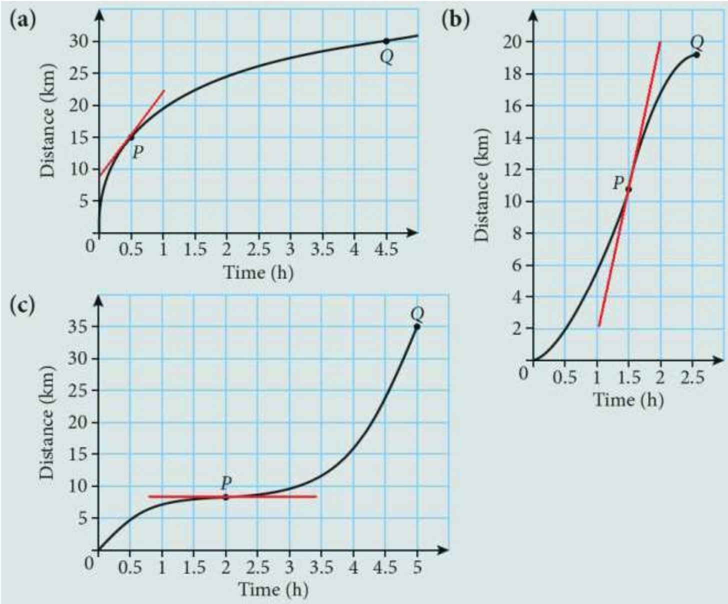
(ii) See graph below. The gradient of the tangent is around 18 km h^{-1}

(iii) No, as there is no point with gradient equal to 0.

(c) **(i)** $\dfrac{35-8}{5-2}=9\text{ km h}^{-1}$

(ii) See graph below.

The tangent is horizontal at P, so the velocity is equal to 0.

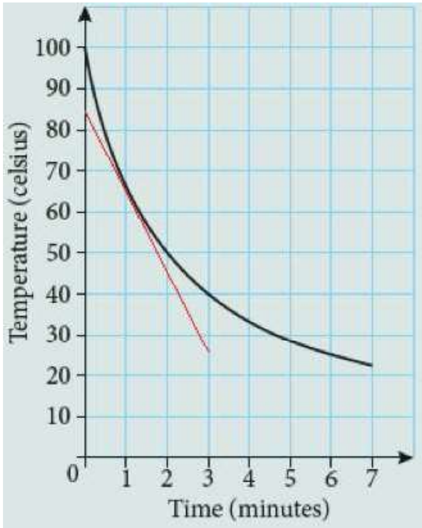


(iii) The gradient is 0 around P (approximately between $t = 1.5$ and $t = 2.5$).

5. **(a)** Reading from the graph we get:

$$\text{average rate of change} \cong \frac{21-100}{7} \cong -11 \text{ C min}^{-1}$$

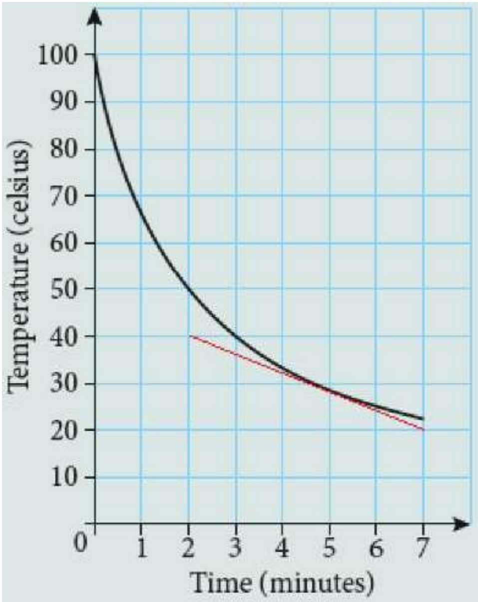
(b) Drawing the tangent at $t = 1$ by eye, one gets, approximately:



$$\text{instanteneous rate of change} = \frac{25-85}{3} \simeq -20 \text{ C min}^{-1}$$

(c) As for **(b)**, drawing the tangent by eye, one gets:

instanteneous rate of change $= \dfrac{20-40}{5} \simeq -4$ C min^{-1}

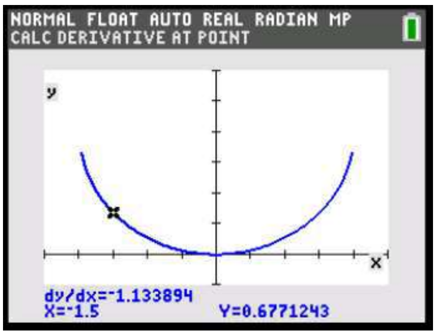


6. **(a)** One can use a GDC to answer this.

The half-pipe can be modelled as a circle centred at (0,2) with radius 2.

The equation of the lower half of the circle is: $y = 2 - \sqrt{4 - x^2}$

Plotting this equations with a GDC and calculating the numerical derivative one gets:



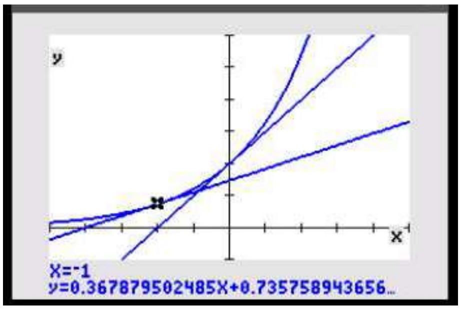
$x = -1.5, \dfrac{dy}{dx} = -1.13;\quad x = -1, \dfrac{dy}{dx} = -0.577;\quad x = 0, \dfrac{dy}{dx} = 0$

$x = 1, \dfrac{dy}{dx} = 0.577;\quad x = 1.5, \dfrac{dy}{dx} = 1.13$

The slope at x is the opposite of the slope at $-x$.

(b) Undefined, or, improperly, $+\infty$ and $-\infty$

7. Using a GDC to draw tangents to $y = e^x$, one gets that, approximately, the gradient at A is $m = 0.368$, at B the gradient is $m = 1$ and at C the gradient is $m = 2.72$.



From a GDC we can also see that $e \approx 2.72$ and $\frac{1}{e} \approx 0.368$. This This seems to suggest that, for $y = e^x$, the value of the derivative at a point is equal to the value of the function at that point.

Exercise 13.2

1. **(a)** $35t^2 + 40t = 400 \Rightarrow 7t^2 + 8t = 80 \Rightarrow 7t^2 + 8t - 80 = 0$

$(7t - 20)(t + 4) = 0$

The only positive solution is $t = \frac{20}{7} = 2.86$ s

(b) $\dfrac{400 - 0}{\frac{20}{7}} = 140 \text{ cm s}^{-1}$

(c)

t	Distance travelled	Velocity
0	0	$70 \cdot 0 + 40 = 40$
1	$35 \cdot 1^2 + 40 \cdot 1 = 75$	$70 \cdot 1 + 40 = 110$
2	$35 \cdot 2^2 + 40 \cdot 2 = 220$	$70 \cdot 2 + 40 = 180$
3	$35 \cdot 3^2 + 40 \cdot 3 = 435$	$70 \cdot 3 + 40 = 250$

(d) The toy car at $t = 3$ s is beyond the end of the ramp so the formulae used are no longer valid.

(e) $70 \cdot \frac{20}{7} + 40 = 240 \text{ cm s}^{-1}$

(f) 40 cm s^{-1} (see table in part **(c)**)

(g) $35t^2 + 40t = 200 \Rightarrow 7t^2 + 8t = 40 \Rightarrow 7t^2 + 8t - 40 = 0$

$t = \dfrac{-8 + \sqrt{8^2 - 4 \cdot 7 \cdot (-40)}}{14} = 1.89$ s

(h) In part **(g)** we calculated that the time to reach half-way down is 1.89 seconds, so the velocity at that point is given by $d'(1.89) = 172 \text{ cm s}^{-1}$

(i) $70t + 40 = 130 \Rightarrow 70t = 90 \Rightarrow t = 1.29 \text{ s}$

2. **(a)** $A = \pi \cdot 10^2 = 100\pi = 314 \; cm^2$

(b) $\text{cm}^2 \; \text{cm}^{-1}$, or equivalently just cm.

(c) The area is equal to $400\pi \,\text{cm}^2$ when r=20 cm.

Therefore the average rate of change is given by:

$$\frac{400\pi - 0}{20} = 20\pi \text{ cm}^2 \text{ cm}^{-1}$$

(d) The instantaneous rate of growth for $r = 20$ is given by:

$$\frac{\mathrm{d}A}{\mathrm{d}r}(20) = 2\pi \cdot 20 = 40\pi \,\text{cm}^2 \text{ cm}^{-1}$$

3. **(a)** $\frac{2}{3} \cdot (2.5)^3 - \frac{35}{6} \cdot (2.5)^2 + \frac{35}{2} \cdot 2.5 = 17.7 \text{ km}$

(b) $\text{d}(5) = \frac{2}{3} \cdot (5)^3 - \frac{35}{6} \cdot (5)^2 + \frac{35}{2} \cdot 5 = 25 \text{ km} \quad \therefore$

(c) $\frac{25-0}{5-0} = 5 \text{ km h}^{-1}$

(d) km h^{-1}

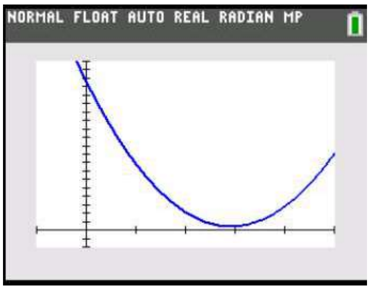
(e) To find the speed we need to use the derivative, so $t = 0$:

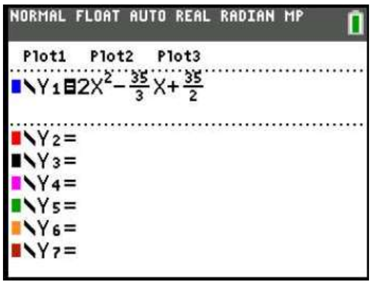
$$d'(0) = 2 \cdot (0)^2 - \frac{35}{3} \cdot (0) + \frac{35}{2} = \frac{35}{2} \text{ km h}^{-1} = 17.5 \text{ km h}^{-1}$$

whilst at $t = 5$.

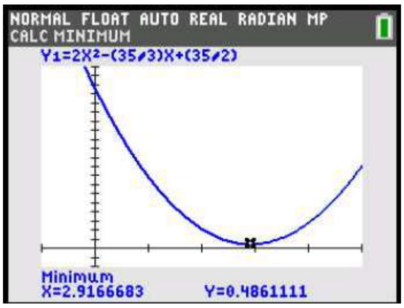
$$d'(5) = 2 \cdot (5)^2 - \frac{35}{3} \cdot (5) + \frac{35}{2} = \frac{35}{2} \text{ km h}^{-1} = 9.17 \text{ km h}^{-1}$$

(f) If you plot the graph of the velocity on a GDC you can see that the velocity (and also the speed) is greater for $t = 0$. From **(e)** we know that the speed at that time is 17.5 km h^{-1}.





(g) Using a GDC to find the minimum, you find that t = 2.92 s and the speed is 0.486 km h^{-1}.



4. **(a)** m s^{-1}

(b) $s(3)=2\cdot\sqrt{9}=6m \Rightarrow$ Average speed $=\dfrac{6-0}{3-0}=2$ m s^{-1}

(c) $\dfrac{ds}{dt}(1)=\sqrt{\dfrac{3}{1}}=$ 1.73 m s^{-1}, $\dfrac{ds}{dt}(2)=\sqrt{\dfrac{3}{2}}=$ 1.22 m s^{-1}, $\dfrac{ds}{dt}(3)=\sqrt{\dfrac{3}{3}}=$ 1 m s^{-1}

(d) $s(1)=2\cdot\sqrt{3}=3.46\ m,\ s(2)=2\cdot\sqrt{3\cdot 2}=4.90\ m,\ s(2)=6\ m$ from (b)

5. **(a)** The rate of change is greater when the function is steeper, so the average rate of change is greatest between A and B.

(b) **(i)** positive rate of change means that the curve is increasing, so at points A, B and F

(ii) negative rate of change means that the curve is decreasing, so at points D and E

(iii) At point C, where the tangent to the curve is horizontal and the curve is neither Increasing or decreasing.

(c) By eye you can see that the lines BD and EF have similar gradients and so the average rate of change between these two pairs of points are approximately equal.

6. The gradient of the function f_1 is negative before 0 and positive after 0. So its derivative corresponds to graph **(d)**.

The gradient of the function f_2 is negative everywhere except at 0, where it is equal to 0.

Therefore its derivative corresponds to graph **(e)**.

The gradient of the function f_3 is equal to 0 at three different points. Its derivative corresponds to graph **(b)**.

The gradient of the function f_4 is constant and negative as it is a straight line. Its derivative corresponds to graph **(a)**.

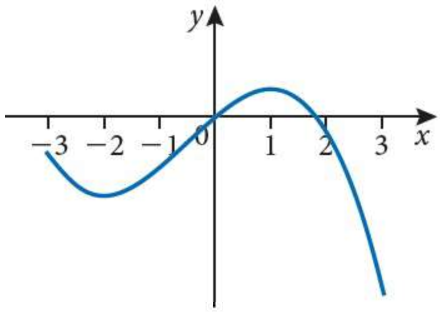
7. A function is increasing when its derivative is positive and decreasing when its derivative is negative.

(a) The function is increasing (positive derivative) for $1 < x < 5$ and decreasing for $0 < x < 1$ or $5 < x < 6$

(b) The function is increasing for $0 < x < 1$ or $3 < x < 5$ and decreasing for $1 < x < 3$ or $5 < x < 6$

8. **(a)** The function is decreasing when the derivative is negative, so for $-3 < x < -2$ or $1 < x < 3$

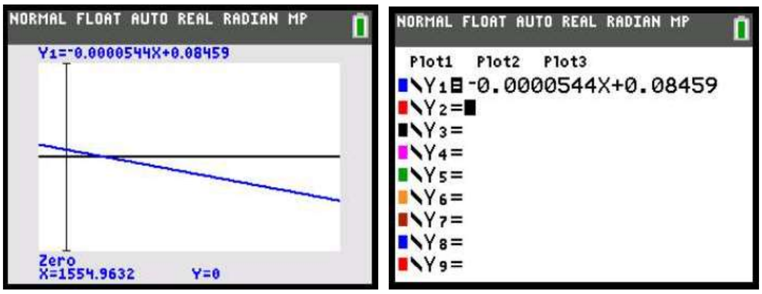
(b) The graph needs to go through (0,0) and it must have a minimum for $x = -2$ and a maximum for $x = 1$. Here there is one possible example, but other graphs with the same features are also correct.



9. **(a)** If $T'(r)$ is positive, it means that at r the torque increases as the RPM increases.

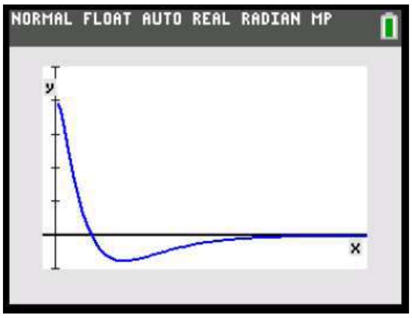
(b) The derivative is given by $T'(r) = -0.0000544r + 0.08459$.

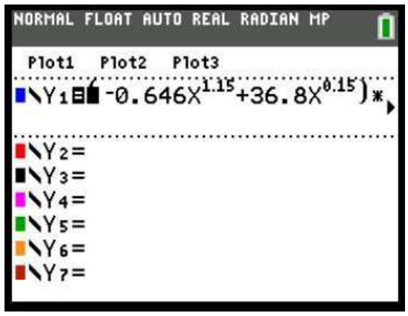
Using a GDC to plot the graph of $T'(r)$ and calculate the x-intercept, the torque is increasing for $0 < r < 1555$ RPM and decreasing for $1555 < r < 10000$ RPM



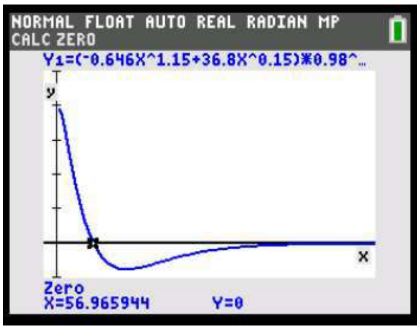
(c) Looking at the graph, in the interval given, the Torque has a maximum when $T'(r) = 0$. Therefore $r = 1555$ RPM and $T(1555) = -0.0000272 \cdot 1555 + 0.08459 \cdot 1555 + 440.0 = 505.8$ N m.

10. **(a)**





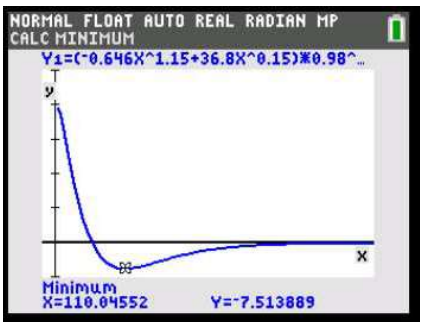
(b) Use a GDC to find where $\frac{\mathrm{d}C}{\mathrm{d}t}=0$, which gives *t=57.0.*



So the concentration of the drug is increasing for $0<t<57.0$ minutes and decreasing for $57.0<t<480$ minutes.

(c) The concentration is decreasing the fastest when $\frac{\mathrm{d}C}{\mathrm{d}t}$ has a minimum.

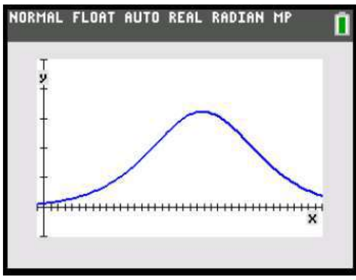
Using a GDC to find the coordinates of the minimum, one gets t = 110 min and $\frac{\mathrm{d}C}{\mathrm{d}t}=-7.51\text{ ng ml}^{-1}\text{ min}^{-1}$.

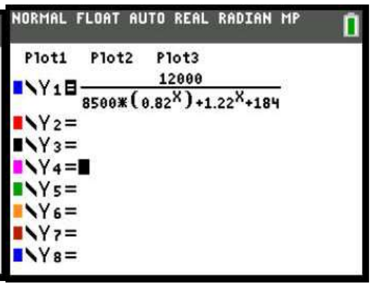


(d) The concentration has a maximum when $\frac{\mathrm{d}C}{\mathrm{d}t}=0$, so at t = 57.0

11. **(a)** Thousands of people per year

(b)

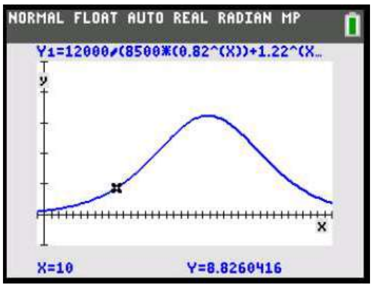


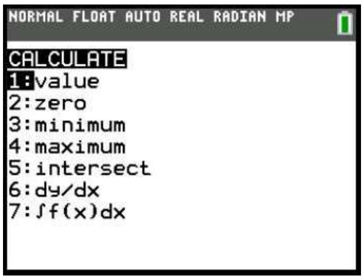


(c) Looking at the graph, and considering that the function contains only exponentials, the derivative is always positive and therefore the population cannot decrease

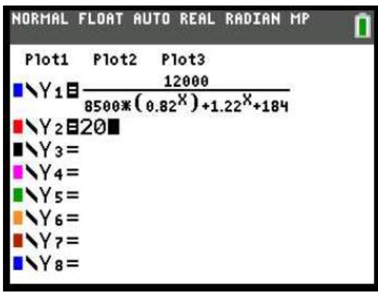
(d) The population grew rapidly from about 1990 to 2010 with a peak of growth around the first years of 2000s. Then it levels off and the population stabilises.

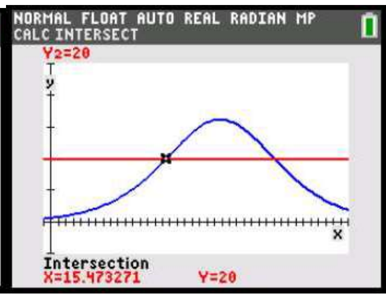
(e) Using the function *value* of a GDC for $t=10$ one gets $\frac{\mathrm{d}P}{\mathrm{d}t} = 8.83$, which, considering the units of P, gives a rate of change of 8830 people per year.

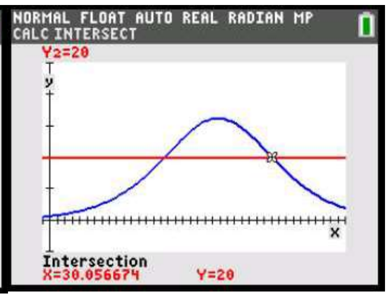




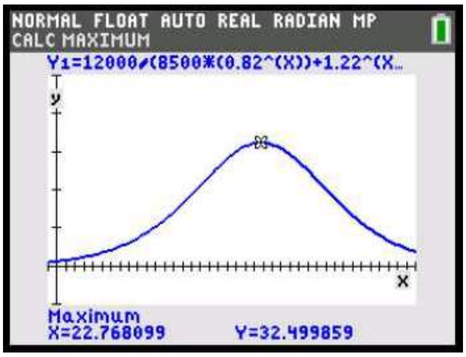
(f) Plotting the line $y=20$ in the same graph as $P'(t)$ and finding the intersections, one gets $15 < t < 30$ which corresponds to the years 1995 to 2010







(g) Use a GDC to find the maximum of $P'(t)$. This gives $t = 22.8$, which means it happens during the year 2002. At this time $P'(t) = 32500$ people per year.



Exercise 13.3

1. **(a)** **(i)** $\frac{dy}{dx} = 3 \cdot 2x - 4 = 6x - 4$

(ii) $6 \cdot 0 - 4 = -4$

(b) **(i)** $\frac{dy}{dx} = -6 - 2x$

(ii) $-6 - 2 \cdot (-3) = 0$

(c) **(i)** $y = 2x^{-3} \Rightarrow \frac{dy}{dx} = 2 \cdot (-3) \cdot x^{-4} = -6x^{-4}$

(ii) $-6 \cdot (-1)^{-4} = -6$

(d) **(i)** $\frac{dy}{dx} = 5x^4 - 3x^2 - 1$

(ii) $5 \cdot 1^4 - 3 \cdot 1^2 - 1 = 1$

(e) **(i)** $y = x^2 - 4x - 12 \Rightarrow \frac{dy}{dx} = 2x - 4$

(ii) $2 \cdot 2 - 4 = 0$

(f) **(i)** $y = 2x + x^{-1} - 3x^{-3} \Rightarrow \frac{dy}{dx} = 2 - x^{-2} - 3 \cdot (-3) \cdot x^{-4} = 2 - x^{-2} + 9x^{-4}$

(ii) $2 - 1^{-2} + 9 \cdot 1^{-4} = 10$

(g) **(i)** $y = x + \frac{1}{x^2} = x + x^{-2} \Rightarrow \frac{dy}{dx} = 1 - 2x^{-3}$

(ii) $1 - 2 \cdot (-1)^{-3} = 1 + 2 = 3$

2. **(a)** $(0, 0)$

$\frac{dy}{dx} = 2x + 3$

$2x + 3 = 3 \Rightarrow 2x = 0 \Rightarrow x = 0 \Rightarrow y = 0^2 + 3 \cdot 0 = 0$

(b) $(2, 8)$ and $(-2, -8)$

$\frac{dy}{dx} = 3x^2$

$3x^2 = 12 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

$y = 2^3 = 8, \quad y = (-2)^3 = -8$

(c) $\left(\frac{5}{2}, -\frac{21}{4}\right)$

$\frac{dy}{dx} = 2x - 5$

$2x - 5 = 0 \Rightarrow 2x = 5 \Rightarrow x = \frac{5}{2} \Rightarrow y = \left(\frac{5}{2}\right)^2 - 5 \cdot \frac{5}{2} + 1 = -\frac{21}{4}$

(d) $(1, -2)$

$\frac{dy}{dx} = 2x - 3$

$2x - 3 = -1 \Rightarrow 2x = 2 \Rightarrow x = 1 \Rightarrow y = 1^2 - 3 \cdot 1 = -2$

3. Using the fact that the point $(2, -4)$ is on the curve we can create one equation:

$-4 = 2^2 + 2a + b \Rightarrow -8 = 2a + b$

Since $\frac{dy}{dx} = 2x + a$ and the gradient at $(2, -4)$ is -1 we can create another equation:

$-1 = 2 \cdot 2 + a \Rightarrow a = -5$

Replacing back in the first equation we can find *b:*

$-8 = 2 \cdot (-5) + b \Rightarrow b = 2$

4. $y = mx + b = mx^1 + bx^0$

$\frac{dy}{dx} = m \cdot 1 \cdot x^{1-1} + b \cdot 0 \cdot x^{0-1} = m \cdot x^0 + 0 = m$

Therefore the gradient is always *m.*

5. **(a)** $g(x)=x^{\frac{1}{3}} \Rightarrow g'(x)=\frac{1}{3}x^{\frac{1}{3}-1}=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3\sqrt[3]{x^2}}$

(b) **(i)** $g'(8)==\frac{1}{3\sqrt[3]{8^2}}=\frac{1}{12}$

(ii) $g(8)=8^{\frac{1}{3}}=2$

Therefore the equation of the tangent is

$$y-2=\frac{1}{12}(x-8)\Rightarrow y=\frac{1}{12}x-\frac{2}{3}+2\Rightarrow y=\frac{1}{12}x+\frac{4}{3}$$

(c) **(i)** $g'(x)=\frac{1}{3\sqrt[3]{x^2}}$ is undefined for $x=0$.

(ii) At (0, 0) the tangent is vertical, so its equation is $x=0$

6. $y=3x^{\frac{1}{2}}-6\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{3}{2}x^{-\frac{1}{2}}=\frac{3}{2\sqrt{x}}\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}(9)=\frac{3}{2\sqrt{9}}=\frac{1}{2}$

Equation of the tangent:

$$y-3=\frac{1}{2}(x-9)\Rightarrow y=\frac{1}{2}x-\frac{3}{2}$$

7. **(a)** $\frac{C(6)-C(2)}{6-2}=\frac{19+\frac{182}{6^{\frac{3}{2}}}-\left(19+\frac{182}{2^{\frac{3}{2}}}\right)}{4}=-13.0\text{ °C min}^{-1}$

(b) $C(t)=19+182t^{-\frac{3}{2}}\Rightarrow C'(t)=182\cdot\left(-\frac{3}{2}\right)t^{-\frac{5}{2}}=-\frac{273}{\sqrt[2]{x^5}}$

$$C'(4)=-\frac{273}{\sqrt[2]{4^5}}=-\frac{273}{32}=-8.53\text{ °C min}^{-1}$$

8. **(a)** Average rate of change $=\frac{2\sqrt{4^3}+17-\left(2\sqrt{1^3}+17\right)}{4-1}=\frac{14}{3}=4.67\text{ °C h}^{-1}$

(b) $C(t)=2t^{\frac{3}{2}}+17\Rightarrow C'(t)=2\cdot\frac{3}{2}t^{\frac{1}{2}}=3\sqrt{t}$

(c) $3\sqrt{t}=\frac{14}{3}\Rightarrow \sqrt{t}=\frac{14}{9}\Rightarrow t=\left(\frac{14}{9}\right)^2=\frac{196}{81}=2.42\text{ h}$

9. **(a)** number of cases per year

(b) $\dfrac{dC}{dt} = -222 \cdot 3 \cdot t^2 + 7260 \cdot 2 \cdot t - 12700 = -666t^2 + 14520t - 12700$

(c) At 1990 t=7. Hence

$$\frac{dC}{dt}(7) = -666 \cdot 7^2 + 14520 \cdot 7 - 12700 = 56306 \text{ cases per year}$$

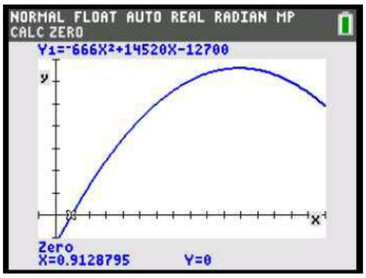
At the beginning of 1990, the cumulative number of cases was increasing at a rate of 56306 cases per year.

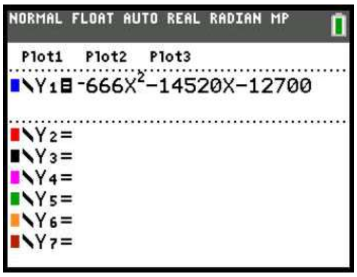
(d) To find the new cases reported in 1990 we need to subtract the cumulative cases at the beginning of 1990 from the ones at the beginning of 1991:

$$\begin{aligned} C(8) - C(7) &= -222 \cdot 8^3 + 7260 \cdot 8^2 - 12700 \cdot 8 + 13500 - (-222 \cdot 7^3 + 7260 \cdot 7^2 - 12700 \cdot 7 + 13500) \\ &= 262876 - 204194 \\ &= 58682 \end{aligned}$$

This is the average rate of change for 1990, whilst the answer of **(c)**is the instantaneous value at the beginning of 1990, so the values are different.

(e) Plotting the graph of $\dfrac{dC}{dt}$ we can see that it is negative for $0 < t < 0.913$ years and positive for $t > 0.913$ years. Therefore for most of 1983 the cumulative number of cases decreased, whilst after than they increased.





10. **(a)** $\dfrac{dP}{dt} = 17 \cdot 2 \cdot t - 3t^2 = 34t - 3t^2$

(b) $34t - 3t^2 > 0 \Rightarrow t(34 - 3t) > 0 \Rightarrow \dfrac{34}{3} < t \le 20$

The population of bacteria is increasing for t between $\dfrac{34}{3} = 11.3$ and 20 minutes and it is decreasing between 0 and 11.3 minutes.

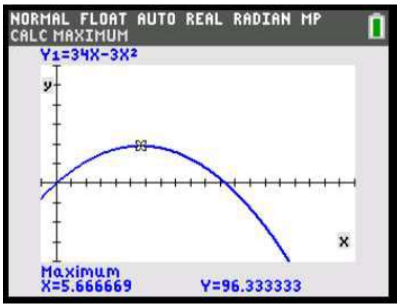
(c) $\dfrac{dP}{dt}(10) = 34 \cdot 10 - 3 \cdot 10^2 = 40$ bacteria per minute

(d) We want to find at what time $\frac{\mathrm{d}P}{\mathrm{d}t} = -165$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 165 \Rightarrow 34t - 3t^2 = -165 \Rightarrow 3t^2 - 34t - 165 = 0$$

$(3t+11)(t-15) = 0 \Rightarrow t = 15$ minutes (the other solution gives negative time)

(f) By plotting the derivative and finding the maximum, one gets that the rate of change is greatest for t = 5.67 min and its value is 96.3 bacteria per minute.



11. **(a)** $h'(t) = -9.8t + 16$, this represents the velocity of the tennis ball in $\mathrm{m\ s^{-1}}$.

(b) $h'(1) = -9.8 \cdot 1 + 16 = 6.2\ \mathrm{m\ s^{-1}}$

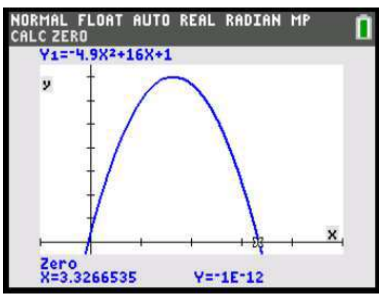
(c) Descending means that the velocity is negative, so we get:

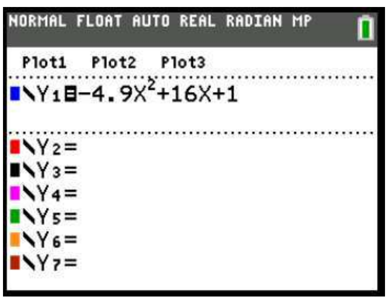
$$-9.8t + 16 = -10 \Rightarrow -9.8t = -26 \Rightarrow t = \frac{-26}{-9.8} = 2.65\ s$$

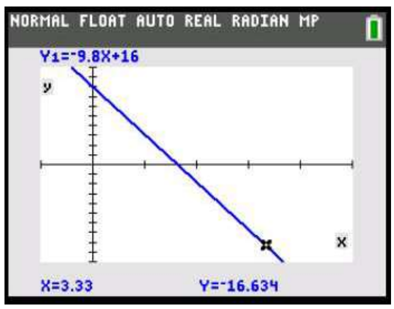
(d) $$-9.8t + 16 = 0 \Rightarrow -9.8t = -16 \Rightarrow t = \frac{-16}{-9.8} = 1.63\ s$$

This is the time at which the ball reaches its maximum height, before it starts to fall.

(e) The equations we are using are valid only before the ball hits the ground. To find the time when the ball hits the ground we can plot $h(t)$ and find the time when $h(t)=0$, *which gives* t = 3.33.







Looking at the graph of $h'(t)$, one can see that the maximum speed (which is the absolute value of the velocity) occurs at t = 3.33 and its value is 16.6 $\mathrm{m\ s^{-1}}$.

Exercise 13.4

1. **(a)** (Chain Rule) $y = u^4, u = (3x-8) \Rightarrow \frac{dy}{dx} = 4u^3 \cdot 3 = 12(3x-8)^3$

(b) (Chain Rule) $y = u^{\frac{1}{2}}, u = 1-x \Rightarrow \frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot (-1) = -\frac{1}{2}(1-x)^{-\frac{1}{2}} = -\frac{1}{2\sqrt{1-x}}$

(c) (Chain Rule) $y = \frac{3}{u} = 3u^{-1}, u = x^3 \Rightarrow \frac{dy}{dx} = 3 \cdot (-1) \cdot u^{-2} \cdot 3x^2 = -\frac{9}{x^6} \cdot x^2 = -\frac{9}{x^4}$

(d) $y = \frac{x^2}{2} + \frac{1}{2x} = \frac{1}{2}x^2 + \frac{1}{2}x^{-1} \Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot 2 \cdot x + \frac{1}{2} \cdot (-1) \cdot x^{-2} = x - \frac{1}{2x^2}$

(e) (Chain Rule) $y = u^{-2}, u = (x^2+4) \Rightarrow \frac{dy}{dx} = -2 \cdot u^{-3} \cdot 2x = -\frac{4x}{(x^2+4)^3}$

(f) (Quotient rule) $\frac{dy}{dx} = \frac{1 \cdot (x+1) - 1 \cdot x}{(x+1)^2} = \frac{1}{(x+1)^2}$

(g) (Chain Rule) $y = u^{-\frac{1}{2}}, u = x+2 \Rightarrow \frac{dy}{dx} = -\frac{1}{2}u^{-\frac{3}{2}} \cdot (1) == -\frac{1}{2\sqrt{(x+2)^3}}$

(h) (Chain Rule) $y = u^3, u = (2x^2-1) \Rightarrow \frac{dy}{dx} = 3u^2 \cdot 4x = 12x(2x^2-1)^2$

(i) (Product rule) $y = x \cdot (1-x)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = x \cdot \frac{1}{2}(1-x)^{-\frac{1}{2}} \cdot (-1) + (1-x)^{\frac{1}{2}} \cdot 1$

$$= -\frac{x}{2\sqrt{1-x}} + \sqrt{1-x} = \frac{-x+2(1-x)}{2\sqrt{1-x}} = \frac{-3x+2}{2\sqrt{1-x}}$$

(j) (Chain Rule) $y = u^{-1}, u = (3x^2-5x+7) \Rightarrow \frac{dy}{dx} = -u^{-2} \cdot (6x-5) = -\frac{6x-5}{(3x^2-5x+7)^2}$

(k) (Chain Rule) $y = u^{\frac{1}{3}}, u = 2x+5 \Rightarrow \frac{dy}{dx} = \frac{1}{3}u^{-\frac{2}{3}} \cdot 2 = \frac{2}{3\sqrt[3]{(2x+5)^2}}$

(l) (Product + chain rule) $\frac{dy}{dx} = (2x-1)^3 \cdot 4x^3 + (x^4+1) \cdot 3 \cdot (2x-1)^2 \cdot 2$

$$= 2(2x-1)^2\left[2x^3(2x-1) + 3(x^4+1)\right]$$

$$= 2(2x-1)^2\left[7x^4 - 2x^3 + 3\right]$$

(m) (Chain Rule) $y = u^{\frac{1}{2}}, u = 3x^2 - 2 \Rightarrow \frac{dy}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \cdot 6x = \frac{3x}{\sqrt{3x^2-2}}$

(n) (Quotient rule) $\frac{dy}{dx} = \frac{2x \cdot (x+2) - 1 \cdot x^2}{(x+2)^2} = \frac{x^2+4x}{(x+2)^2}$

(o) (Quotient rule) $\frac{dy}{dx} = \frac{1 \cdot (x-1) - 1 \cdot (x+1)}{(x-1)^2} = -\frac{2}{(x-1)^2}$

2. **(a)** $y = x^{-4} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -4 \cdot x^{-5} = -\frac{4}{x^5}$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = -4 \cdot x^{-5} \Rightarrow \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -4 \cdot -5 \cdot x^{-6} = \frac{20}{x^6}$

3. **(a)** $(x-1)(4x^2+4x+1) = 4x^3+4x^2+x-4x^2-4x-1 = 4x^3-3x-1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 12x^2 - 3$$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = (x-1)\cdot 2(2x+1)\cdot 2 + (2x+1)^2 \cdot 1 = (x-1)(8x+4) + (2x+1)^2$

$$= 8x^2 - 4x - 4 + 4x^2 + 4x + 1 = 12x^2 - 3$$

4. **(a)** Using the quotient rule:

$$f'(x) = \frac{(2x-3)\cdot(x+1)^2 - 2(x+1)(x^2-3x+4)}{(x+1)^4}$$

$$= \frac{(2x-3)\cdot(x+1) - 2(x^2-3x+4)}{(x+1)^3}$$

$$= \frac{2x^2-x-3-2x^2+6x-8}{(x+1)^3} = \frac{5x-11}{(x+1)^3}$$

(b) Using the quotient rule again:

$$f''(x) = \frac{5\cdot(x+1)^3 - 3(x+1)^2\cdot(5x-11)}{(x+1)^6}$$

$$= \frac{5(x+1) - 3(5x-11)}{(x+1)^4}$$

$$= \frac{5x+5-15x+33}{(x+1)^4} = \frac{-10x+38}{(x+1)^4}$$

5. $f'(x) = \frac{1\cdot(x+a) - 1\cdot(x-a)}{(x+a)^2}$

$$= \frac{2a}{(x+a)^2}$$

$$f''(x) = \frac{0\cdot(x+a)^2 - 2a\cdot 2\cdot(x+a)}{(x+a)^4}$$

$$= -\frac{4a}{(x+a)^3}$$

6. Using the product rule:

$$y = x\cdot(x+1)^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = x\cdot\frac{1}{2}(x+1)^{-\frac{1}{2}} + (x+1)^{\frac{1}{2}}\cdot 1 =$$

$$= \frac{x}{2\sqrt{x+1}} + \sqrt{x+1} = \frac{x+2(x+1)}{2\sqrt{x+1}} = \frac{3x+2}{2\sqrt{x+1}}$$

7. We can see that $y = x^2(x^2-6)$ and the graphs of this function is negative for $-\sqrt{6} < x < \sqrt{6}$, and therefore it is negative for $0 < x < 1$, which is contained in the previous one.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 12x = 4x(x^2-3)$ is negative for $0 < x < \sqrt{3}$

and therefore in the interval requested.

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 12 = 12(x^2-1)$ which is also negative for $0 < x < 1$.

However, $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 24x$ is positive for all positive x and thus for $0 < x < 1$.

8. **(a)** Using Pythagoras' theorem we get:

$$h = \sqrt{(b+1)^2 + b^2} = \sqrt{2b^2 + 2b + 1}$$

(b) Using the chain rule:

$$h = \sqrt{u} = u^{\frac{1}{2}}, u = 2b^2 + 2b + 1 \Rightarrow \frac{\mathrm{d}h}{\mathrm{d}b} = \frac{1}{2}u^{-\frac{1}{2}}\cdot(4b+2) = \frac{2b+1}{\sqrt{2b^2+2b+1}}$$

Exercise 13.5

1. **(a)** (Product) $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2\cdot \mathrm{e}^x + \mathrm{e}^x\cdot 2x = x^2\mathrm{e}^x + \mathrm{e}^x 2x$

(b) (Chain) $y = \cos u, u = 4x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\sin u\cdot 4 = -4\sin(4x)$

(c) (Chain) $y = e^u, u = 1-2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = e^u\cdot(-2) = -2e^{1-2x}$

(d) (Quotient) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1\cdot(1+\sin x) - \cos x\cdot x}{(1+\sin x)^2} = \frac{1+\sin x - x\cos x}{(1+\sin x)^2}$

(e) (Quotient) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{e}^x\cdot x - 1\cdot \mathrm{e}^x}{x^2} = \frac{\mathrm{e}^x(x-1)}{x^2}$

(f) (Chain) $y = \frac{1}{2}u^2, u = \sin 2x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = u\cdot\cos 2x\cdot 2 = 2\sin 2x\cos 2x = \sin(4x)$

(g) (Quotient) $\frac{dy}{dx}=\frac{1\cdot(e^x-1)-e^x\cdot x}{(e^x-1)^2}=\frac{-xe^x+e^x-1}{(e^x-1)^2}$

(h) $y=\cos x\cdot\frac{\sin x}{\cos x}=\sin x\Rightarrow\frac{dy}{dx}=\cos x$

(i) (Chain) $y=u^{\frac{1}{2}}, u=3-\cos x\Rightarrow\frac{dy}{dx}=\frac{1}{2}u^{-\frac{1}{2}}\cdot(\sin x)=\frac{\sin x}{2\sqrt{3-\cos x}}$

2. **(a)** When $x=6e, y=\ln(2e)=1+\ln(2)$.

$y=\ln(x)-\ln(3)\Rightarrow\frac{dy}{dx}=\frac{1}{x}\Rightarrow\frac{dy}{dx}(6e)=\frac{1}{6e}$

Equation of the tangent:

$y-(1+\ln(2))=\frac{1}{6e}(x-6e)\Rightarrow y=\frac{1}{6e}x+\ln(2)$

(b) When $x=1, y=\ln(1^3+1)=\ln(2)$.

$\frac{dy}{dx}=\frac{1}{x^3+1}\cdot 3x^2=\frac{3x^2}{x^3+1}\Rightarrow\frac{dy}{dx}(1)=\frac{3}{2}$

Equation of the tangent:

$y-\ln(2)=\frac{3}{2}(x-1)\Rightarrow y=\frac{3}{2}x-\frac{3}{2}+\ln(2)$

(c) When $x=\frac{\pi}{4}, y=\ln(\frac{1}{\sqrt{2}})=-\frac{1}{2}\ln(2)$.

$\frac{dy}{dx}=\frac{1}{\sin x}\cdot\cos x=\cot x\Rightarrow\frac{dy}{dx}(\frac{\pi}{4})=1$

Equation of the tangent:

$y+\frac{1}{2}\ln(2)=1(x-\frac{\pi}{4})\Rightarrow y=x-\frac{\pi}{4}-\frac{1}{2}\ln(2)$

(d) When $x=0, y=\ln(1)=0$.

$\frac{dy}{dx}=\frac{1}{\sqrt{x+1}}\cdot\frac{1}{2\sqrt{x+1}}=\frac{1}{2(x+1)}\Rightarrow\frac{dy}{dx}(0)=\frac{1}{2}$

Equation of the tangent:

$y-0=\frac{1}{2}(x-0)\Rightarrow y=\frac{1}{2}x$

(e) When $x = \mathrm{e}, y = \mathrm{e} \cdot \ln(\mathrm{e}) - \mathrm{e} = 0$.

Using the product rule for the first component

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x \cdot \frac{1}{x} + 1 \cdot \ln(x) - 1 = \ln(x) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}(\mathrm{e}) = 1$$

Equation of the tangent:

$$y - 0 = (x - \mathrm{e}) \Rightarrow y = x - \mathrm{e}$$

(f) When $x = 3, y = \dfrac{1}{\ln(\mathrm{e})} = 1$.

Using the chaing rule with $y = u^{-1}, u = \ln(x)$

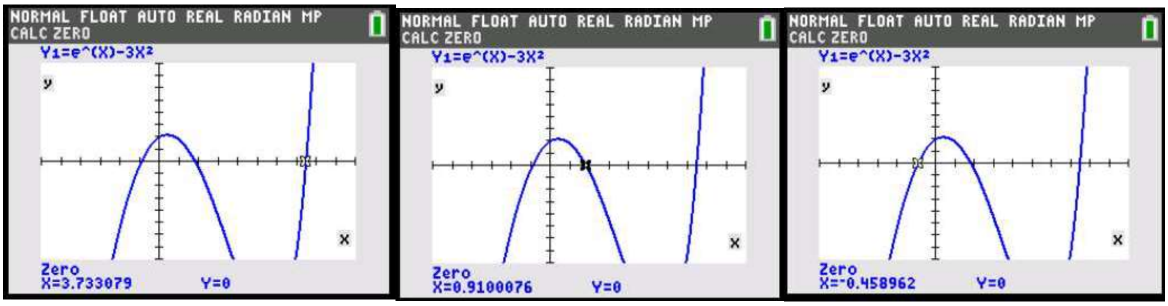
$$\frac{\mathrm{d}y}{\mathrm{d}x} = -u^{-2} \cdot \frac{1}{x} = -\frac{1}{(\ln(x))^2 x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}(\mathrm{e}) = -\frac{1}{(\ln(\mathrm{e}))^2 \mathrm{e}} = -\frac{1}{\mathrm{e}}$$

Equation of the tangent:

$$y - 1 = -\frac{1}{\mathrm{e}}(x - \mathrm{e}) \Rightarrow y = -\frac{1}{\mathrm{e}}x + 2$$

3. **(a)** $f'(x) = \mathrm{e}^x - 3x^2; f''(x) = \mathrm{e}^x - 6x$

(b) Using a GDC so solve $\mathrm{e}^x - 3x^2$ graphically, one gets $x = -0.459, 0.910, 3.73$



(c) Looking at the graph of the derivative above, one gets that the function is decreasing in the in the intervals $(-\infty, -0.459)$ and $(0.910, 3.73)$ and increasing in the intervals $(-0.459, 0.910)$ and $(3.73, \infty)$

4. **(a)** Let's first find the points in common:

$$\mathrm{e}^{-x} = \mathrm{e}^{-x}\cos x \Rightarrow 1 = \cos x \Rightarrow x = 2\pi k, k \in \mathbb{Z}$$

If the two curves are tangent to each other at these points, it means they have the same gradient. Looking at their derivatives at these points:

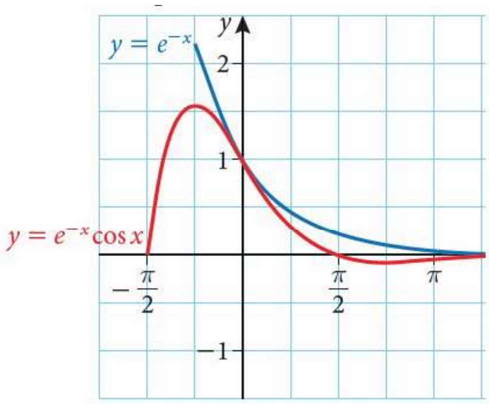
For the lhs: $y = \mathrm{e}^{-x}, y' = -\mathrm{e}^{-x}; y'(2\pi k) = -\mathrm{e}^{-2\pi k}$.

For the rhs: $y = \mathrm{e}^{-x}\cos x, y' = \mathrm{e}^{-x} \cdot (-\sin x) - \mathrm{e}^{-x}\cos x = -\mathrm{e}^{-x}(\sin x + \cos x);$

$$y'(2\pi k) = -\mathrm{e}^{-2\pi k}(\sin 2\pi k + \cos 2\pi k) = -\mathrm{e}^{-2\pi k}(0 + 1) = -\mathrm{e}^{-2\pi k}$$

So, at each intersection, the two curves have the same gradient, so they must be tangent.

(b)



5. The particle comes to rest when the velocity is equal to 0.

$$v(t) = \frac{ds}{st} - 4\sin t + 2\sin 2t = -4\sin t + 4\sin t\cos t = 4\sin t(\cos t - 1)$$

$4\sin t(\cos t - 1) = 0 \Rightarrow t = \pi$ (and later at 2π, 3π, etc.)

$$a(t) = \frac{dv}{dt} = -4\cos t + 4\cos(2t) \Rightarrow a(\pi) = -4\cos\pi + 4\cos 2\pi = 8 \text{ m s}^{-1}$$

6. $$h'(x) = \frac{2x \cdot e^x - e^x \cdot (x^2 - 3)}{e^{2x}} = \frac{2x - (x^2 - 3)}{e^x} = \frac{-x^2 + 2x + 3}{e^x}$$

7. **(a)** $\frac{dy}{dx} = -e^{-x}; \frac{d^2y}{dx^2} = -e^{-x} \cdot (-1) = e^{-x}$

(b) $\frac{dy}{dx} = \frac{1}{\frac{1}{x}} \cdot -\frac{1}{x^2} = -\frac{x}{x^2} = -\frac{1}{x}; \frac{d^2y}{dx^2} = -\left(-\frac{1}{x^2}\right) = \frac{1}{x^2}$

(c) $\frac{dy}{dx} = 2 \cdot \sin x \cdot \cos x = \sin 2x; \frac{d^2y}{dx^2} = \cos(2x) \cdot 2 = 2\cos 2x$

(d) $y = \sin u,\ u = x^2$

$$\frac{dy}{dx} = \cos u \cdot 2x$$
$$= 2x\cos x^2$$

$$\frac{d^2y}{dx^2} = 2x \cdot (-\sin x^2 \cdot 2x) + 2 \cdot \cos x^2$$
$$= -4x^2 \sin x^2 + 2\cos x^2$$

8. **(a)** $\sin\theta = \dfrac{a}{a+3} \Rightarrow \sin\theta(a+3) = a \Rightarrow 3\sin\theta = a - a\sin\theta$

$a(1-\sin\theta) = 3\sin\theta \Rightarrow a = \dfrac{3\sin\theta}{1-\sin\theta}$

(b) Using the quotient rule:

$$\frac{\mathrm{d}a}{\mathrm{d}\theta} = \frac{3\cos\theta\cdot(1-\sin\theta)-(-\cos\theta)\cdot 3\sin\theta}{(1-\sin\theta)^2}$$

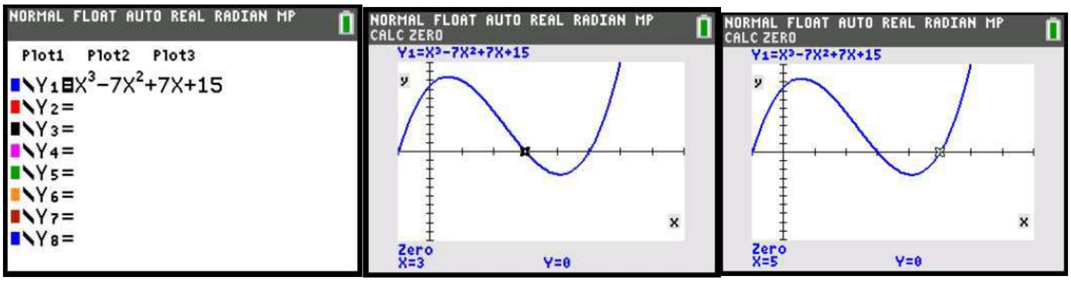
$$= \frac{3\cos\theta - 3\cos\theta\sin\theta + 3\cos\theta\sin\theta}{(1-\sin\theta)^2}$$

$$= \frac{3\cos\theta}{(1-\sin\theta)^2}$$

Chapter 13 practice questions

1. **(a)** We need to solve the equation $t^3 - 7t^2 + 7t + 21 = 6 \Rightarrow t^3 - 7t^2 + 7t + 15 = 0$

Using a GDC we find two solutions, $t = 3$ and $t = 5$.



(b) $v(t) = h'(t) = 3t^2 - 14t + 7$

(c) $3t^2 - 14t + 7 = 0 \Rightarrow t = \dfrac{14 \pm \sqrt{14^2 - 4 \cdot 3 \cdot 7}}{6} = 0.569$ or 4.10 s

(d) $a(t) = v'(t) = h''(t) = 6t - 14$

(e) **(i)** $a(0) = 6 \cdot 0 - 14 = -14$ cm s^{-2}

(ii) $a(5.5) = 6 \cdot 5.5 - 14 = 19$ cm s^{-2}

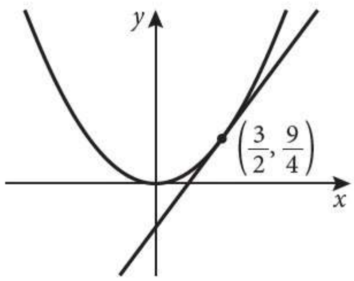
2. **(a)** $f'(x) = 2x \Rightarrow f'(1.5) = 3$

(b) $f(1.5) = 1.5^2 = \dfrac{9}{4}$

$$y - \frac{9}{4} = 3\left(x - \frac{3}{2}\right)$$

$$y = 3x - \frac{9}{4}$$

(c)



(d) x-intercept $\Rightarrow y = 0;\quad 0 = 3x - \dfrac{9}{4} \Rightarrow x = \dfrac{3}{4} \Rightarrow Q: \left(\dfrac{3}{4}, 0\right)$

y-intercept $\Rightarrow x = 0;\quad y = -\dfrac{9}{4} \Rightarrow R: \left(0, -\dfrac{9}{4}\right)$

(e) Midpoint of PR: $\left(\dfrac{\frac{3}{2}+0}{2}, \dfrac{\frac{9}{4}+\left(-\frac{9}{4}\right)}{2}\right) = \left(\dfrac{3}{4}, 0\right) \Rightarrow$ point Q

(f) $f'(a) = 2a \Rightarrow y - a^2 = 2a(x-a) \Rightarrow y = 2ax - a^2$

(g) Repeating what done for part **(d)** one gets $T:\left(\dfrac{a}{2}, 0\right)$; $U:\left(0, -a^2\right)$

(h) Midpoint of SU: $\left(\dfrac{a+0}{2}, \dfrac{a\ +\left(-a^2\right)}{2}\right) = \left(\dfrac{a}{2}, 0\right) \Rightarrow$ point T

3. $\dfrac{dy}{dx} = 3ax^2 - 4x - 1$

We know that $\dfrac{dy}{dx} = 3$ when $x = 2$, so:

$3 = 3a\cdot 2^2 - 4\cdot 2 - 1 \Rightarrow 3 = 12a - 8 - 1 \Rightarrow 12 = 12a \Rightarrow a = 1$

4. **(a)** **(i)** $x = 0$

(ii) $y = \lim\limits_{x\to\infty} \dfrac{3x-2}{x} = \lim\limits_{x\to\infty} \dfrac{3-\frac{2}{x}}{1} = 3$

(b) $y = 3 - \dfrac{2}{x} = 3 - 2x^{-1} \Rightarrow \dfrac{dy}{dx} = -2\cdot(-1)\cdot x^{-2} = \dfrac{2}{x^2}$

(c) The derivative is always positive, so, within the domain $x \in \mathbb{R}\setminus\{0\}$, the curve is always increasing.

5. The derivative of the function is $f'(x) = 2x - 3b$. From this we can find out b:

$0 = 2\cdot 3 - 3b \Rightarrow b = 2$

Then, to find c we can use $f(1)=0$:

$0 = 1 - 6 + c + 2 \Rightarrow c = 3$

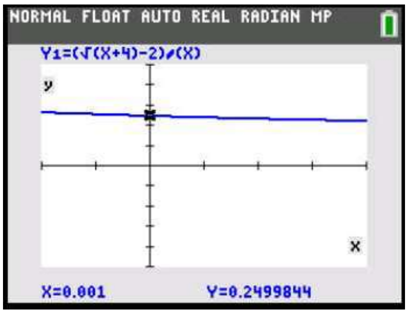
6. **(a)** $s'(t) = 10 - t \Rightarrow s'(0) = 10 \text{ m s}^{-1}$

(b) $10 - t = 0 \Rightarrow t = 10 \text{ s}$

(c) $s(10) = 10\cdot 10 - \dfrac{1}{2}\cdot 10^2 = 50 \text{ m}$

7. **(a)** $\lim_{x\to\infty}\dfrac{\frac{2}{x^2}-\frac{3}{x}+5}{\frac{8}{x^2}-3}=\dfrac{5}{-3}=-\dfrac{5}{3}$

(b) Using a GDC to graph the curve as it approaches 0 one sees that the limit is $\dfrac{1}{4}$



(c) $\lim_{x\to 1}\dfrac{x^3-1}{x-1}=\lim_{x\to 1}\dfrac{(x-1)(x^2+x+1)}{(x-1)}=\lim_{x\to 1}x^2+x+1=3$

(d) $\lim_{x\to 1^-}\ln(1-x)=\lim_{a\to 0^+}\ln a=-\infty$

8. **(a)** $f(x)=x^{\frac{3}{2}}-4x^{\frac{1}{2}}\Rightarrow f'(x)=\dfrac{3}{2}x^{\frac{1}{2}}-4\cdot\dfrac{1}{2}x^{-\frac{1}{2}}=\dfrac{3\sqrt{x}}{2}-\dfrac{4}{2\sqrt{x}}=\dfrac{3x-4}{2\sqrt{x}}$

(b) $f(x)=x^{-1}-\dfrac{1}{2}x\Rightarrow f'(x)=-x^{-2}-\dfrac{1}{2}=-\dfrac{1}{x^2}-\dfrac{1}{2}$

(c) $f(x)=\dfrac{7}{3}x^{-13}\Rightarrow f'(x)=\dfrac{7}{3}\cdot -13x^{-14}=-\dfrac{91}{3x^{14}}$

9. $v(t)=\dfrac{\mathrm{d}s}{st}=3t^2-18t+24;\quad a(t)=\dfrac{\mathrm{d}v}{\mathrm{d}t}=6t-18$

(a) $3t^2-18t+24=0\Rightarrow 3(t-2)(t-4)=0;\ t=2$ or $t=4$

$s(2)=2^3-9\cdot 2^2+24\cdot 2=20;\quad s(4)=4^3-9\cdot 4^2+24\cdot 4=16$

(b) $6t-18=0;\ \ t=3$

$s(3)=3^3-9\cdot 3^2+24\cdot 3=18$

10. **(a)** **(i)** $v(0)=66-66=0\text{ m s}^{-1}$

(ii) $v(10)=66-66\cdot e^{-1.5}\approx 51.3\text{ m s}^{-1}$

(b) **(i)** $a(t)=\dfrac{\mathrm{d}v}{\mathrm{d}t}=-66\cdot(-0.15)e^{-0.15t}=9.9e^{-0.15t}$

(ii) $a(0)=9.9e^0=9.9\ \text{ m s}^{-1}$

(c) **(i)** The exponential term tends to 0 so the velocity approaches 66 m s^{-1}

(ii) 0

(iii) If the velocity tends to a constant value, then there is less and less change in velocity, therefore the acceleration tends to 0 (the object reaches its terminal velocity)

11. **(a)** **(i)** $g'(x) = -3e^{-3x} = -\frac{3}{e^{3x}}$

(ii) the exponential is always positive, so $g'(x) = -\frac{3}{e^{3x}}$ is always negative and the function is always decreasing.

(b) **(i)** $y = g\left(-\frac{1}{3}\right) = 2 + \frac{1}{e^{-1}} = 2 + e$

(ii) $g'\left(-\frac{1}{3}\right) = -\frac{3}{e^{3\cdot -\frac{1}{3}}} = -3e$

12. **(a)** $y = u^{-2}; u = 2x+3 \Rightarrow y' = -2u^{-3}\cdot 2 = -\frac{4}{(2x+3)^3}$

(b) $y' = e^{\sin 5x}\cdot\cos 5x\cdot 5 = 5\cos(5x)e^{\sin(5x)}$

(c) $y' = 2\tan(x^2)\cdot\sec^2(x^2)\cdot 2x = 4x\tan(x^2)\sec^2(x^2)$

(d) $\frac{dy}{dx} = \frac{1\cdot(e^x-1) - e^x\cdot x}{(e^x-1)^2} = \frac{-xe^x + e^x - 1}{(e^x-1)^2}$

(e) $y' = e^x\cdot 2\cos 2x + e^x\cdot\sin(2x) = e^x(2\cos 2x + \sin 2x)$

(f) $y' = (x^2-1)\cdot\frac{1}{3x}\cdot 3 + \ln(3x)\cdot 2x = \frac{x^2-1}{x} + 2x\ln(3x)$

13. **(a)** E, because f is decreasing and the curvature is negative

(b) A, because f is decreasing and the curvature is positive

(c) C, because f is increasing and the curvature is negative

14. Using the product rule,

$$f'(x) = x^2\cdot\frac{1}{x} + \ln(x)\cdot 2x = x + 2x\ln(x)$$

15. $f'(x) = \frac{1}{2}\cos(2x)\cdot 2 - \sin(x) = \cos(2x) - \sin(x) = 1 - 2\sin^2(x) - \sin(x)$

$$1 - 2\sin^2(x) - \sin(x) = 0 \Rightarrow -(2\sin(x)-1)(\sin(x)+1) = 0$$

$$\Rightarrow \sin x = \frac{1}{2} \quad \text{or} \quad \sin x = -1$$

16. **(a)** $f'(x)=\frac{1}{2x-1}\cdot 2=\frac{2}{2x-1}$

(b) $\frac{2}{2x-1}=x \Rightarrow 2=2x^2-x \Rightarrow 2x^2-x-2=0$

$x=\frac{1+\sqrt{1-4\cdot 2\cdot(-2)}}{4}=\frac{1+\sqrt{17}}{4}\cong 1.28$

17. **(a)** $\frac{\mathrm{d}y}{\mathrm{d}x}=\sec^2(x)-8\cos(x)$

(b) $\sec^2(x)-8\cos(x)=0$

$\Rightarrow \frac{1}{\cos^2(x)}-8\cos(x)=0$

$\Rightarrow \frac{1}{\cos^2(x)}=8\cos(x)$

$\Rightarrow \frac{1}{8}=\cos^3(x)$

$\Rightarrow \cos(x)=\frac{1}{2}$

18. Using the product rule for the second term:

$\frac{\mathrm{d}y}{\mathrm{d}x}=k\cos(kx)-\left(kx\cdot(-k\sin kx)-k\cdot\cos(kx)\right)=k^2x\sin(kx)$

19. $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{2x-1}=2\left(2x-1\right)^{-1};\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2\cdot(-1)\left(2x-1\right)^{-2}\cdot 2=-\frac{4}{\left(2x-1\right)^2}$

20. **(a)** $h(0)=650, \quad h(20)=2\cdot h(0)=1300$

$1300=650\mathrm{e}^{20k} \Rightarrow \frac{1300}{650}=\mathrm{e}^{20k} \Rightarrow 2=\mathrm{e}^{20k}$

$\ln(2)=20k \Rightarrow k=\frac{\ln(2)}{20}=0.0347$

(b) $h(t)=650\mathrm{e}^{0.0347t} \Rightarrow h'(t)=650\cdot 0.0347\mathrm{e}^{0.0347t}$

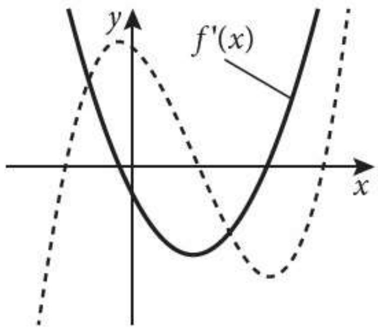
$h'(90)=650\cdot 0.0347\mathrm{e}^{0.0347\cdot 90}=512 \text{ bacteria min}^{-1}$

21. **(a)** $g'(x)=\frac{2\cdot\left(x^2+6\right)-2x\cdot 2x}{\left(x^2+6\right)^2}=\frac{2x^2+12-4x^2}{\left(x^2+6\right)^2}=\frac{12-2x^2}{\left(x^2+6\right)^2}$

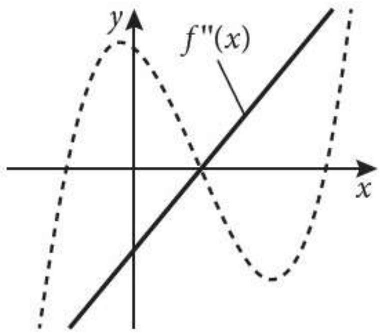
(b) When the derivative changes sign the function has a maximum or a minimum, and this makes the function many-to-one and therefore the inverse does not exist. Since for large value of x the derivative is negative, we want to find the smallest b such that the derivative is negative for $x>b$. To find this we need to find the biggest value for which $g'(x)=0$.

$$\frac{12-2b^2}{\left(b^2+6\right)^2}=0 \Rightarrow 12-2b^2=0 \Rightarrow b=\sqrt{6}$$

22. **(a)** Looking at the graph of the function *f* we predict that the derivative is equal to zero when *f* has a maximum and a miminum, and it's negative between these two. Therefore, the graph will look like:



(b) The second derivative will have only one zero, in correspondence of the point of inflection of *f*. The graph will look like:



23. **(a)** $c'(x)=6x^2-24x+30$

(b) Since the units for x is hundreds of caps, for 100 caps $x = 1$

$c'(1)=6\cdot 1^2-24\cdot 1+30=12$,

which corresponds to 12000 THB per hundreds of caps.

This means that the marginal cost after producing 100 caps is 12000 THB.

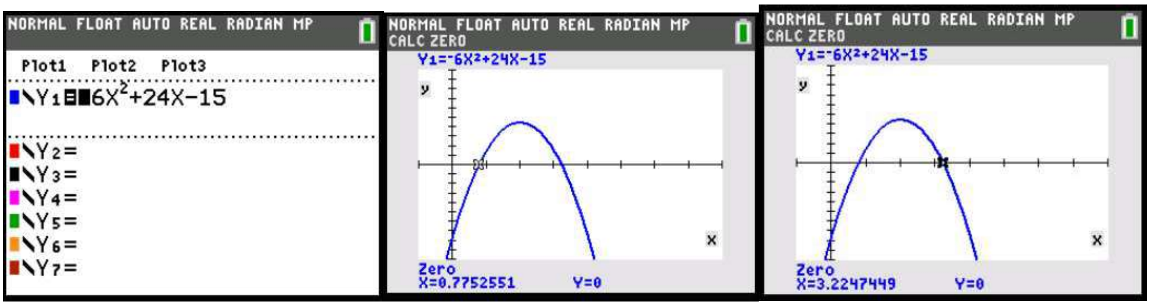
(c) $c'(x)=6x^2-24x+30=6(x^2-4x+5)=6\left(\left(x-2\right)^2+1\right)>0$ for all $x\in\mathbb{R}$

Since the derivative of the cost function is always positive, the cost function is always increasing.

(d) $p(x)=r(x)-c(x)=15x-\left(2x^3-12x^2+30x\right)=-2x^3+12x^2-15x$

(e) $p^{'}(x) = -6x^2 + 24x - 15$

(f) Plotting the derivative $p^{'}(x)$ with a GDC we see that it is positive for $0.775 < x < 3.22$, the function $p(x)$ is increasing in this interval and decreasing elsewhere.



(g) The marginal profit $p^{'}(x)$ is equal to zero at $x = 0.775$ or at $x = 3.22$.

Checking the value of $p(x)$ for these two we get $p(0.755) = -5.35$ and $p(3.22) = 9.35$.

Therefore the optimum production level happens for $x = 3.22$ (322 caps) and the expected profit is 9350 THB.

24. **(a)** $Q(10) = 300 \cdot (20-10)^2 = 30000, \quad Q(0) = 300 \cdot (20-0)^2 = 120000$

$$\text{Average rate} = \frac{Q(10)-Q(0)}{10} = \frac{30000-120000}{10} = -9000 \text{ l min}^{-1}$$

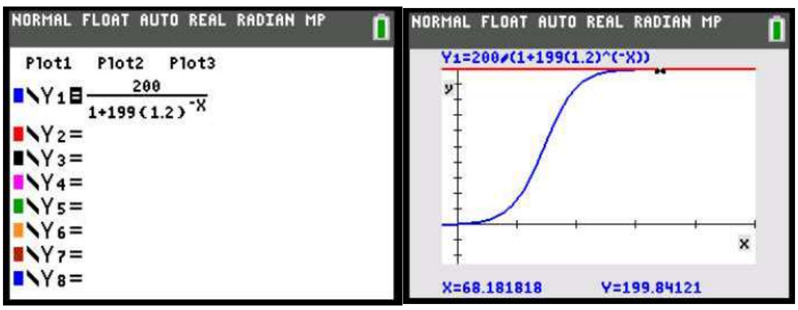
(b) $Q(t) = 300 \cdot \left(400 - 40t + t^2\right) = 120000 - 12000t + 300t^2$

$Q^{'}(t) = -12000 + 600t \Rightarrow Q^{'}(10) = -12000 + 600 \cdot 10 = -6000 \text{ l min}^{-1}$

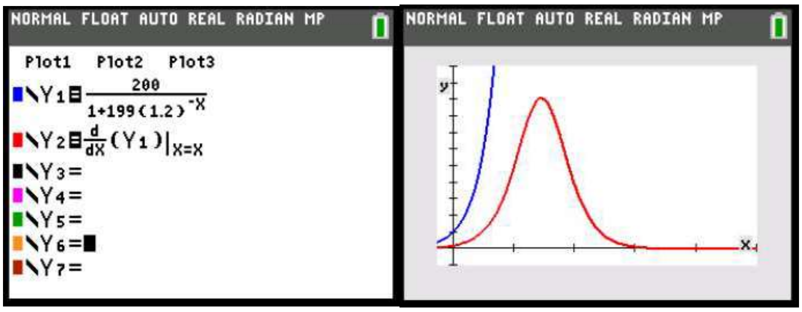
(c) $Q^{'}(0) = -12000, Q^{'}(10) = -6000, Q^{'}(20) = 0$

The rate of change is increasing with time, and this means that the rate at which the water is draining is slowing down.

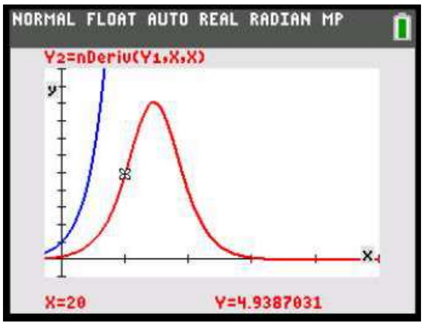
25. **(a)** Using a GDC, you can see that the graph is always increasing for $t > 0$ and it approaches 200 as t increases, so that's the maximum number of students infected.



(b)

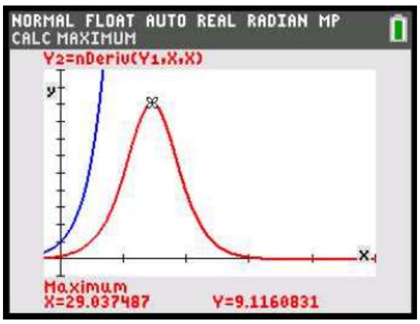


(c) Using a GDC to find the value of $\frac{\mathrm{d}Y_1}{\mathrm{d}x}(20)$ one gets 4.94 students per day

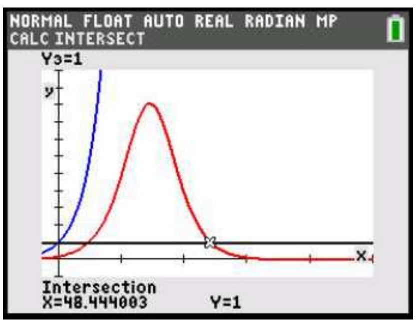


(d) $\frac{P(20)-P(0)}{20}=\frac{32.3-1}{20}=1.57$ students per day

(e) Using a GDC to find the maximum one gets 9.12 students per day on day 29



(f) We need to plot the line $y=1$ and find its intersection with the graph of the derivative. This gives that the infection rate drops to less than 1 during day 49.



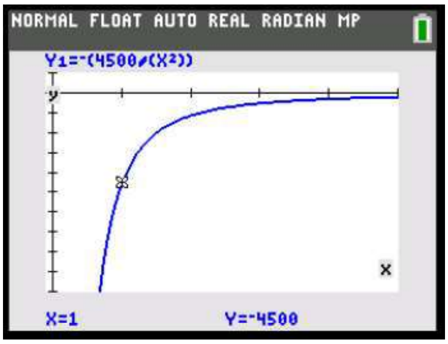
26. **(a)** $M(t) = 4500t^{-1} + 750 \Rightarrow M'(t) = -4500t^{-2} = -\frac{4500}{t^2}$

(b) $M'(t) = -\frac{4500}{t^2}$ which is negative for all the values of t.

Thus $M(t)$ is a decreasing function.

(c) $M'(t) = -1000 \Rightarrow -\frac{4500}{t^2} = -1000 \Rightarrow \frac{-4500}{-1000} = t^2 \Rightarrow t^2 = 4.5 \Rightarrow t = 2.12$ years

(d) From the graph of $M'(t)$, the most negative value within the domain $t \geq 1$ is reached at $t = 1$



(e) $-\frac{4500}{t^2} > -50 \Rightarrow -4500 > -50t^2 \Rightarrow \frac{-4500}{-50} < t^2$

$t^2 > 90 \Rightarrow t = 9.49$ years

$M(9.49) = \frac{4500}{9.49} + 750 = 1224.18 = \$1220$ (to 3 s.f.)

27. **(a)** $P'(n) = -6n^2 + 12n + 1 \Rightarrow -6n^2 + 12n + 1 = 0$

$n = \frac{-12 \pm \sqrt{12^2 - 4\cdot(-6)\cdot 1}}{-12} \Rightarrow n = -0.0801$ or 2.08

Looking at the two roots and considering that n is a positive quantity, $P'(n) > 0$ in the interval $0 \leq n < 2.08$, an in this interval the profit is increasing.

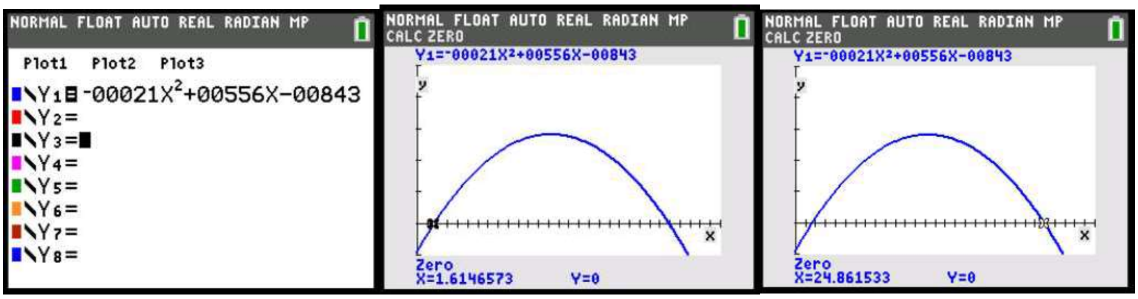
(b) Considering what we found in **(a)** $P'(n)$ changes from positive to negative at $n = 2.08$, so the optimum production level is 2080 t-shirts and the expected profit at this level is

$P(2.08) = \$10,000$

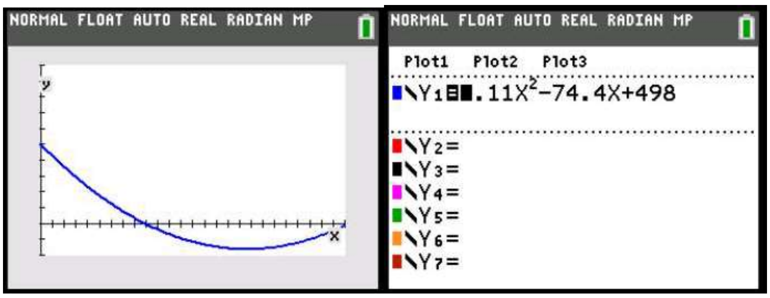
28. **(a)** $E'(t) = -0.0007 \cdot 3 \cdot t^2 + 0.0278 \cdot 2 \cdot t - 0.0843 = -0.0021t^2 + 0.0556t - 0.0843$

(b) We need to use a GDC to find when $E'(t)$ is positive.

So a=1972 and b=1995.



29. **(a)** Using a GDC to plot $S'(t)$ we can see that CD sales increased sharply at the beginning, kept increasing for the first 9 year at a lower rate of change and then started decreasing from about 1999.

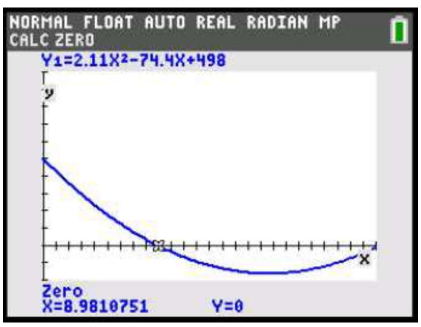


(b) Use GDC to find the intersection between $y = S'(t)$ and $y = 250$.

This gives t = 3 and therefore the sales increase by 250 million in 1993.

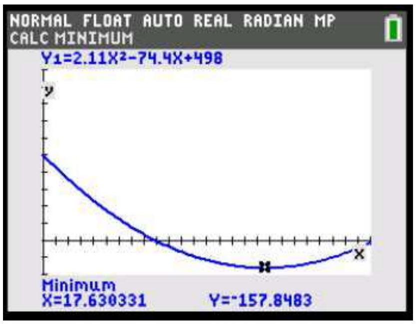
(c) The sales begin to decline when $S'(t)$ changes from positive to negative.

A GDC gives t = 8.98, so this happened at the very end of 1998.



(d) To answer this question we need to find the minimum of $S'(t)$.

A GDC gives t = 17.6, so the biggest decrease is in 2007 and the sales decrease by 158 million.



(e) Considering the graph in **(a)** the sales initially increase and have a peak around the 9th year. The only graph with these features is A.

30. **(a)** $A = \dfrac{8.84}{2} = 4.42 \text{ cm}$

(b)

$$v(t) = \frac{\mathrm{d}s}{\mathrm{d}t} = -4.42\sin\left(\frac{\pi}{30}bt\right)\cdot\frac{\pi}{30}b = -0.463b\sin\left(\frac{\pi}{30}bt\right)$$

$$a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -0.463b\cos\left(\frac{\pi}{30}bt\right)\cdot\frac{\pi}{30}b = -0.0485b^2\cos\left(\frac{\pi}{30}bt\right)$$

(c) Max for both quantities is when cos/sin are equal to 1.

$v_{\max} = -0.463\cdot 550 = 255 \text{ cm s}^{-1}$

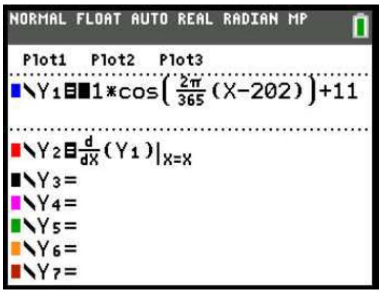
$a_{\max} = -0.0485\cdot 550^2 = 14700 \text{ cm s}^{-2}$

(d) $a \propto b^2$ so if b increases by a factor m, then a increases by a factor m^2.

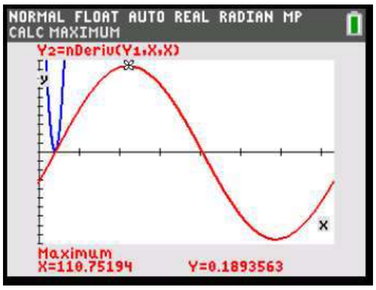
Thus $k = m^2$

(e) Taking into account the relation found in **(d)**, large RPMs means that the engine has to handle significantly high accelerations (and hence forces). And since the maximum acceleration increases as the square of the RPMs, as the RPMs grow the acceleration grow much quicker.

31. In this question we can use a GDC to plot the function and its derivative to answer all the questions.



(a) **(i)** using a GDC to find the maximum of $T'(d)$ we get d=110.75=111th day



(ii) To find what month this is we can do $\frac{110}{30} = 3.7$ which corresponds to the 4th month April, and therefore mid spring

(iii) $T(112) - T(111) = 0.189 = 0.2$ °C (to the nearest tenth)

(b) **(i)** Use the DC to find the minimum. This gives d =293.25 = 294$^{\text{th}}$ day.

(ii) $\frac{294}{30} = 9.8$, so it's the 10$^{\text{th}}$ month, late October, in the middle of the autumn

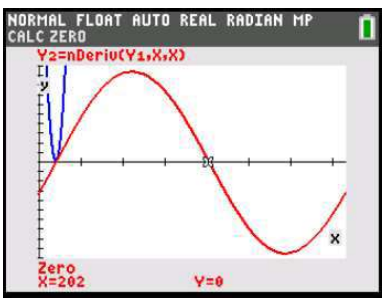
(iii) $T(295) - T(294) = -0.189 = -0.2$ °C (to the nearest tenth)

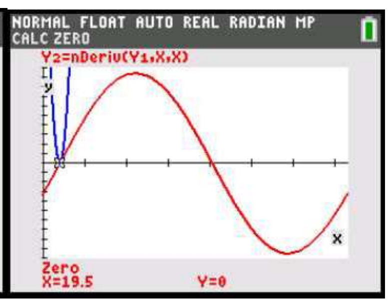
(c) **(i)** This happens when $T'(d) = 0$.

Again, using a GDC we find d =19.5 (20th day) and *d*=202

(see graphs below)

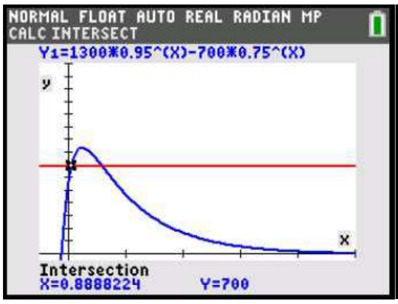
(ii) The first one is mid January (winter) and the second is in the 7th month (July, mid-summer)

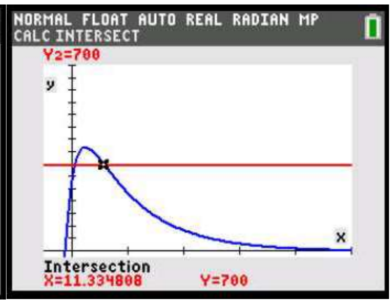




32. $V = 20\pi x^2 - \frac{\pi}{3}x^3 \Rightarrow \frac{dV}{dx} = 40\pi x - \pi x^2 = \pi x(40 - x)$ cm^3 per cm of depth

33. **(a)** Using a GDC we find that the level is dangerous in the interval $0.889 < n < 11.3$, i.e. from late in the first year until the beginning of the 12th year.

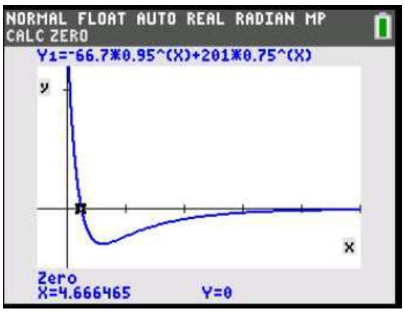




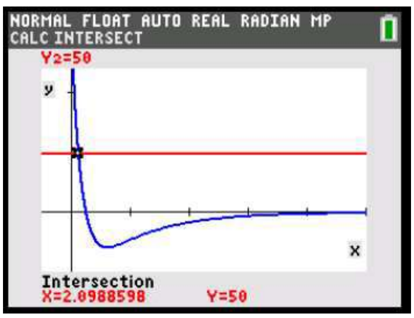
(b) $\frac{dC}{dn} = 1300 \cdot (0.95)^n \cdot \ln(0.95) - 700 \cdot (0.75)^n \cdot \ln(0.75)$

$= -66.7(0.95)^n + 201(0.75)^n$

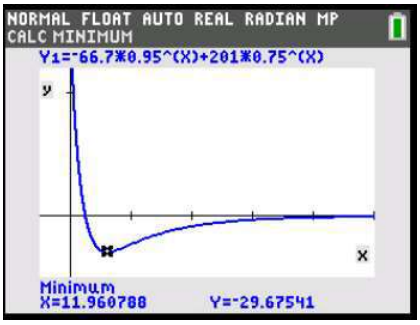
(c) Using the graph of $\frac{dC}{dn}$ we can see that concentration is increasing for $0 \le n < 4.67$ and decreasing for $n > 4.67$



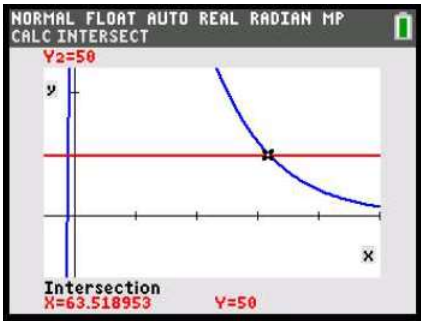
(d) From a GDC we get $0 \le n < 2.10$, thus from the beginning until early in the 3rd year.



(e) Using the GDC to find the minimum of the derivative, one gets n = 11.9, so late in the 12th Year



(f) **(i)** From the intersection between $C(n)$ and $n = 50$, we obtain $n = 63.5$ (64th year)



(ii) Because $\frac{\mathrm{d}C}{\mathrm{d}n}$ is negative for all values of $n > 63.5$ therefore the concentration will keep decreasing.

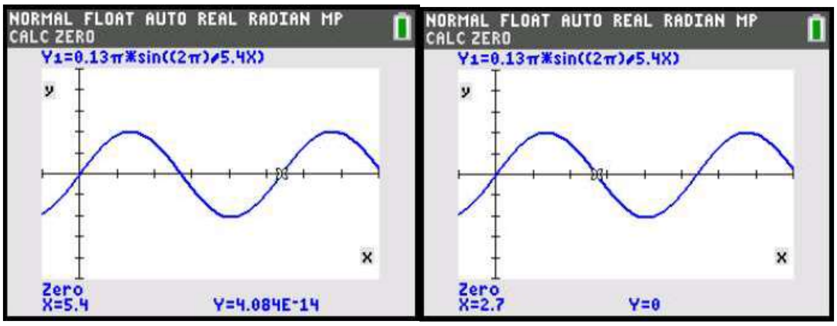
34. **(a)** $|a| = \frac{4.35 - 3.65}{2} = 0.35 \Rightarrow a = -0.35$ (since we want a minimum at $t = 0$)

$b = \frac{2\pi}{T} = \frac{2\pi}{5.4}$; $c = \frac{4.35 + 3.65}{2} = 4$

$M(t) = -0.35\cos\left(\frac{2\pi}{5.4}t\right) + 4$

(b) $\frac{\mathrm{d}M}{\mathrm{d}t} = 0.35\sin\left(\frac{2\pi}{5.4}t\right) \cdot \frac{2\pi}{5.4} = 0.130\pi\sin\left(\frac{2\pi}{5.4}t\right)$

(c) Plotting $\frac{\mathrm{d}M}{\mathrm{d}t}$ on a GDC, we can find that the derivative is negative in the interval $2.7 < t < 5.4$. Therefore the function is decreasing in this interval and increasing in the intervals $0 < t < 2.7$ and $5.4 < t < 7$.



35. **(a)** $\frac{\mathrm{d}C}{\mathrm{d}t} = 3 - \frac{12}{\pi} \cdot -\sin\left(\frac{\pi}{12}t\right) \cdot \frac{\pi}{12} = 3 + \sin\left(\frac{\pi}{12}t\right)$

(b) $3 + \sin\left(\frac{\pi}{12}t\right) > 3.5 \Rightarrow \sin\left(\frac{\pi}{12}t\right) > 0.5 \Rightarrow \frac{\pi}{6} < \frac{\pi}{12}t < \frac{5\pi}{6} \Rightarrow 2 < t < 10$

(c) The number of chirps per hour is maximum when sin=1, so when $\frac{\pi}{12}t = \frac{\pi}{2}$.

This happens when $t = 6$ (6 hours after sunset) and the number of chirps per hour are 4000.

Exercise 14.1

1. **(a)** $\frac{dy}{dx} = 2x - 2$

$\frac{dy}{dx} = 0 \Leftrightarrow 2x - 2 = 0 \Leftrightarrow x = 1$

When $x = 1$, $y = (1)^2 - 2(1) - 6 = -7$ so vertex at $(1, -7)$

(b) $\frac{dy}{dx} = 8x + 12$

$\frac{dy}{dx} = 0 \Leftrightarrow 8x + 12 = 0 \Leftrightarrow x = -1.5$

When $x = -1.5$, $y = 4(-1.5)^2 + 12(-1.5) + 17 = 8$ so vertex at $(-1.5, 8)$

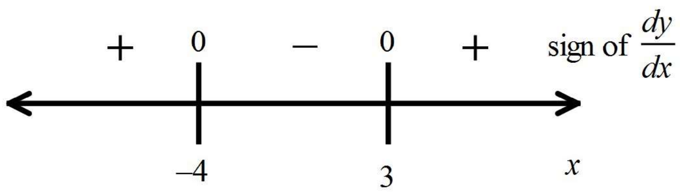
(c) $\frac{dy}{dx} = -2x + 6$

$\frac{dy}{dx} = 0 \Leftrightarrow -2x + 6 = 0 \Leftrightarrow x = 3$

When $x = 3$, $y = -(3)^2 + 6(3) - 7 = 2$ so vertex at $(3, 2)$

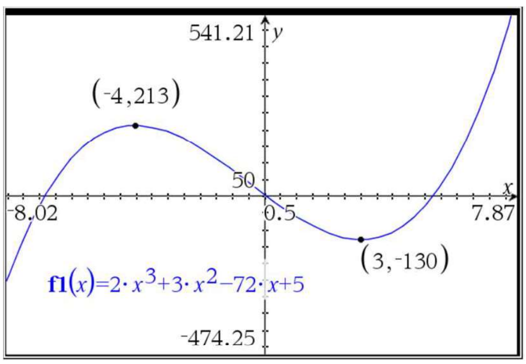
2. **(a)** **(i)** $\frac{dy}{dx} = 6x^2 + 6x - 72 = 0 \Rightarrow (-4, 213), (3, -130)$ are stationary points

(ii) Sign diagram:



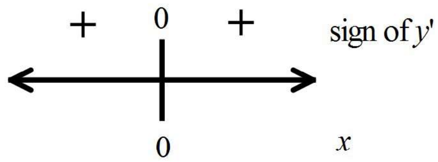
Therefore $(-4, 213)$ is local maximum and $(3, -130)$ is a local minimum.

(iii)



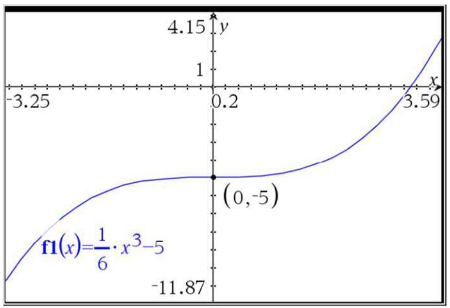
(b) **(i)** $\frac{dy}{dx} = \frac{1}{2}x^2 = 0 \Rightarrow (0, -5)$ is stationary

(ii) Sign diagram:



Therefore $(0, 0)$ is neither a minimum nor a maximum.

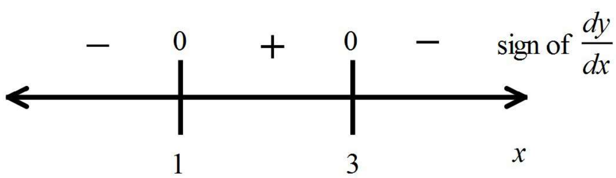
(iii)



(c) **(i)** First re-arrange (or use the product rule) to get: $y = -x^3 + 6x^2 - 9x$

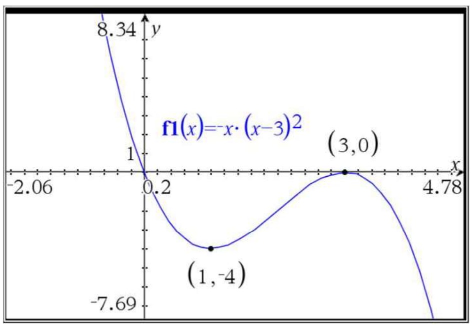
Then $\frac{dy}{dx} = -3x^2 + 12x - 9 = 0 \Rightarrow (1, -4), (3, 0)$ are stationary points

(ii) Sign diagram:



Therefore $(1, -4)$ is a local minimum and $(3, 0)$ is a local maximum.

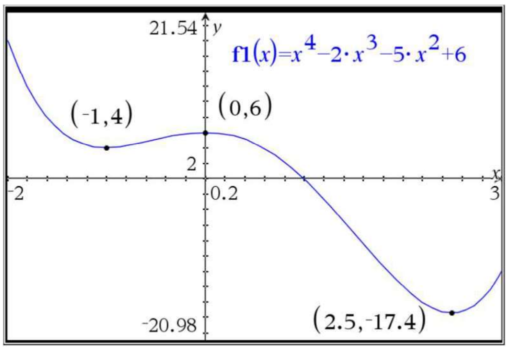
(iii)



(d) **(i)** $\frac{dy}{dx} = 4x^3 - 6x^2 - 10x = 0 \Rightarrow (-1,4), (0,6), \left(\frac{5}{2}, -\frac{279}{16}\right)$ are stationary points

(ii) $(-1,4)$ is a local minimum since $f'(x)$ changes from negative to positive at $x = -1$; $(0,6)$ is a local maximum since $f'(x)$ changes from positive to negative at $x = 0$; $\left(\frac{5}{2}, -\frac{279}{16}\right)$ is a local minimum since $f'(x)$ changes from negative to positive at $x = \frac{5}{2}$.

(iii)

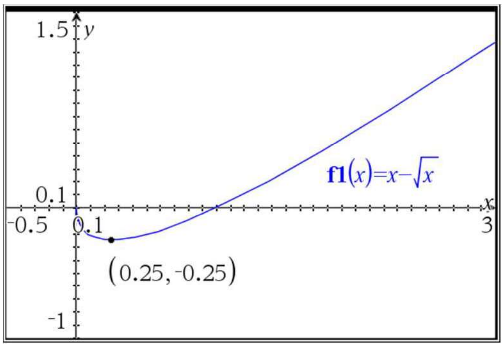


(e) **(i)** $y = x - x^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{2}x^{-\frac{1}{2}} = 1 - \frac{1}{2\sqrt{x}} = 0 \Rightarrow x = \frac{1}{4}$,

$\therefore$ stationary point t $\left(\frac{1}{4}, -\frac{1}{4}\right)$

(ii) $\frac{dy}{dx}$ changes sign from - to + at $x = \frac{1}{4}$ so $\left(\frac{1}{4}, -\frac{1}{4}\right)$ is a local minimum.

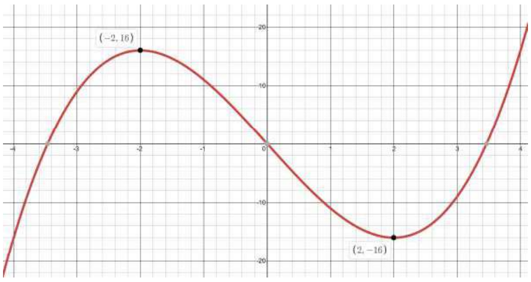
(iii)



3. **(a)** $f'(x) = 3x^2 - 12 = 0 \Rightarrow x = \pm 2$

Using software or GDC

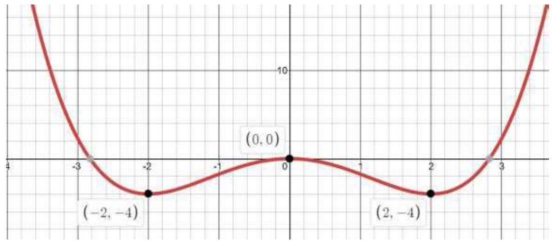
Local max $(-2, 16)$; local min $(2, -16)$



(b) $f'(x) = x^3 - 4x \Rightarrow x = 0, \pm 2$

Using software or GDC

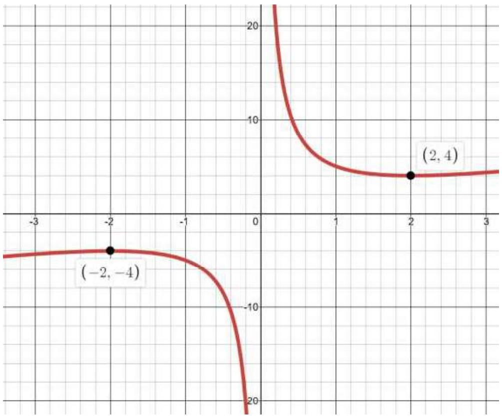
Local max $(0, 0)$, local min $(-2, -4)$ and $(2, -4)$



(c) $f'(x) = 1 - \dfrac{4}{x^2} = 0 \Rightarrow x = \pm 2$

Using Software or GDC

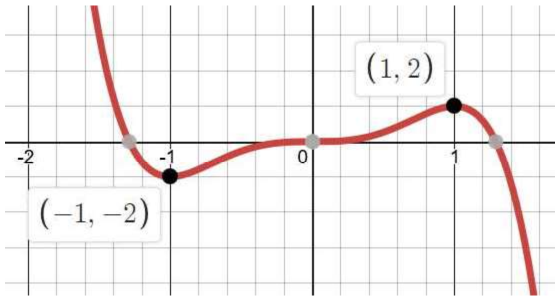
Local min $(2, 4)$, local max $(-2, -4)$



(d) $f'(x) = -15x^5 + 15x^2 = 0 \Rightarrow x = 0, \pm 1$

Using Software or GDC

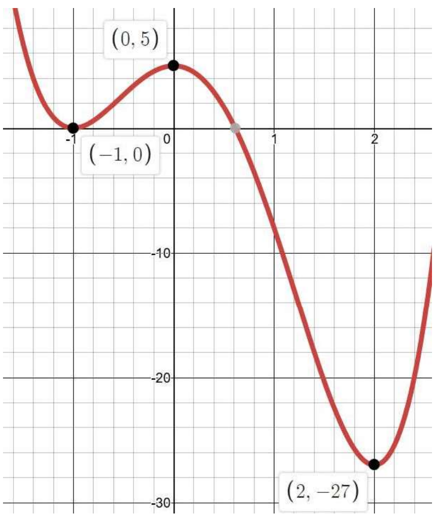
Local max $(1, 2)$, local min $(-1, -2)$



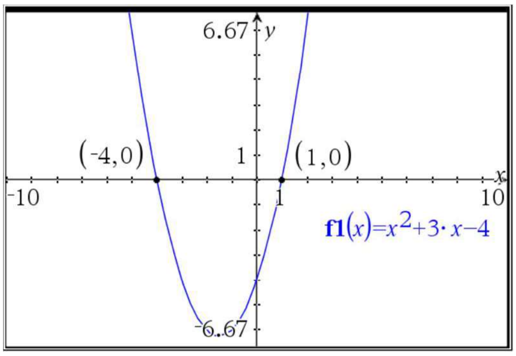
(e) $f'(x) = 12x^3 - 12x^2 - 24x = 0 \Rightarrow x = 0, -1, 2$

Using Software or GDC

Local max $(0, 5)$, local minima $(-1, 0)$ and $(2, -27)$

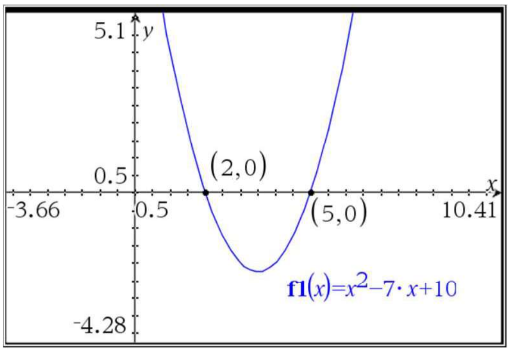


4. **(a)** **(i)**



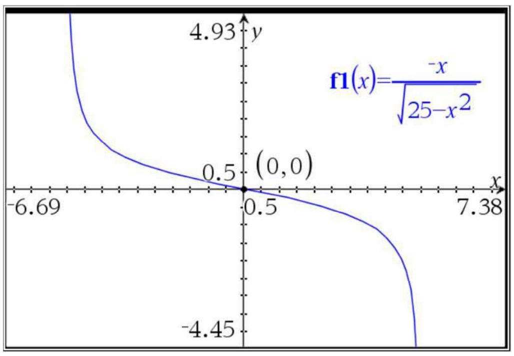
(ii) Local maximum at $x = -4$ since $f'(x)$ changes from positive to negative; Local minimum at $x = 1$ since $f'(x)$ changes from negative to positive.

(b) **(i)**



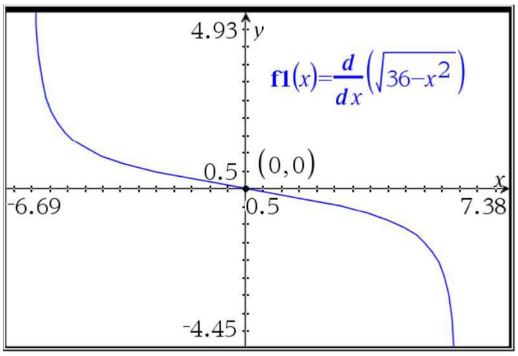
(ii) Local maximum at $x = 2$ since $f'(x)$ changes from positive to negative.
Local minimum at $x = 5$ since $f'(x)$ changes from negative to positive.

(c) **(i)**



(ii) Local maximum at $x = 0$ since $f'(x)$ changes from positive to negative.

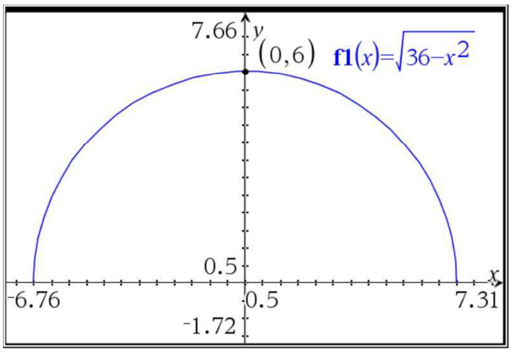
5. **(a)** **(i)**



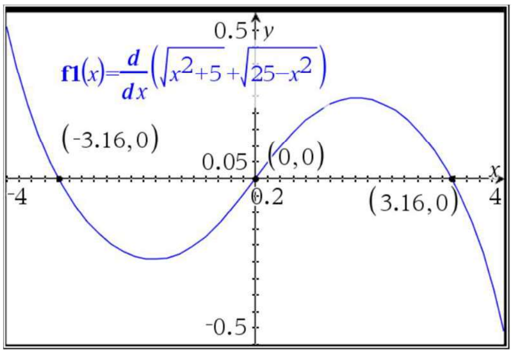
(ii) Local maximum at $x = 0$

Since $f'(x)$ changes from positive to negative; maximum is at $(0, 6)$

Graph of function:



(b) **(i)**



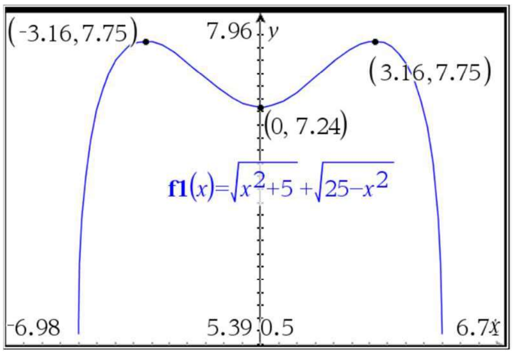
(ii)

Local maximum at $x=-3.16$ since $f'(x)$ changes from positive to negative;

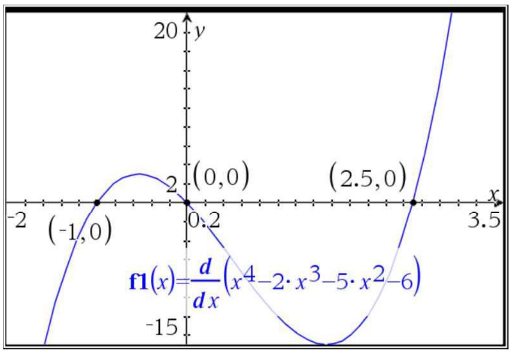
Local minimum at $x=0$ since $f'(x)$ changes from negative to positive;

Local maximum at $x=3.16$ since $f'(x)$ changes from positive to negative.

Graph of function:



(c) **(i)**



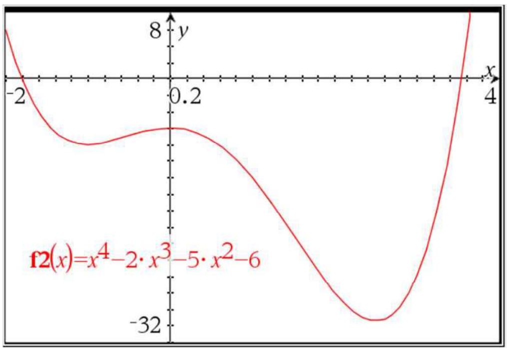
(ii)

Local minimum at $x=-1$ since $f'(x)$ changes from negative to positive;

Local maximum at $x=0$ since $f'(x)$ changes from positive to negative;

Local minimum at $x=2.5$ since $f'(x)$ changes from negative to positive.

Graph of function:



6. **(a)** **(i)** Increasing on $1 < x < 5$; decreasing elsewhere.

(ii) Local min at $x = 1$, local max at $x = 5$.

(b) **(i)** Increasing on $0 < x < 1$ or $3 < x < 5$, decreasing on $1 < x < 3$ or $x > 5$

(ii) Local maxima at $x = 1$ and $x = 5$; local min at $x = 3$

7. **(a)** $v(t) = 3t^2 - 8t + 1$; $a(t) = 6t - 8$

(b) Using GDC: max displacement is -6.88 m when $t = 2.54$ s $\left(t = \dfrac{4 + \sqrt{13}}{3} \right)$

(c) Using GDC: Min velocity $= -4.33\text{ m s}^{-1}$; $t = 1.33\text{ s}^{-1}$

(d) Starting from -6 m (at $t = 1$ s), the object moves toward the origin, passing the origin at $t = 0$ s, before reaching its maximum positive displacement at 0.131 s, then moving in the negative direction reaching minimum displacement of -6.88 m at 2.54 seconds, where it changes directions again.

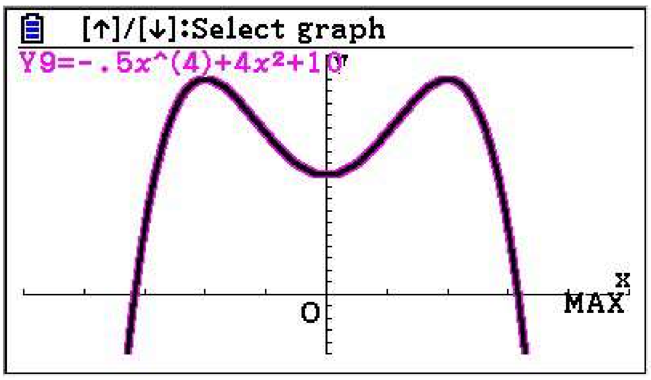
8. **(a)** $f'(x) = -2x^3 + 8x = 0 \Rightarrow -2x(x^2 - 4) = 0$ so $x = 0, 2, -2$

Check the changes of signs by substituting any values in the correct intervals, for example:

$f'(-3) = 30 > 0; f'(-1) = -6 < 0; f'(1) = 6 > 0; f'(3) = -30 < 0$

(alternatively, sketch the cubic to visualise the changes of signs)

So, Max at $x = -2, 2$, Min at $x = 0$
Substitute into $f(x)$ to find the corresponding y-values: $f(-2), f(0), f(2)$ to obtain: Max: $(-2, 18)$, $(2, 18)$; rel. Min: $(0, 10)$



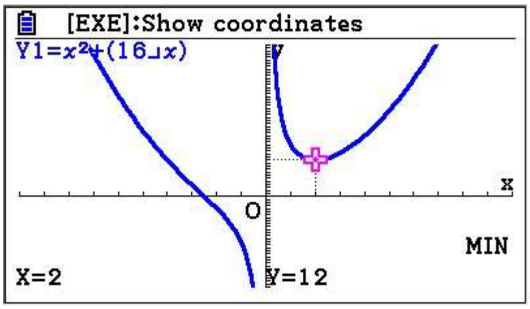
(b) $f'(x) = 2x - \dfrac{16}{x^2} = 0 \Rightarrow \dfrac{2x^3 - 16}{x^2} = 0$ so $x = 2$

Check the changes of signs by substituting any values in the correct intervals, for example:

$f'(1) = -14 < 0; f'(3) = \dfrac{38}{9} > 0$

So, it is a relative minimum at $x = 2$

Substitute into $f(x)$ to find the corresponding *y*-value: $f(2) = 12$ so min at $(2,\ 12)$

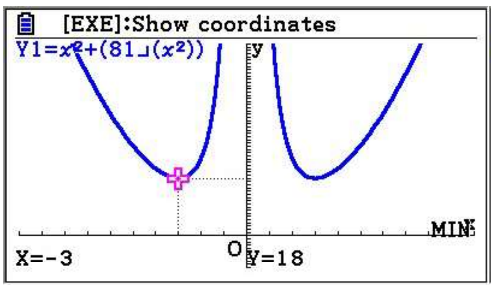


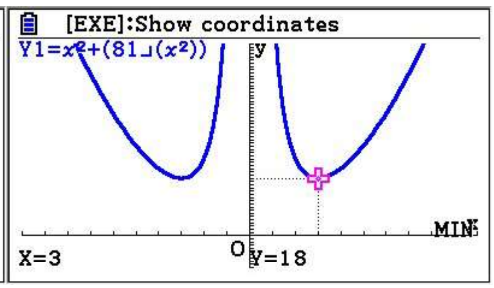
(c) $f'(x) = 2x - \dfrac{162}{x^3} = 0 \Rightarrow \dfrac{2x^4 - 162}{x^3} = 0$ so $x = -3, 3$

Check the changes of signs by substituting any values in the correct intervals (note that there $y = f'(x)$ has a discontinuity at $x = 0$), for example:

$f'(-4) = -5.469 < 0; f'(-1) = 160 < 0; f'(1) = -160 < 0; f'(4) = 5.469 > 0$

So, both Min at $x = -3$ and 3
Substitute into $f(x)$ to find the corresponding *y*-values: $f(-3), f(3)$ to obtain:
Min at $(-3,\ 18)$ and at $(3,\ 18)$





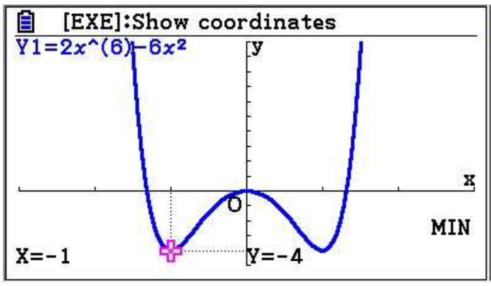
(d) $f'(x) = 12x^5 - 12x = 0 \Rightarrow 12x(x^4 - 1) = 0$ so $x = 0, 1, -1$

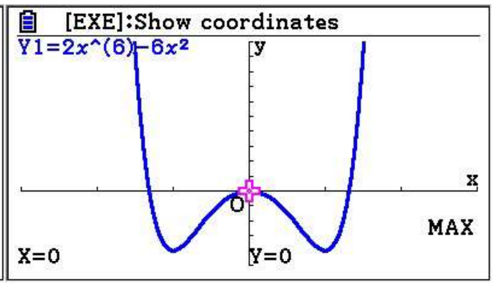
Check the changes of signs by substituting any values in the correct intervals, for example:

$f'(-2) = -360 < 0; f'(-0.5) = 5.625 > 0; f'(0.5) = -5.625 < 0; f'(2) = 360 > 0$

(alternatively, sketch the quintic to visualise the changes of signs)

So, Max at $x = 0$, Min at $x = -1$, or 1
Substitute into $f(x)$ to find the corresponding y-values: $f(-1), f(0), f(1)$ to obtain: Max: (0,0); Min: (−1, −4) and (1, −4)



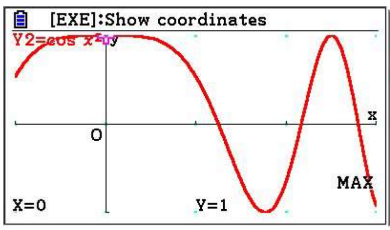


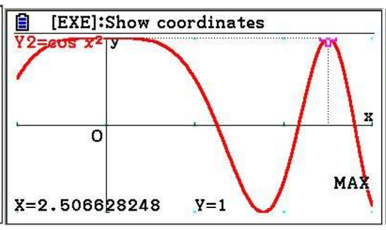
(e) $f'(x) = -2x\sin(x^2) = 0$ so $x = 0$ or $x^2 = 0, \pi$ so $x = \sqrt{\pi} \approx 1.77$ or $\sqrt{2\pi} \approx 2.51$
(other solutions are out of the domain)

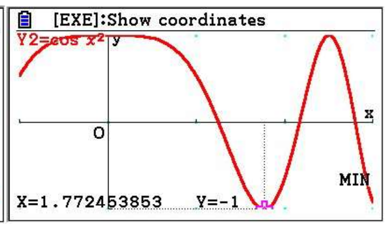
Check the changes of signs by substituting any values in the correct intervals, for example:

$f'(-1) = -1.491 < 0; f'(1) = 1.6829 > 0; f'(2) = -1.6829 < 0; f'(3) = -2.473 < 0$

So, Max at $x = 0, \sqrt{2\pi}$, Min at $x = \sqrt{\pi}$
Substitute into $f(x)$ to find the corresponding y-values:
$f(0), f(\sqrt{\pi}), f(\sqrt{2\pi})$ to obtain: Max: (0,1) and $\left(\sqrt{2\pi}, 1\right)$; Min: $\left(\sqrt{\pi}, -1\right)$







(f) $f'(x) = 1\mathrm{e}^x + x\mathrm{e}^x$ using the product rule.

$f'(x) = 0 \Rightarrow \mathrm{e}^x(1+x) = 0$ so $x = -1$

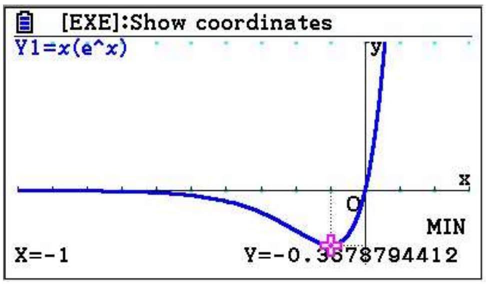
Check the changes of signs by substituting any values in the correct intervals, for example:

$f'(-2) = \mathrm{e}^{-2}(-1) < 0; f'(0) = \mathrm{e}^0(1) > 0$

So, it is a min at $x = -1$

Substitute into $f(x)$ to find the corresponding y-value: $f(-1) = (-1)\mathrm{e}^{-1} = \frac{-1}{\mathrm{e}}$

So, min at $\left(-1, \frac{-1}{\mathrm{e}}\right)$



(g) $f'(x) = 1\sin(x) + x\cos(x)$ using the product rule.

Solve $f'(x) = 0$ using GDC to obtain so $x = 0, 2.03, 4.91$

(other solutions are out of the domain)

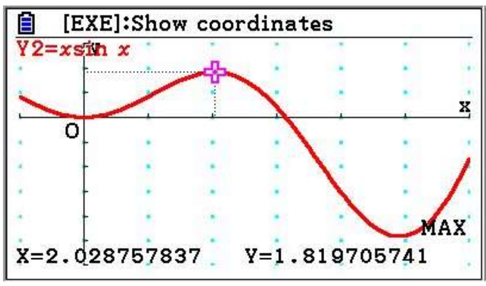
Check the changes of signs by substituting any values in the correct intervals, for example:

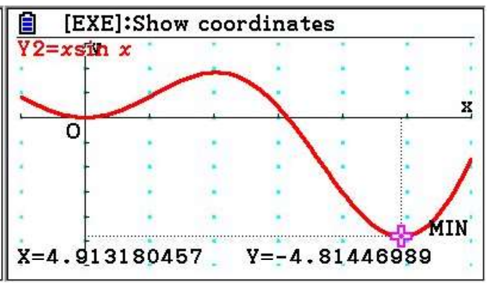
$f'(-1) = -1.382 < 0; f'(1) = 1.3818 > 0; f'(3) = -2.829 < 0; f'(5) = 0.4594 > 0$

so Max: $x = 2.03$; Min: $x = 0, 4.91$

Substitute into $f(x)$ to find the corresponding y-values:

$f(0), f(2.03), f(4.91)$ to obtain: Max: (2.03, 1.82); Min: (0, 0), (4.91, −4.81)





9. **(a)** $f'(x)$ is positive,0, negative so $f(x)$ is increasing, stationary, decreasing hence local maximum

(b) $f'(x)$ is negative,0, negative so $f(x)$ is decreasing, stationary, decreasing hence neither max not min (we will later call this a point of inflection)

(c) $f'(x)$ is negative,0, positive so $f(x)$ is decreasing, stationary, increasing hence local minimum

(d) $f''(x)$ is negative so $f(x)$ is concave up hence local minimum

(e) $f''(x)$ is positive so $f(x)$ is concave down hence local maximum

(f) cannot be determined

10. **(a)** $v(t) = s'(t) = 3(8t^2 - 22t + 9); a(t) = 6(8t - 11)$

$v(0) = 27 \text{ m s}^{-1}$; $a(0) = -66 \text{ m s}^{-1}$

(b) $v(3) = 45 \text{ m s}^{-1}$; $a(3) = 78 \text{ m s}^{-1}$

(c) Changing directions means that $v(t)$ changes sign.

Solve $v(t) = \frac{ds}{dt} = 0 \Leftrightarrow 24t^2 - 66t + 27 = 0$ for $t = 0.5\,s$ *or* $2.25\,s$.(by GDC) and check that $v(t)$ changes sign at these values.

These are times when the displacement is at a relative maximum or minimum.

(d) For minimum velocity, look for zeros of $a(t) = \frac{dv}{dt} = \frac{d}{dt}(24t^2 - 66t + 27)$

so $48t - 66 = 0$ and $t = \frac{11}{8} = 1.375$ s

Check that sign changes: $a(1) = 48(1) - 66 < 0$ and $a(2) = 48(2) - 66 > 0$

so it is indeed a minimum.

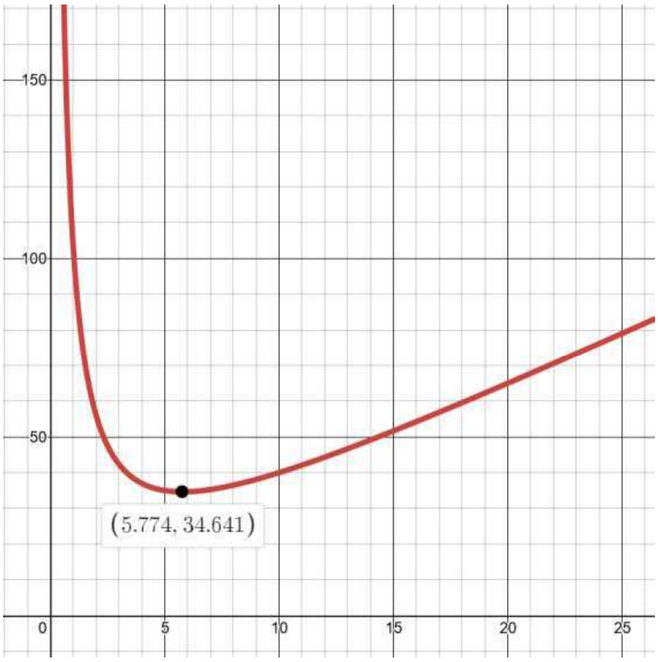
Significance: this time is where acceleration is zero.

11. **(a)-(b)** $\frac{dD}{dx} = 3 - \frac{100}{x^2} = \frac{3x^2 - 100}{x^2} = 0$ which gives $x = 5.77$ tonnes.

Second derivative = $\frac{200}{x^3}$ is positive at $x = 5.77 \Rightarrow$ local minimum.

$D_{\text{min}} = 34.6 = 34600$ dollars.

(c) GDC or Software:



12. **(a)** $v(t)=\dfrac{\mathrm{d}}{\mathrm{d}t}\left(10t-\dfrac{1}{2}t^2\right)=10-t$ m/s

(b) $10-t=0 \Leftrightarrow t=10$ s

(c) $s(10)=10(10)-\dfrac{1}{2}(10)^2$ = 50 m

13. **(a)** $v(t)=\dfrac{\mathrm{d}h}{\mathrm{d}t}=14-9.8t$

(b) Max height is when $v(t)=0$ so $14-9.8t=0 \Rightarrow t\approx 1.43$ s

We then have $h(1.43)=10$ m

(c) Velocity at max height is zero.

14. **(a)** Vertical velocity is given by :

$$h'(t)=-50\cos\left(\frac{\pi}{10}(t+5)\right)\cdot\frac{\pi}{10}=-5\pi\cos\left(\frac{\pi}{10}(t+5)\right),$$

$$h'(t)=0 \Leftrightarrow \cos\left(\frac{\pi}{10}(t+5)\right)=0 \Leftrightarrow \frac{\pi}{10}(t+5)=0,\pi,2\pi,\ldots$$

so $t+5=0,\pi\cdot\dfrac{10}{\pi},2\pi\cdot\dfrac{10}{\pi},\ldots$ or $t=-5,5,15,\ldots$

Considering $t\geq 0$, we have $t=5,15$

(b) Vertical acceleration is given by $h''(t) = \frac{\pi^2}{2}\sin\left(\frac{\pi}{10}(t+5)\right)$,

sine is maximised when $\frac{\pi}{10}(t+5) = \frac{\pi}{2}$ or $\frac{\pi}{10}(t+5) = \frac{5\pi}{2} \Rightarrow t = 0, 20$,

sine is minimised when $\frac{\pi}{10}(t+5) = \frac{3\pi}{2} \Rightarrow t = 10$

15. $P(x) = R(x) - C(x) = -0.00016x^3 + 0.1x^2 + 20x - \left(0.004x^2 + 5x + 2000\right)$

So, $P(x) = -0.00016x^3 + 0.096x^2 + 15x - 2000$

$P'(x) = -0.00048x^2 + 0.192x + 15 = 0$ for $x = -66.9, 466.9$

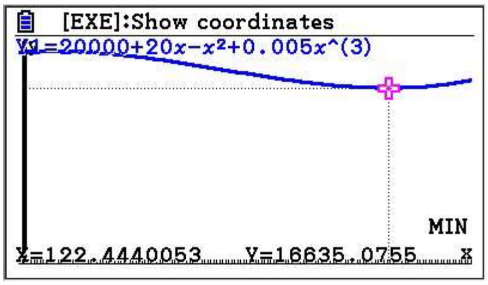
Reject the negative value and round to 467 toys (technically, we should verify that it does indeed correspond to a maximum, but since the question implies that there is a max, it can only be this one).

Now $P(467) = -0.00016 \cdot 467^3 + 0.096 \cdot 467^2 + 15 \cdot 467 - 2000 \approx 9645.93$

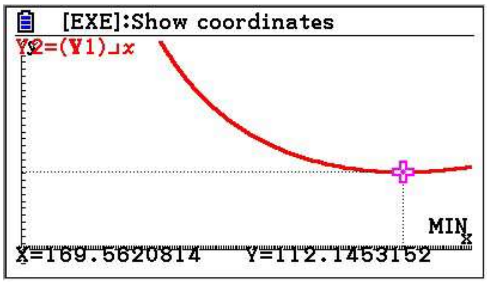
So, the maximum profit of 9646 (or 59650 to 3 s.f.) Chinese Renminbi is reached when 467 toys are produced.

16. **(a)** The derivative of the cost function, also called marginal cost is $C'(n) = 0.015n^2 - 2n + 20$ and it is equal to zero at $n \approx 11$, or $n \approx 122$

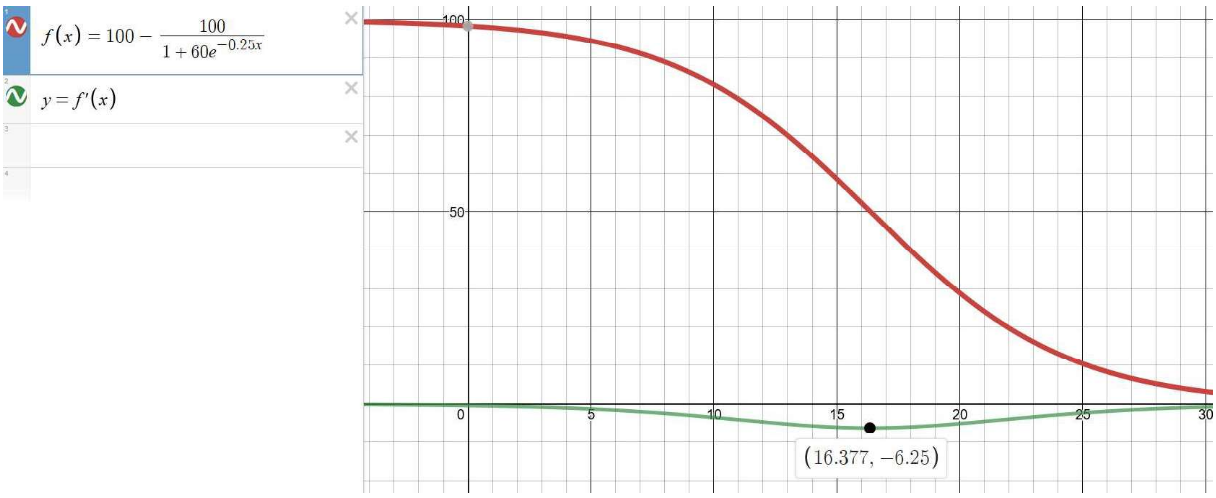
The minimum is 16630 EUR when $n \approx 122$



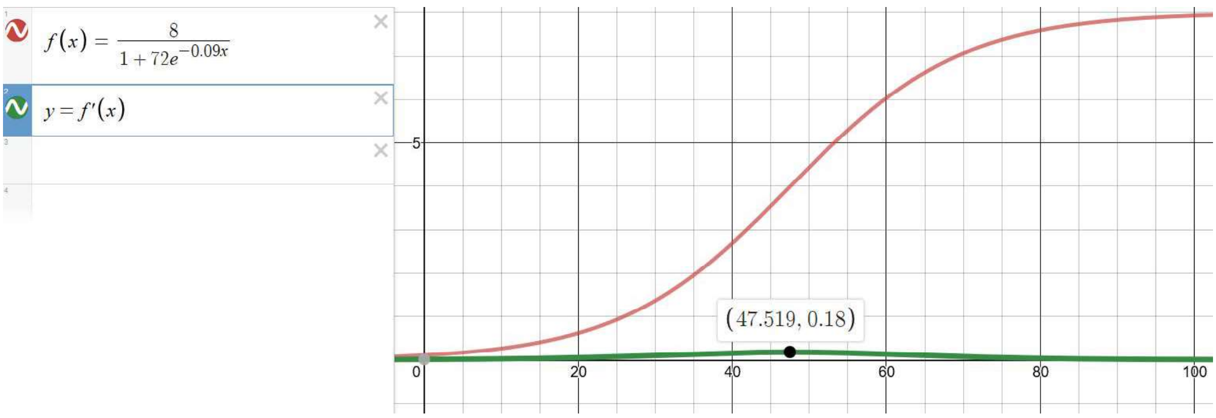
(b) Again, using GDC with the new function leads to 112 EUR for 170 wheels



17. "Fastest" means when the change in percentage of the reactant is a maximum.
So, we want $\frac{\mathrm{d}R}{\mathrm{d}t}$ to be an absolute maximum (here a minimum since the reactant is consumed). By GDC or Software, we find $t \approx 16.4$ min.



18. Similarly, to the previous question, we use technology to find the maximum rate of change of the given function to find $t \approx 47.5$ s



Notice that this time, the rate of change remains positive, indicating that the pH is increasing throughout the experiment.

19. **(a)** Using the quotient rule: $C'(t) = \dfrac{0.13(t^2+1.5) - 0.13t(2t)}{(t^2+1.5)^2} = \dfrac{0.195 - 0.13t^2}{(t^2+1.5)^2}$

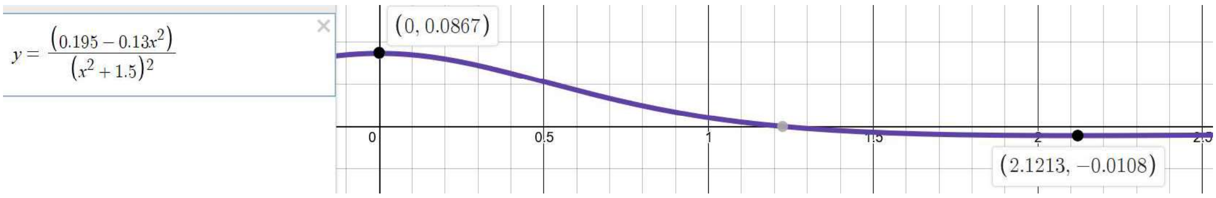
(b) $C'(t) = 0 \Leftrightarrow 0.195 - 0.13t^2 = 0 \Leftrightarrow t = \sqrt{1.5} \approx 1.22$

Check if it is a Max:

$C'(1) = 0.0104 > 0$ while $C'(2) = -0.002 < 0$ so $t = \sqrt{1.5}$ correspond to a max

Finally, $C\left(\sqrt{1.5}\right) \approx$ 0.0531 mg cm^{-3}

(c) Using a GDC or graphing package to graph $y = C'(t)$



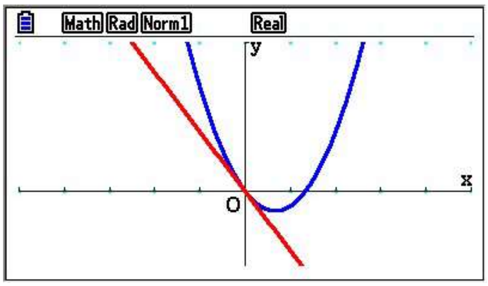
The bloodstream concentration is increasing the fastest at time $t = 0$ (rate of change: $\dfrac{13}{150} \approx 0.0867$ mg cm^{-3} h^{-1}) and decreasing the fastest at time $t \approx 2.12$ hours (rate of change: $-\dfrac{13}{1200} \approx 0.0108$ mg cm^{-3} h^{-1})

Exercise 14.2

1. The question asks for GDC verification. Some GDCs have the option of drawing tangents and normals. We will demonstrate this in one question. The rest is left for you to use your GDC. If your GDC does not have this option, then use to results found to visually see that the lines you found are tangent and normal at the specific points.

(a) **(i)** $\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2-4x\right)=6x-4$ so when $x=0, \frac{\mathrm{d}y}{\mathrm{d}x}=-4$ so $y=-4x+c$

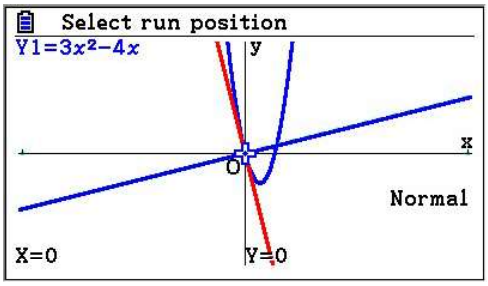
At $(0,0)$: $0=-4(0)+c \Rightarrow c=0$ so the tangent has equation $y=-4x$



(ii) The slope of the perpendicular is the negative reciprocal so

$m=\frac{-1}{-4}=\frac{1}{4}$

At $(0,0)$: $0=\frac{1}{4}(0)+c \Rightarrow c=0$ so the normal has equation $y=\frac{1}{4}x$



(b) **(i)** $\frac{\mathrm{d}y}{\mathrm{d}x}=-6-2x$ so when $x=-3, \frac{\mathrm{d}y}{\mathrm{d}x}=-6-2(-3)=0$ so $y=c$ (horizontal)

At $(-3,10)$: $10=c$ so the tangent has equation $y=10$

(ii) The normal will be a vertical line with equation $x=k$

At $(-3,10)$: $-3=k$ so the normal has equation $x=-3$

(c) **(i)** $\frac{dy}{dx}=2x+2$ so when $x=-3, \frac{dy}{dx}=2(-3)+2=-4$ so $y=-4x+c$

When $x=-3, y=(-3)^2+2(-3)+1=4$

At $(-3,4): 4=-4(-3)+c \Rightarrow c=-8$ so the tangent has equation $y=-4x-8$

(ii) The slope of the perpendicular is the negative reciprocal so $m=\frac{-1}{-4}=\frac{1}{4}$

At $(-3,4)$: $4=\frac{1}{4}(-3)+c \Rightarrow c=\frac{19}{4}$ so, the normal has equation

$y=\frac{1}{4}x+\frac{19}{4}$

(d) **(i)** $\frac{dy}{dx}=3x^2+2x$ so, when $x=\frac{-2}{3}, \frac{dy}{dx}=3\left(\frac{-2}{3}\right)^2+2\left(\frac{-2}{3}\right)=0$ so $y=c$

When $x=\frac{-2}{3}, y=\left(\frac{-2}{3}\right)^3+\left(\frac{-2}{3}\right)^2=\frac{4}{27}$

At $\left(\frac{-2}{3},\frac{4}{27}\right)$: $\frac{4}{27}=-4\left(\frac{-2}{3}\right)+c \Rightarrow c=\frac{4}{27}$

So, the tangent has equation $y=\frac{4}{27}$

(ii) The normal will be a vertical line with equation $x=k$

At $\left(\frac{-2}{3},\frac{4}{27}\right)$: $\frac{-2}{3}=k$ so, the normal has equation $x=\frac{-2}{3}$

(e) **(i)** $\frac{dy}{dx}=6x-1$ so when $x=0, \frac{dy}{dx}=6(0)-1=-1$ so $y=-x+c$

When $x=0, y=1$

At $(0,1): 1=-(0)+c \Rightarrow c=1$ so, the tangent has equation $y=-x+1$

(ii) The slope of the perpendicular is the negative reciprocal so $m=\frac{-1}{-1}=1$

At $(0,1)$: $1=1(0)+c \Rightarrow c=1$ so the normal has equation $y=x+1$

(f) **(i)** $\frac{dy}{dx}=2-\frac{1}{x^2}$ so when $x=\frac{1}{2}, \frac{dy}{dx}=2-\frac{1}{\left(\frac{1}{2}\right)^2}=-2$ so $y=-2x+c$

When $x=\frac{1}{2}, y=2\left(\frac{1}{2}\right)+\frac{1}{\frac{1}{2}}=3$

At $\left(\frac{1}{2},3\right): 3=-2\left(\frac{1}{2}\right)+c \Rightarrow c=4$ so the tangent has equation $y=-2x+4$

(ii) The slope of the perpendicular is the negative reciprocal so $m = \frac{-1}{-2} = \frac{1}{2}$

At $\left(\frac{1}{2}, 3\right)$: $3 = \frac{1}{2}\left(\frac{1}{2}\right) + c \Rightarrow c = \frac{11}{4}$ so the normal has equation

$y = \frac{1}{2}x + \frac{11}{4}$

(g) **(i)** $\frac{dy}{dx} = 2\cos(2x)$ so when

$x = \frac{\pi}{8}, \frac{dy}{dx} = 2\cos\left(2 \cdot \frac{\pi}{8}\right) = 2\cos\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$ so $y = \sqrt{2} \cdot x + c$

When $x = \frac{\pi}{8}, y = \sin\left(2 \cdot \frac{\pi}{8}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

At $\left(\frac{\pi}{8}, \frac{\sqrt{2}}{2}\right)$: $\frac{\sqrt{2}}{2} = \sqrt{2}\left(\frac{\pi}{8}\right) + c \Rightarrow c = \frac{\sqrt{2}(4-\pi)}{8}$

So, the tangent has equation $y = \sqrt{2} \cdot x + \frac{\sqrt{2}(4-\pi)}{8}$

(ii) The slope of the perpendicular is the negative reciprocal so $m = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$

At $\left(\frac{\pi}{8}, \frac{\sqrt{2}}{2}\right)$: $\frac{\sqrt{2}}{2} = \frac{-\sqrt{2}}{2}\left(\frac{\pi}{8}\right) + c \Rightarrow c = \frac{\sqrt{2}(8+\pi)}{16}$

So, the normal has equation $y = \frac{-\sqrt{2}}{2}x + \frac{\sqrt{2}(8+\pi)}{16}$

(h) **(i)** $\frac{dy}{dx} = e^x$ so when $x = 2, \frac{dy}{dx} = e^2$ so $y = e^2x + c$

When $x = 2$, $y = e^2$

At $(2, e^2)$: $e^2 = e^2(2) + c \Rightarrow c = -e^2$ so the tangent has equation $y = e^2x - e^2$

(ii) The slope of the perpendicular is the negative reciprocal so $m = \frac{-1}{e^2}$

At $(2, e^2)$: $e^2 = \frac{-1}{e^2}(2) + c \Rightarrow c = \frac{e^4 + 2}{e^2}$

So, the normal has equation $y = -\frac{1}{e^2}x + \frac{e^4 + 2}{e^2}$

2. **(a)** $\frac{dy}{dx}=3 \Leftrightarrow 2x+3=3 \Leftrightarrow x=0$ and when $x=0, y=0$ so $(0,0)$

(b) $\frac{dy}{dx}=12 \Leftrightarrow 3x^2=12 \Leftrightarrow x=\pm 2$ and when $x=\pm 2, y=(\pm 2)^3=\pm 8$

so $(-2,-8)$ and $(2,8)$

(c) $\frac{dy}{dx}=e^2 \Leftrightarrow e^x=e^2 \Leftrightarrow x=2$ and when $x=2, y=e^2$ so $(2,\ e^2)$

3. $\frac{dy}{dx}=1+\tan^2(x) \Leftrightarrow 1+\tan^2(x)=1 \Leftrightarrow \tan(x)=0$ so, on $0 \le x \le 2\pi, x=0,\pi,2\pi$

and when $x=0,\pi,2\pi, y=0$ so $(0,0),(0,\pi),(0,2\pi)$

4. The curve intersects the x-axis when

$x^3-3x^2+2x=0 \Leftrightarrow x(x^2-3x+2)=0 \Leftrightarrow x(x-1)(x-2)=0 \Leftrightarrow x=0,1,2$

The values of $\frac{dy}{dx}=3x^2-6x+2$ at $x=0,1,2$ are $\frac{dy}{dx}=2,-1,2$ respectively, giving the slopes.

at $(0,0)$, $y=2x$

at $(1,0)$, $y=-x+c$ and $0=-1+c \Rightarrow c=1$ so $y=-x+1$

at $(2,0)$, $y=2x+c$ and $0=2(2)+c \Rightarrow c=-4$ so $y=2x-4$

5. Re-arrange $x-2y=1$ into $y=\frac{1}{2}x-\frac{1}{2}$

Perpendicular to $y=\frac{1}{2}x-\frac{1}{2}$ so we need a gradient of $\frac{-1}{\frac{1}{2}}=-2$

We need to find the points on $y=x^2-2x$ such that $\frac{d}{dx}(x^2-2x)=-2$

so $2x-2=-2 \Rightarrow x=0$

When $x=0, y=(0)^2-2(0)=0$

At $(0,0): 0=-2(0)+c \Rightarrow c=0$ so the tangent has equation $y=-2x$

6. At $(3,-1)$, we have $\frac{d}{dx}(x^2+ax+b)=4$ so $2(3)+a=4 \Rightarrow a=-2$

Using $a=-2$ and point $(3,-1)$ into $y=x^2+ax+b$, we have $-1=(3)^2+(-2)(3)+b$ which leads to $b=-4$

7. Perpendicular to $y = \frac{-1}{3}x + 3$ so we need a gradient of $\frac{-1}{\frac{-1}{3}} = 3$

At $(3,\ 2)$, we need $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + ax + b) = 3$ so $2(3) + a = 3 \Rightarrow a = -3$

Using $a = -3$ and point $(3,2)$ into $y = x^2 + ax + b$, we have $2 = (3)^2 + (-3)(3) + b$ which leads to $b = 2$

8. We need $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - x) = 5$ so $2x - 1 = 5 \Rightarrow x = 3$

When $x = 3, y = (3)^2 - 3 = 6$ so point $(3,6)$

9. Re-arrange $x + 7y = 38$ into $y = \frac{-1}{7}x + \frac{38}{7}$

Perpendicular to $y = \frac{-1}{7}x + \frac{38}{7}$ so we need a gradient of $\frac{-1}{\frac{-1}{7}} = 7$

So, the tangent(s) have equation(s) $y = 7x + c$ for some value(s) of *c*.

To determine the value(s) of *c*, we need to find the points on $y = x^3 - 5x$ such that $\frac{\mathrm{d}}{\mathrm{d}x}(x^3 - 5x) = 7$ so $3x^2 - 5 = 7 \Rightarrow x^2 = \frac{12}{3} = 4 \Rightarrow x = \pm 2$

When $x = 2, y = (2)^3 - 5(2) = -2$ so point $(2,-2)$

When $x = -2, y = (-2)^3 - 5(-2) = 2$ so point $(-2,2)$

At $(2,-2): -2 = 7(2) + c \Rightarrow c = -16$ so the tangent has equation $y = 7x - 16$

At $(-2,2): 2 = 7(-2) + c \Rightarrow c = 16$ so the tangent has equation $y = 7x + 16$

10. $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 4$ so when $x = -3, \frac{\mathrm{d}y}{\mathrm{d}x} = 2(-3) + 4 = -2$

We want the normal so we need a gradient of $\frac{-1}{-2} = \frac{1}{2}$

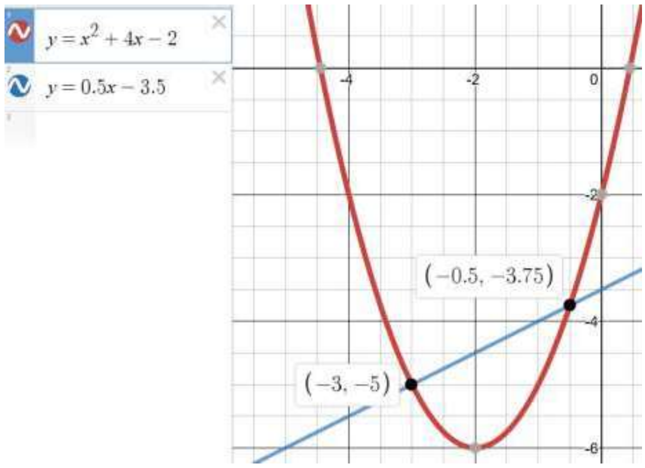
When $x = -3, y = (-3)^2 + 4(-3) - 2 = -5$ so point $(-3,-5)$

At $(-3,-5): -5 = \frac{1}{2}(-3) + c \Rightarrow c = \frac{-7}{2}$ so the tangent has equation $y = \frac{1}{2}x - \frac{7}{2}$

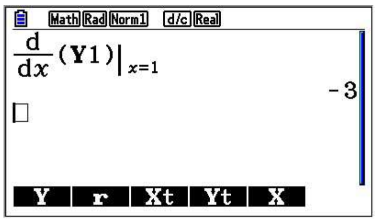
The other point must verify $x^2 + 4x - 2 = \frac{1}{2}x - \frac{7}{2}$

Using a GDC or graphing package, the other point has coordinates $(-0.5,-3.75)$

Q10 cont.



11. **(a)** Below is a sample of GDC output $g'(1) = -3$



(b) Using a gradient of -3 for the tangent and $\frac{-1}{-3} = \frac{1}{3}$ for the normal, we need to find the corresponding values of c.

For the tangent, using point $(1,0)$: $0 = -3(1) + c \Rightarrow c = 3$ so the tangent has equation $y = -3x + 3$

For the normal, using point $(1,0)$: $0 = \frac{1}{3}(1) + c \Rightarrow c = \frac{-1}{3}$ so the normal has equation $y = \frac{1}{3}x - \frac{1}{3}$

12. **(a)** $f'(-1) = 3(-1)^2 + (-1) = 2$ so the tangent has equation $y = 2x + c$

At $\left(-1, \frac{1}{2}\right)$: $\frac{1}{2} = 2(-1) + c \Rightarrow c = \frac{5}{2}$ so the tangent has equation $y = 2x + \frac{5}{2}$

(b) The other point must verify $f'(x) = 2$ so $3x^2 + x = 2$

Using a GDC, we get $x = -1, \frac{2}{3}$ so the other point has $x = \frac{2}{3}$

When $x = \frac{2}{3}, y = \left(\frac{2}{3}\right)^3 + \frac{1}{2}\left(\frac{2}{3}\right)^2 + 1 = \frac{41}{27}$ (use **Table** on a GDC)

so point $\left(\frac{2}{3}, \frac{41}{27}\right)$

13. $y=\sqrt{x}\left(1-\sqrt{x}\right)=x^{\frac{1}{2}}-x$ so $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}-1$

At $x=4$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2\sqrt{4}}-1=\frac{-3}{4}$ so the tangent has equation $y=\frac{-3}{4}x+c$

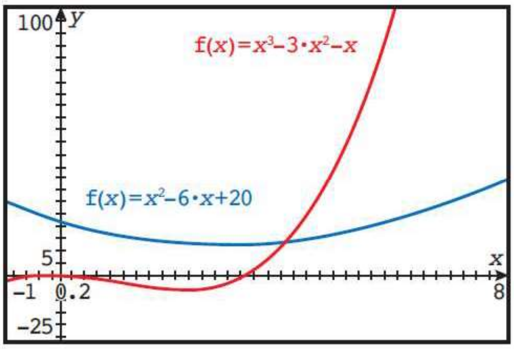
At $x=4$, $y=\sqrt{4}\left(1-\sqrt{4}\right)=2(1-2)=-2$ so $-2=\frac{-3}{4}(4)+c\Rightarrow c=1$

So, the tangent has equation $y=\frac{-3}{4}x+1$

For the normal, we need a gradient of $\frac{-1}{\frac{-3}{4}}=\frac{4}{3}$

At $(4,-2):-2=\frac{4}{3}(4)+c\Rightarrow c=\frac{-22}{3}$ so the normal has equation $y=\frac{4}{3}x-\frac{22}{3}$

14. **(a)**



(b) Solve $\frac{\mathrm{d}}{\mathrm{d}x}(x^2-6x+20)=\frac{\mathrm{d}}{\mathrm{d}x}(x^3-3x^2-x)\Leftrightarrow 2x-6=3x^2-6x-1$

leading to the quadratic $3x^2-8x+5=0$

Using a GDC, the 2 solutions are $x=1,\frac{5}{3}$. We want an integer, so keep $x=1$

(c) Using $x=1$ in the quadratic, we obtain $y=(1)^2-6(1)+20=15$

Using $x=1$ in the cubic, we obtain $y=(1)^3-3(1)^2-(1)=-3$

The slope of both tangent at $x=1$ is $\frac{\mathrm{d}y}{\mathrm{d}x}=2(1)-6=-4$

(we could also have used the derivative of the cubic for this)

Both tangents have equation $y=-4x+c$ for their respective values of c.

For the quadratic, $15=-4(1)+c\Rightarrow c=19$ and the equation is $y=-4x+19$

For the cubic, $-3=-4(1)+c\Rightarrow c=1$ and the equation is $y=-4x+1$

15. **(a)** L has equation $y = mx$ for some value of *m*.

L intersects the parabola when $-x^2 - 6x - 4 = mx$ leading to the quadratic equation $x^2 + (m+6)x + 4 = 0$

If L is tangent to the curve, there is only 1 point of intersection, hence 1 solution to this quadratic equation, so the discriminant $\Delta = b^2 - 4ac$ must be 0.

$(m+6)^2 - 4(1)(4) = 0 \Rightarrow m^2 + 12m + 20 = 0$ leading to $m = -10, -2$

When $m = -10$, the quadratic equation becomes $x^2 - 4x + 4 = 0$ giving $x = 2$

When $m = -2$, the quadratic equation becomes $x^2 + 4x + 4 = 0$ giving $x = -2$

The corresponding values of *y* are $y = -(2)^2 - 6(2) - 4 = -20$ and $y = 4$

so the points of tangency are $(2, -20)$ and $(-2, 4)$

(b) The possible equations of L are $y = -10x$ and $y = -2x$

16. A line that passes through point $(2, -3)$ has equation $y - (-3) = m(x - 2)$

For a given value of *m*, we can find the points of intersection of this line with the curve $y = x^2 + x$ by solving simultaneously , by substituting *y*: $x^2 + x + 3 = m(x-2)$

Re-arranging the quadratic for *x*: $x^2 + (1-m)x + 3 + 2m = 0$

We want the general line through $(2, -3)$ to be a tangent to $y = x^2 + x$, which means that we only want one point of intersection, hence one solution to the previous quadratic. This means that the discriminant must be 0.

$\Delta = b^2 - 4ac = (1-m)^2 - 4(1)(3+2m) = m^2 - 10m - 11 = 0$

so $(m-11)(m+1) = 0$ leading to $m = -1, 11$

For $m = -1$: $y = -x + c$ passes through $(2, -3)$ so $-3 = -(2) + c \Rightarrow c = -1$ so the first tangent has equation $y = -x - 1$

For $m = 11$: $y = 11x + c$ passes through $(2, -3)$ so $-3 = 11(2) + c \Rightarrow c = -25$ so the second tangent has equation $y = 11x - 25$

17. Using the GDC (the instruction to give a value to 3 s.f. is a clue to using the GDC)

At $x = 0$, $\frac{\mathrm{d}y}{\mathrm{d}x} \approx 1.367879$ so the normal has a gradient of -0.731

At $x = 0$, $y = 1$ so $1 = -0.731(0) + c \Rightarrow c = 1$

The normal has equation $y = -0.731x + 1$

18. The use of the word "Exact" means that we should not be using a GDC here.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x \cdot \cos 2x + \sin x \cdot -2\sin 2x$ so when $x = \frac{\pi}{6}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sqrt{3}}{2} \cdot \frac{1}{2} - 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} = \frac{-\sqrt{3}}{4}$

When $x = \frac{\pi}{6}$, $y = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ so $\frac{1}{4} = \frac{-\sqrt{3}}{4} \cdot \frac{\pi}{6} + c \Rightarrow c = \frac{6 + \pi\sqrt{3}}{24}$

and the equation of the tangent is $y = \frac{-\sqrt{3}}{4}x + \frac{6 + \pi\sqrt{3}}{24}$

Exercise 14.3

1. This time, the length of the box will be $11-2x$, the width $8.5-2x$ and the height x
So, the volume is $V(x)=(11-2x)(8.5-2x)x=4x^3-39x^2+93.5x$

$\frac{\mathrm{d}V}{\mathrm{d}x}=0 \Leftrightarrow 12x^2-78x+93.5=0$

Using GDC $x\approx 1.59, 4.91$

With the 2nd derivative or a sign table, we find that the max corresponds to $x\approx 1.59$

$V(1.59)\approx 66.1$ in^3

The dimensions are $1.59\times 5.33\times 7.83$ inches

2. The first part of the rope (hypotenuse of the triangle with sides 8 and x) has length $\sqrt{64+x^2}$ while the second part (hypotenuse of the triangle with sides 14 and $30-x$) has length $\sqrt{14^2+(30-x)^2}=\sqrt{1096-60x+x^2}$

So, the total length is $T=\sqrt{64+x^2}+\sqrt{1096-60x+x^2}$
Using the GDC, we have $T'=0$ when $x\approx 10.9$ m
which gives a total length of $T(10.9)\approx 37.2$ m

3. Running: rate of 3 min 40 sec so 220 sec for 1000 m so, on a straight distance of c m

time running $T_R=\frac{c}{\frac{1000}{220}}=\frac{11c}{50}$

Swimming: rate of 70 sec for 100 m, on a distance of $\sqrt{(300-c)^2+50^2}$

So, time swimming $T_S=\frac{\sqrt{(300-c)^2+50^2}}{\frac{100}{70}}=\frac{7}{10}\sqrt{(300-c)^2+50^2}$

The total time taken is $T=\frac{11c}{50}+\frac{7}{10}\sqrt{(300-c)^2+50^2}$

Using the GDC, we have $T'=0$ when $c\approx 283$ m (that is the running distance)
which gives a total time of $T(283)\approx 99.2$ sec (1 min 39 sec) to save the swimmer.

4. Let x be the abscissa of the bottom right vertex of the rectangle, so the width is $2x$ cm.

The equation of the semicircle is $y=\sqrt{1-x^2}$, which represents the height of the rectangle.

The area is therefore given by $A = 2x\sqrt{1-x^2}$

From the GDC, $A' = 0$ for $x \approx 0.707$ cm so dimensions are $2(0.707) \approx 1.41$ cm on $\sqrt{1-0.707^2} \approx 0.707$ cm.

You may recognise these rounded values which correspond to the exact values $\sqrt{2}$ and $\frac{\sqrt{2}}{2}$, which you will obtain by solving algebraically. Try it!

5. Let x be the width of the rectangle, in which case the length y must be such that $2x+2y=40$ to respect the constraint of the perimeter. So the length is $20-x$

The volume of a cylinder is given by $V=\pi r^2 h$ where h can be taken, without loss of generality, to be the width, so $h = x$ here, while the length $20-x$ will be rolled to give a circular cross section of circumference $2\pi r$, so $r = \frac{20-x}{2\pi}$.

Therefore, $V = \pi\left(\frac{20-x}{2\pi}\right)^2 \cdot x$

Using a GDC gives $V' = 0$ for $x \approx 6.67$ cm (the width) and $20-6.67 \approx 13.3$ cm (the height).

Note: Solving algebraically would lead to the exact values $6\frac{2}{3}$ and $13\frac{1}{3}$ cm

6. Let $(x, \sqrt{x})$ be a general point on the graph of the function. Its distance to the point $\left(\frac{3}{2}, 0\right)$ can be given (using the distance formula) by

$$D=\sqrt{\left(x-\frac{3}{2}\right)^2+\left(\sqrt{x}-0\right)^2}=\sqrt{x^2-2x+\frac{9}{4}}$$

We can use a GDC, we solve $D' = 0$ or simply argue that D will be minimum when $x^2-2x+\frac{9}{4}$ is minimum. Since $x^2-2x+\frac{9}{4}=(x-1)^2+\frac{5}{4}$ (by completing the square), whose minimum value is $\frac{5}{4}$, we conclude that the minimum distance is $\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$

7. Given that the area of the rectangle is 100 cm², its width must be $\frac{100}{x}$ cm

The perimeter of the entire figure is $P = 2x +$ circumference of a circle of radius $\frac{\frac{100}{x}}{2} = \frac{50}{x}$ (2 semicircles) so $P = 2x + 2\pi \cdot \frac{50}{x} = 2x + \frac{100\pi}{x}$

Now $P'(x) = 2 - \frac{100\pi}{x^2}$ and $P'(x) = 0 \Leftrightarrow x = \sqrt{50\pi} = 5\sqrt{2\pi}$ (reject the negative value).

Technically, one should check that this corresponds to a minimum, but the phrasing of the question indicated that there is a minimum, which can only be this point.

8. Similar to Q3 above.
Again, we will use time as the ratio of distance and speed.
Let T_R be the time spend walking the x km on the road, at a speed of 5 km h^{-1},
so $T_R = \frac{x}{5}$.
Let T_D be the time spend walking the hypotenuse of the triangle of sides 10 and $7 - x$ in the desert, at a speed of 2 km h^{-1}, o $T_D = \frac{\sqrt{10^2 + (7-x)^2}}{2}$

We want to minimise the total time $T = \frac{x}{5} + \frac{\sqrt{10^2 + (7-x)^2}}{2}$

Using a GDC, $x \approx 2.64$ km

9. We want to maximise the volume of a cylinder $V = \pi r^2 h$.
Taking a cross section of the figure, one can extract a right angle triangle with sides R, r and $\frac{h}{2}$ cm, so $10^2 = r^2 + \left(\frac{h}{2}\right)^2$ (Pythagoras) that we rearrange to $r^2 = 100 - \frac{h^2}{4}$.
Substituting into the formula for the volume, we obtain
$V = \pi \cdot \left(100 - \frac{h^2}{4}\right) \cdot h = \frac{\pi h(400 - h^2)}{4}$

Using a GDC, we find that $V_{\min}$ is reached when $h \approx 11.5$ cm

so $r \approx \sqrt{100 - \frac{11.5^2}{4}} \approx 8.16$ cm

10. Total surface is made of a curved surface ($2\pi rh$) and 2 disks (each πr^2)
so $A = 2\pi rh + 2\pi r^2$ so, $54\pi = 2\pi rh + 2\pi r^2$ and, dividing by 2π : $27 = rh + r^2$

(a) Solving for *h*, we have $h = \frac{27 - r^2}{r}$

Volume of a cylinder $V = \pi r^2 h$ so here: $V = \pi r^2\left(\frac{27 - r^2}{r}\right) = \pi r\left(27 - r^2\right)$.

(b) $\frac{dV}{dr} = \pi\left(27 - 3r^2\right) = 0 \Rightarrow r = 3\text{ cm}$

11. Using point $(0,10)$, we immediately obtain $c = 10$
Maximum at $x = 2$ so $\frac{dy}{dx} = 2a(2) + b = 0 \Leftrightarrow 4a + b = 0$
Using point $(2,18)$, $18 = a(2)^2 + b(2) + 10 \Leftrightarrow 2a + b = 4$
Solving simultaneously, we obtain $a = -2$ and $b = 8$
(Try the alternative solution of using the vertex form)

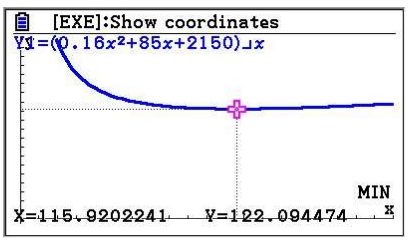
12. **(a)** Let *EM* be the cost of Equipment maintenance and *x* the number of devices produced each day
So $EM \propto x^2$ or $EM = k \cdot x^2$
Substituting: $4 = k \cdot 5^2 \Rightarrow k = \frac{4}{25}$ we can write $EM = \frac{4}{25}x^2$

(b) Adding the 3 sources of costs, we obtain $C(x) = 2150 + 85x + \frac{4}{25}x^2$

(c) Differentiating: $C'(x) = 85 + \frac{8}{25}x$
This is positive for all $x > 0$ hence production costs increase for all $x > 0$.

(d) Average cost is total cost divided by number of devices produced
so $\bar{C}(x) = \frac{0.16x^2 + 85x + 2150}{x}$

(e) Using a GDC, we find that the minimum average manufacturing cost per device is \$112.
116 devices should be produced to minimise average cost per device.



13. We wish to maximise the volume of a cone $V = \frac{1}{3}\pi r^2 h$

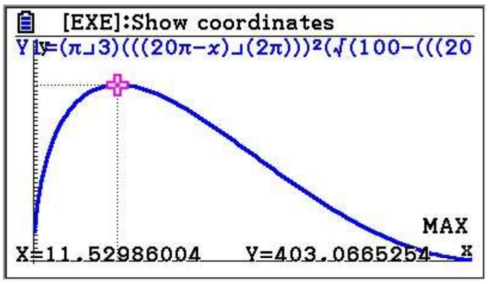
The circumference of the base of the cone is the same as the major arc of the original circle, so $2\pi r = 2\pi(10) - x$, simplifying to $r = \frac{20\pi - x}{2\pi}$

Using the diagram on the right: $h^2 + r^2 = 10^2$ so $h = \sqrt{100 - r^2} = \sqrt{100 - \left(\frac{20\pi - x}{2\pi}\right)^2}$

Therefore, the volume of the cone can be written as

$$V = \frac{1}{3}\pi\left(\frac{20\pi - x}{2\pi}\right)^2 \sqrt{100 - \left(\frac{20\pi - x}{2\pi}\right)^2}$$

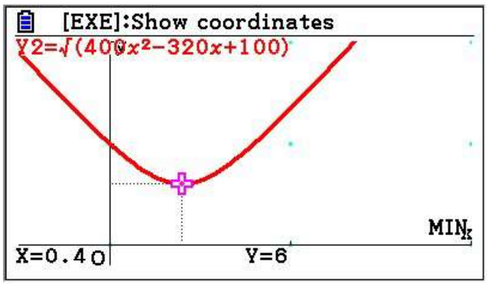
Using a GDC, we find that the max volume of 403 cm^3 is reached when $x \approx 11.5$ cm



14. Placing the origin at the initial position of the 2nd boat, the positions of the boat after t hours (1 knot is 1 nautical mile per hour) is given by:
$(0, 10 - 16t)$ for boat 1 and $(-12t, 0)$ for boat 2.

The distance is therefore: $\sqrt{(0 + 12t)^2 + (10 - 16t - 0)^2} = \sqrt{400t^2 - 320t + 100}$

Using a GDC, we find the minimum distance is 6 nautical miles
(when $t \approx 0.400$ h, so after 24 minutes approximately).



Exercise 14.4

1. Given: $\frac{dV}{dt} = -2$, $H = 8$ and $D = 6 \Rightarrow R = 3$

(a) Find $\frac{dh}{dt}$ when $h = 5$

Model involving V and h: $V = \frac{1}{3}\pi r^2 h$

where r and h are the radius and height at time t.

Using similar triangle, we have $\frac{r}{h} = \frac{3}{8} \Rightarrow r = \frac{3}{8}h$

and $V = \frac{1}{3}\pi\left(\frac{3}{8}h\right)^2 \cdot h = \frac{3\pi h^3}{64}$

Differentiating with respect to time: $\frac{dV}{dt} = \frac{3\pi}{64} \cdot 3h^2 \cdot \frac{dh}{dt}$

Substituting the given values, we have $-2 = \frac{3\pi}{64} \cdot 3(5)^2 \cdot \frac{dh}{dt}$ leading to

$\frac{dh}{dt} = -0.181$ m min^{-1} which is a decrease in height of 18.1 cm every minute.

(b) If instead of re-arranging $\frac{r}{h} = \frac{3}{8}$ for r we do it for h: $\frac{r}{h} = \frac{3}{8} \Rightarrow h = \frac{8}{3}r$

We would obtain $V = \frac{1}{3}\pi r^2 \cdot \frac{8}{3}r = \frac{8\pi r^3}{9}$

Differentiating with respect to time: $\frac{dV}{dt} = \frac{8\pi}{9} \cdot 3r^2 \frac{dr}{dt}$

When $h = 5$, we have $r = \frac{15}{8}$

Substituting the given values, we have $-2 = \frac{8\pi}{9} \cdot 3\left(\frac{15}{8}\right)^2 \frac{dr}{dt}$ leading to

$\frac{dr}{dt} = -0.0679$ m min^{-1} which is a decrease in radius of 6.79 cm every minute.

2. **(a)** Given: $\frac{dV}{dt} = 240$, find $\frac{dr}{dt}$

Model involving V and r: $V = \frac{4}{3}\pi r^3$

Differentiating with respect to time: $\frac{dV}{dt} = \frac{4\pi}{3} \cdot 3r^2 \frac{dr}{dt}$

Substituting the given values, we have $240 = \frac{4\pi}{3} \cdot 3(8)^2 \frac{dr}{dt}$

leading to $\frac{dr}{dt} = 0.298$ cm s^{-1}

(b) After 5 seconds, the volume is equal to $240 \times 5 = 1200$ (assuming we started with an empty balloon) so $1200 = \frac{4}{3}\pi r^3$ and $r = \sqrt[3]{\frac{1200}{\frac{4}{3}\pi}} \approx 6.59$ cm

Substituting the given values, we have $240 = \frac{4\pi}{3} \cdot 3(6.59)^2 \frac{dr}{dt}$

leading to $\frac{dr}{dt} = 0.439$ cm s^{-1}

3. Given: $\frac{dr}{dt} = 1$

(a) Find $\frac{dC}{dt}$ when $r = 4$

Model involving C and r: $C = 2\pi r$

Differentiating with respect to time: $\frac{dC}{dt} = 2\pi \cdot \frac{dr}{dt}$

Substituting the given values, we have $\frac{dC}{dt} = 2\pi \cdot 1 = 2\pi$ cm h^{-1}

(b) Model involving A and r: $A = \pi r^2$

Differentiating with respect to time: $\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$

Substituting the given values, we have $\frac{dC}{dt} = \pi \cdot 2(4) \cdot 1 = 8\pi$ cm^2 h^{-1}

4. Given: $\frac{dh}{dt} = 50$

Find $\frac{d\theta}{dt}$ when $h = 250$

Model involving h and θ, using the right angle triangle with perpendicular sides 150 and h: $\tan\theta = \frac{h}{150}$

Differentiating with respect to time: $\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{150} \cdot \frac{dh}{dt}$

Using the triangle again, we find that, when $h = 250$, $\theta = \arctan\left(\frac{250}{150}\right) \approx 1.03$ so

$\sec^2(1.03) \approx 3.78$

Substituting the given values, we have $3.78 \cdot \frac{d\theta}{dt} = \frac{1}{150} \cdot 50$

leading to $\frac{d\theta}{dt} = 0.0882$ rad min^{-1}

5. Let x be the length of the string and h be the horizontal movement of the kite at a given time.

We are given $\frac{\mathrm{d}h}{\mathrm{d}t} = 6$ and asked to find $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $x = 120$ (not $h = 120$)

From the triangle with sides 72, h and hypotenuse x we have: $h^2 + 72^2 = x^2$

Differentiating with respect to time: $2h \cdot \frac{\mathrm{d}h}{\mathrm{d}t} + 0 = 2x \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$

Using the triangle again, we find that, when $x = 120$, $h = \sqrt{120^2 - 72^2} = \sqrt{9216} = 96$

Substituting the given values, we have $2(96) \cdot 6 = 2(120) \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$

leading to $\frac{\mathrm{d}x}{\mathrm{d}t} = 4.8$ m sec^{-1}

6. Let x be the distance between the feet of the girl and the foot of the lamppost and let s be the length of the shadow.

We are given $\frac{\mathrm{d}x}{\mathrm{d}t} = -2$ (since the girl is getting closer to the lamppost, the distance x decreases.

The quantity $\frac{\mathrm{d}s}{\mathrm{d}t}$ represents the change in the size of the shadow, so from the tip of the shadow to the feet of the girl which moves themselves at a speed of $\frac{\mathrm{d}x}{\mathrm{d}t} = -2$ m s^{-1}

First, let us determine the value of $\frac{\mathrm{d}s}{\mathrm{d}t}$:

Using similar triangle, we have $\frac{s + x}{6} = \frac{s}{1.5}$

Differentiating with respect to time: $\frac{\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{\mathrm{d}x}{\mathrm{d}t}}{6} = \frac{\frac{\mathrm{d}s}{\mathrm{d}t}}{1.5}$

Substituting the given values, we have $\frac{\frac{\mathrm{d}s}{\mathrm{d}t} + (-2)}{6} = \frac{\frac{\mathrm{d}s}{\mathrm{d}t}}{1.5}$

leading to $\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{-2}{3}$ m s^{-1} (indeed, the shadow gets smaller as the girls gets closer to the lamppost).

Finally, the speed of the tip of the shadow is $\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-2}{3} - 2 = \frac{-8}{3} \approx -2.67$ m s^{-1}

7. Let A be the origin, so we are given $\frac{\mathrm{d}x}{\mathrm{d}t} = -60$ (West is "left", so negative x's) and $\frac{\mathrm{d}y}{\mathrm{d}t} = 35$ (North is "up").

The distance D is the hypotenuse, so $D = \sqrt{x^2 + y^2}$

Differentiating with respect to time: $\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{1}{2}\left(x^2 + y^2\right)^{\frac{-1}{2}} \cdot \left(2x \cdot \frac{\mathrm{d}x}{\mathrm{d}t} + 2y \cdot \frac{\mathrm{d}y}{\mathrm{d}t}\right)$

After 3 hours, $x = -60 \cdot 3 = -180$ km and $y = 35 \cdot 3 = 105$

Substituting the given values, we have

$$\frac{\mathrm{d}D}{\mathrm{d}t} = \frac{1}{2}\left((-180)^2 + (105)^2\right)^{\frac{-1}{2}} \cdot \left(2(-180) \cdot (-60) + 2(105) \cdot (35)\right) = 69.5 \text{ km h}^{-1}$$

8. Differentiating with respect to time: $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}\left(x^2 + 1\right)^{\frac{-1}{2}} \cdot \left(2x \cdot \frac{\mathrm{d}x}{\mathrm{d}t}\right)$

Substituting the given values, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left((3)^2 + 1\right)^{\frac{-1}{2}} \cdot \left(2(3) \cdot 4\right) = \frac{12}{\sqrt{10}} \approx 3.79$

9. Given: $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.03$, find $\frac{\mathrm{d}h}{\mathrm{d}t}$ where h is the height of the water (water level) at $t = 25$

Model involving V and h: $V = \frac{1}{2} \cdot w \cdot h \cdot 4.5$ where w is the width of the water in the trough. Using similar triangle, we have $\frac{w}{h} = \frac{1.5}{1} \Rightarrow r = 1.5h$ and

$V = \frac{1}{2} \cdot (1.5h) \cdot h \cdot 4.5 = \frac{27}{8}h^2$

Differentiating with respect to time: $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{27}{8} \cdot 2h\frac{\mathrm{d}h}{\mathrm{d}t}$

After 25 seconds, the volume is equal to $0.03 \times 25 = 0.75$ (assuming we started with an empty trough) so $0.75 = \frac{27}{8}h^2$ and $h = \sqrt{\frac{0.75}{\frac{27}{8}}} \approx 0.471$ cm

Substituting the given values, we have $0.03 = \frac{27}{8} \cdot 2(0.471)\frac{\mathrm{d}h}{\mathrm{d}t}$

leading to $\frac{\mathrm{d}h}{\mathrm{d}t} = 0.00943$ m s^{-1} or 0.94 cm s^{-1}

10. Given: $\frac{\mathrm{d}r}{\mathrm{d}t} = 3$, find $\frac{\mathrm{d}A}{\mathrm{d}t}$ when $A = 10$

Model involving V and r: $V = \frac{4}{3}\pi r^3$

Differentiating with respect to time: $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{4\pi}{3} \cdot 3r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$

When $A = 10$, $10 = 4\pi r^2$ so $r = \sqrt{\frac{10}{4\pi}} \approx 0.892$ mm

Substituting the given values, we have $\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{4\pi}{3} \cdot 3(0.892)^2 \cdot 3 = 30\ \text{mm}^3\ \text{s}^{-1}$

11. Let D be the distance between the 2 cars. We wish to find $\frac{\mathrm{d}D}{\mathrm{d}t}$.

Let a and b be the distances from the intersection at a certain time t.
Model involving D, a and b: $D^2 = a^2 + b^2 - 2ab\cos(60)$ (using the cosine rule)

Differentiating with respect to time: $2D\frac{\mathrm{d}D}{\mathrm{d}t} = 2a\frac{\mathrm{d}a}{\mathrm{d}t} + 2b\frac{\mathrm{d}b}{\mathrm{d}t} - 2\cos(60)\left(a\frac{\mathrm{d}b}{\mathrm{d}t} + b\frac{\mathrm{d}a}{\mathrm{d}t}\right)$

(using the product rule in the last part).

We are given $\frac{\mathrm{d}a}{\mathrm{d}t} = -40$ and $\frac{\mathrm{d}b}{\mathrm{d}t} = -50$ (as they both travel towards the intersection)
When $a = b = 2$ we must also have $D = 2$ since there is an angle of 60° so the triangle is equilateral at that specific time (you can also substitute in the cosine rule)
Since $\cos(60) = \frac{1}{2}$, we obtain

$$2(2)\frac{\mathrm{d}D}{\mathrm{d}t} = 2(2)\cdot(-40) + 2(2)\cdot(-50) - ((2)\cdot(-50) + (2)\cdot(-40))$$

leading to $\frac{\mathrm{d}D}{\mathrm{d}t} = -45$ km h^{-1} (negative since the cars are getting closer to one another)

12. Let x be the length of the side of the cube. Then the length of diagonal is $\sqrt{x^2 + x^2 + x^2} = \sqrt{3x^2} = x\sqrt{3}$

We are given that $\frac{\mathrm{d}}{\mathrm{d}t}\left(x\sqrt{3}\right) = 8$

so $\sqrt{3}\frac{\mathrm{d}x}{\mathrm{d}t} = 8$ and $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{8}{\sqrt{3}} = \frac{8\sqrt{3}}{3} \approx 4.62$ cm s^{-1}

13. P moving at a speed of 3 units s^{-1} means that the arc length $r\theta$ is increasing at a rate of 3 units every second or $\frac{\text{d}}{\text{d}t}(r\theta) = 3$

Since $r = 10$ units is a constant; $10\frac{\text{d}\theta}{\text{d}t} = 3 \Rightarrow \frac{\text{d}\theta}{\text{d}t} = 0.3$ rad s^{-1}

Now, the projection of P on the x-axis is $r\cos\theta = 10\cos\theta$ and its rate of change is

$\frac{\text{d}}{\text{d}t}(10\cos\theta) = 10\cdot(-\sin\theta)\frac{\text{d}\theta}{\text{d}t} = -10\cdot\sin\theta\cdot 0.3 = -3\sin\theta$

At point P, $\sin\theta = \frac{5}{10} = \frac{1}{2}$ so the projection of P on the x-axis moves at a rate of -1.5 units s^{-1}.

14. We are given $\frac{\text{d}\theta}{\text{d}t} = \frac{1}{60}$ and asked to find $\frac{\text{d}x}{\text{d}t}$ where x is the horizontal displacement.

Consider the right-angled triangle with sides x and 10 000 m. We have $\tan\theta = \frac{10000}{x}$

Differentiating with respect to time: $\sec^2\theta\frac{\text{d}\theta}{\text{d}t} = 10000\cdot(-1)x^{-2}\frac{\text{d}x}{\text{d}t}$

When $\theta = \frac{\pi}{3}$, $\tan\left(\frac{\pi}{3}\right) = \frac{10000}{x}$ and therefore $x = \frac{10000}{\sqrt{3}}$

Substituting the given values, we have $\sec^2\left(\frac{\pi}{3}\right)\cdot\frac{1}{60} = 10000\cdot(-1)\left(\frac{10000}{\sqrt{3}}\right)^{-2}\frac{\text{d}x}{\text{d}t}$

leading to $\frac{\text{d}x}{\text{d}t} = -222$ m s^{-1} or about 800 km h^{-1} towards the observer.

15. Let x be the position of the automobile along the straight section of the track.

We are given that $\frac{\text{d}x}{\text{d}t} = 288$ km h^{-1} so 288000 m h^{-1} and required to find $\frac{\text{d}\theta}{\text{d}t}$

Consider the right-angled triangle with sides x and 40 m. We have $\tan\theta = \frac{x}{40}$

Differentiating with respect to time (for which the angle will be initially measured in radians): $\sec^2\theta\frac{\text{d}\theta}{\text{d}t} = \frac{1}{40}\frac{\text{d}x}{\text{d}t}$

(a) "the car is directly in front of the camera" means that $\theta = 0$

so $1\frac{\text{d}\theta}{\text{d}t} = \frac{1}{40}\cdot 288000$

$\frac{\text{d}\theta}{\text{d}t} = 7200$ rad h^{-1} so $\frac{7200}{3600} = 2$ rad s^{-1} or $2\cdot\frac{180}{\pi} \approx 115$ deg s^{-1}

(b) at $t=\frac{1}{2}$, $x=288000\cdot\frac{1}{3600}\cdot\frac{1}{2}=40$ m and $\tan\theta=\frac{40}{40}=1$ so $\theta=\frac{\pi}{4}$ and

$$\sec^2\left(\frac{\pi}{4}\right)=\frac{1}{\cos^2\left(\frac{\pi}{4}\right)}=\frac{1}{\left(\frac{\sqrt{2}}{2}\right)^2}=2$$

so $2\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{1}{40}\cdot 288000$; so, half the speed found in **(a)**, that is 57.2 deg s^{-1}

16. Let x be the distance due West of the point directly below the plane (so its horizontal displacement) and y be the altitude. The distance D is $D=\sqrt{x^2+y^2}$.

We can also use $D^2=x^2+y^2$

Differentiating with respect to time: $2D\frac{\mathrm{d}D}{\mathrm{d}t}=2x\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+2y\cdot\frac{\mathrm{d}y}{\mathrm{d}t}$

Change 180 m min^{-1} into 10.8 km h^{-1} for consistency of units

So we want to find $\frac{\mathrm{d}D}{\mathrm{d}t}$ given $\frac{\mathrm{d}x}{\mathrm{d}t}=-640$ and $\frac{\mathrm{d}y}{\mathrm{d}t}=10.8$

When $x=-6$ and $y=5$, $D=\sqrt{(-6)^2+5^2}=\sqrt{61}$

Substituting the given values, we have $2\sqrt{61}\frac{\mathrm{d}D}{\mathrm{d}t}=2(-6)\cdot(-640)+2(5)\cdot(10.8)$

leading to $\frac{\mathrm{d}D}{\mathrm{d}t}\approx$ 499 km h^{-1}

Chapter 14 practice questions

1. **(a)** The slant height is equal to the radius of the semicircle.

Using $A_{\text{semicircle}} = \frac{1}{2}\pi r^2$ we get:

$$39.27 = \frac{1}{2}\pi l^2$$
$$25 = l^2$$
$$5 = l$$

(b) **(i)** The circumference of the base of the cone is equal to the arc length of the semicircle. Since $r = l = 5$, we have

$$C = \frac{1}{2}(2\pi r) = \pi(5) = 5\pi = 15.7 \text{ m.}$$

(ii) For the distance C to be formed into a circle, it must satisfy

$$C = 2\pi r \Rightarrow 5\pi = 2\pi r \Rightarrow r = \frac{5}{2}.$$

(iii) $r^2 + h^2 = l^2 \Rightarrow \left(\frac{5}{2}\right)^2 + h^2 = 5^2 \Rightarrow h = \sqrt{5^2 - \left(\frac{5}{2}\right)^2} = \frac{5\sqrt{3}}{2} = 4.33$

(c) $h + 2r = 9.33 \Rightarrow h = 9.33 - 2r$

(d) Using the general model for the volume of a cone, $V = \frac{1}{3}Bh$, we have

$$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi r^2(9.33 - 2r) = 3.11\pi r^2 - \frac{2}{3}\pi r^3$$

(e) $\frac{dV}{dr} = 3.11\pi(2)r - \frac{2}{3}\pi(3)r^2 = 6.22\pi r - 2\pi r^2$

(f) **(i)** We need to find where $\frac{dV}{dr} = 0$ hence $6.22\pi r - 2\pi r^2 = 0$

$\Rightarrow r = 0$ or $r = 3.11$. We discard $r = 0$.

Sign analysis shows that $\frac{dV}{dr}$ is positive to the left of $r = 3.11$ and negative to the right, so the volume is maximised at $r = 3.11$ m.

(ii) $V = 3.11\pi(3.11)^2 - \frac{2}{3}\pi(3.11)^3 = 31.5 \text{ m}^3$

2. **(a)** $f'(x) = 5x^4$

(b) $f'(2) = 5 \cdot 2^4 = 80$

(c) $y - 16 = -\frac{1}{80}(x - 2) \Rightarrow 80y - 1280 = -x + 2 \Rightarrow x + 80y - 1268 = 0$

3. **(a)** $\frac{dy}{dx} = 3x^2 + 2kx$

(b) $\frac{dy}{dx} = 3 \cdot 3^2 + 2k \cdot 3 = 0 \Rightarrow k = -\frac{9}{2}$

(c) $y = (3)^3 - \frac{9}{2}(3)^2 = -\frac{27}{2}$

4. **(a)** (0, 7)

(b) (−2, 3)

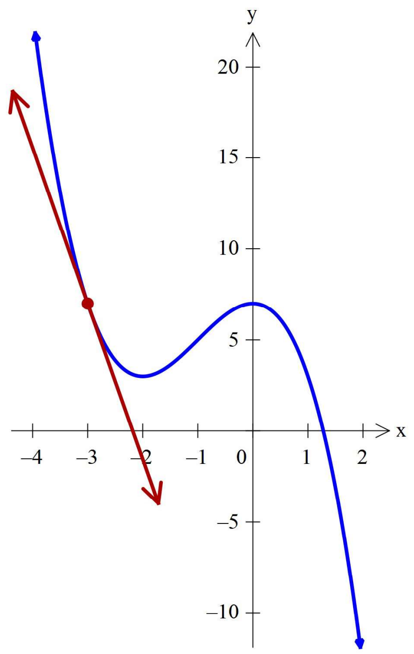
(c) $]-2, 0[$

(d) $f'(x) = -3x^2 - 6x \quad \Rightarrow \quad f'(-3) = -9$

$f(-3) = 7$

$\therefore\ y - 7 = -9(x+3)$ or $y = -9x - 20$

(e)



5. **(a)** $f'(x) = 9x^2 - 7 - \frac{4}{x^2}$

(b) (1.08, 9.92)

(c) (−1.08, 10.1)

6. **(a)** $T = 2\pi r + 4r + 4l$

(b) $V = Bh = \frac{1}{2}\pi r^2 l \quad \Rightarrow \quad 0.75 = \frac{1}{2}\pi r^2 l \quad \Rightarrow \quad 1.5 = \pi r^2 l$

(c) From **(b)**, $1.5 = \pi r^2 l \quad \Rightarrow \quad l = \frac{1.5}{\pi r^2}$. From **(a)**,

$$T = 2\pi r + 4r + 4l \quad \Rightarrow \quad T = 2\pi r + 4r + 4\left(\frac{1.5}{\pi r^2}\right) \quad \Rightarrow \quad T = 2\pi r + 4r + \frac{6}{\pi r^2}$$

$$\Rightarrow \quad T = (2\pi + 4)r + \frac{6}{\pi r^2}$$

(d) $\frac{\mathrm{d}T}{\mathrm{d}r} = 2\pi + 4 - \frac{12}{\pi r^3}$

(e) We solve $\frac{\mathrm{d}T}{\mathrm{d}r} = 2\pi + 4 - \frac{12}{\pi r^3} = 0 \Rightarrow r = \sqrt[3]{\frac{12}{\pi(2\pi+4)}} = 0.719$ m

(f) $l = \frac{1.5}{\pi r^2} = \frac{1.5}{\pi(0.719)^2} = 0.924$ m

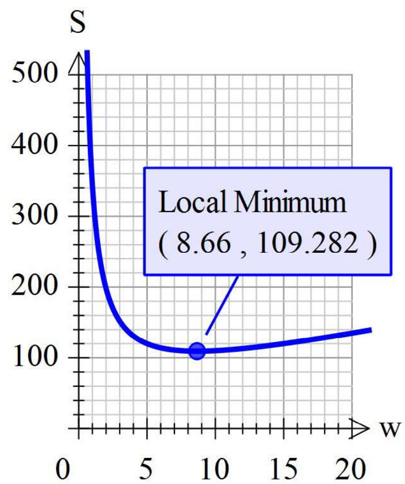
(g) $T = (2\pi+4)(0.719) + \frac{6}{\pi(0.719)^2} = 11.1$ m

7. **(a)** $V = lwh \Rightarrow V = 20lw$

(b) $V = 20lw \Rightarrow 3000 = 20lw \Rightarrow l = \frac{150}{w}$

(c) $S = 2(20) + 4w + 2l = 40 + 4w + 2\left(\frac{150}{w}\right) = 40 + 4w + \frac{300}{w}$

(d)



(e) $\frac{\mathrm{d}S}{\mathrm{d}W} = 4 - \frac{300}{w^2}$

(f) $\frac{\mathrm{d}S}{\mathrm{d}W} = 4 - \frac{300}{w^2} = 0 \Rightarrow w = \sqrt{75} = 8.66$ cm

(g) $l = \frac{150}{w} = \frac{150}{\sqrt{75}} = 17.3$ cm

(h) $S = 40 + 4w + \frac{300}{w} = 40 + 4\left(\sqrt{75}\right) + \frac{300}{\sqrt{75}} = 40 + 40\sqrt{3} = 110$ cm

8. **(a)** $16x^4 - 27x$

(b) $f'(x) = 64x^3 - 27$

(c) $f'(x) = 64x^3 - 27 = 0 \Rightarrow x = \frac{3}{4}$

9. **(a)** $f(-2) = \frac{3}{4}(-2)^4 - (-2)^3 - 9(-2)^2 + 20 = 4$

(b) $f'(x) = 3x^3 - 3x^2 - 18x$

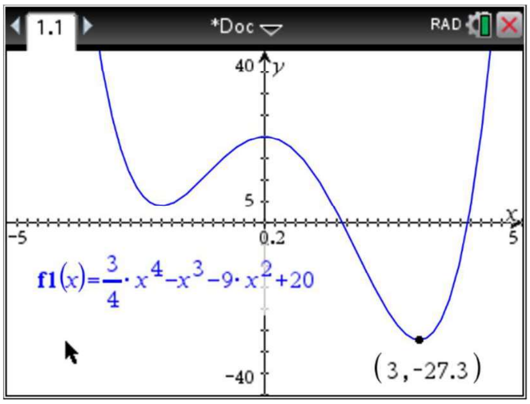
(c) $f'(3) = 3(3)^3 - 3(3)^2 - 18(3) = 0$ therefore $x = 3$ is a stationary point.

Now, we must show that this stationary point is in fact a local minimum.
We can do this by using the first derivative test or sketching the graph on our GDC.

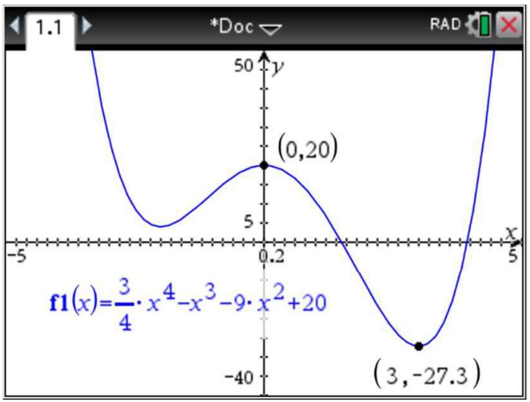
First derivative test:

$f'(2) < 0$, $f'(4) > 0$ therefore there is a local minimum at $x = 3$.

GDC graph:



(d)



(e) $(0, 20)$

(f) $f'(2) = 3(2)^3 - 3(2)^2 - 18(2) = -24$

(g) **(i)** $\frac{1}{24}$

(ii) $y + 12 = \frac{1}{24}(x - 2) \Rightarrow 24y + 288 = x - 2$

$\therefore b = -24, c = -290$

10. **(a)** $f'(x) = 3ax^2 - 3$

(b) $f'(0) = -3$

(c) Local maximum implies $f'(x) = 0 \Rightarrow f'(-2) = 3a(-2)^2 - 3 = 0 \Rightarrow a = \frac{1}{4}$.

11. **(a)** $f(1) = \dfrac{14}{(1)} + (1) - 6 = 9$

(b) $f'(x) = -\dfrac{14}{x^2} + 1$

(c) $f'(x) = -\dfrac{14}{x^2} + 1 = 0 \Rightarrow x^2 = 14 \Rightarrow x = \sqrt{14} = 3.7$

We see from the given graph that the stationary point at $x = 3.7$ must be a local minimum.

(d) From the graph, minimum occurs at $x = \sqrt{14}$.
Therefore, range is $1.48 \le f(x) \le 9$.

(e) $f(7) = \dfrac{14}{(7)} + (7) - 6 = 3$

(f) $m = \dfrac{9-3}{1-7} = -1$

(g) $M = \left(\dfrac{1+7}{2}, \dfrac{9+3}{2}\right) = (4, 6)$

(h) $f'(4) = -\dfrac{14}{4^2} + 1 = \dfrac{1}{8}$

(i) Point on function is $(4, f(4))$, $f(4) = \dfrac{14}{(4)} + (4) - 6 = 1.5 \Rightarrow (4, 1.5)$

Therefore equation of L is $y - 1.5 = \dfrac{1}{8}(x-4) \Rightarrow y = \frac{1}{8}x + 4$.

12. **(a)** $y = -\dfrac{75^2}{10} + \dfrac{27}{2} \times 75 = 450$ therefore A is on the track.

(b) $\dfrac{dy}{dx} = -\dfrac{2x}{10} + \dfrac{27}{2} = -0.2x + 13.5$

(c) Stationary point(s) occur at $\dfrac{dy}{dx} = -0.2x + 13.5 = 0 \Rightarrow x = 67.5$. Since A has

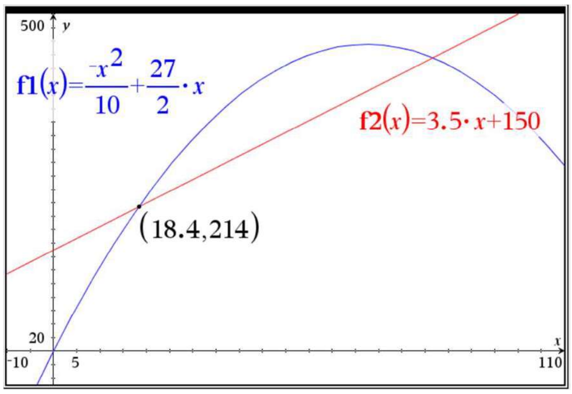
x- coordinate 75, it cannot be the farthest point north.

(d) **(i)** $M = \left(\dfrac{0+100}{2}, \dfrac{0+350}{2}\right) = (50, 175)$

(ii) $m = \dfrac{350-0}{100-0} = 3.5$

(e) $y - 150 = 3.5(x - 0) \Rightarrow 3.5x - y = -150$

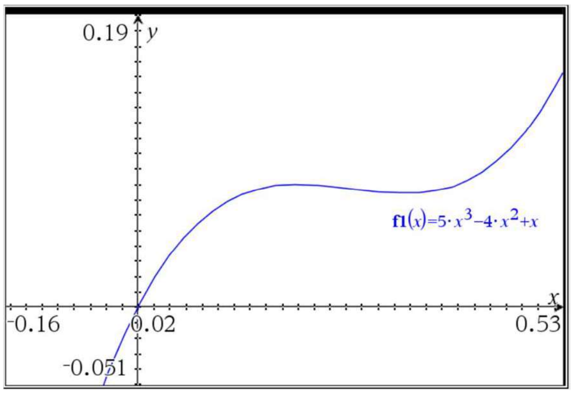
(f) Use your GDCs Intersection feature to find the first point of intersection is at $(18.4, 214)$.



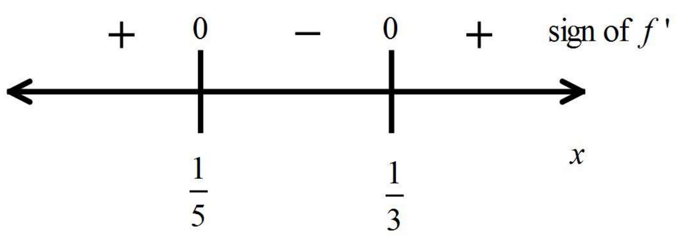
13. **(a)** $f'(x) = 15x^2 - 8x + 1$

(b) $f'(x) = 15x^2 - 8x + 1 = 0 \Rightarrow x = \frac{1}{5}, \frac{1}{3}$.

Use GDC graph or first derivative test to find which stationary point is a local minimum and which is a local maximum. GDC graph:



Or, via first derivative test, using sign diagram:



Therefore,

(i) local maximum is at $x = \frac{1}{5}$

(ii) local minimum is at $x = \frac{1}{3}$

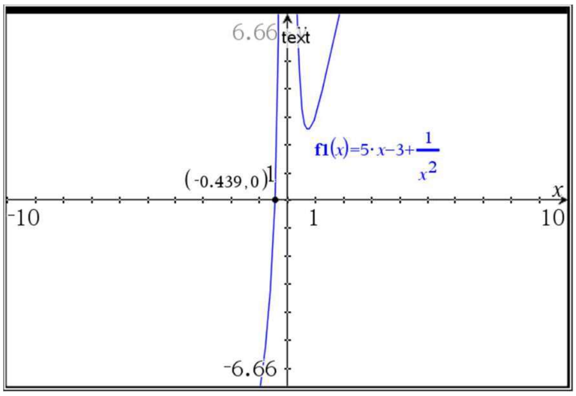
14. **(a)** Vertical asymptote occurs when denominator equals zero; therefore, $x^2 = 0 \;\Rightarrow\; x = 0$.

(b) $g'(x) = b - \dfrac{2}{x^3}$

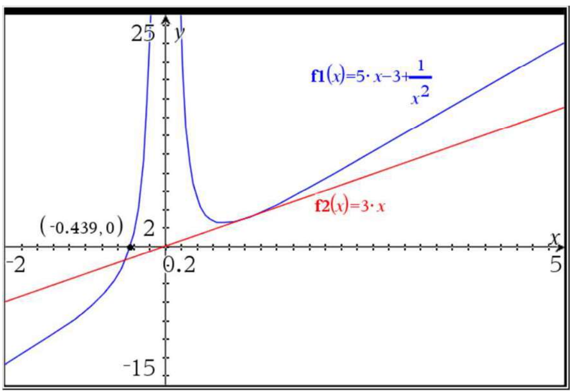
(c) $3 = b - \dfrac{2}{(1)^3} \;\Rightarrow\; b = 5$

(d) Point of tangency is at $g(1) = 3 \;\Rightarrow\; (1, 3)$, gradient is 3 (given), equation of T is $y - 3 = 3(x - 1) \;\Rightarrow\; y = 3x$.

(e) x-intercept is at $(-0.439, 0)$. Graph:



(f) **(i)** and **(ii)** shown on graph below:



(g) $(0.737, 2.53)$

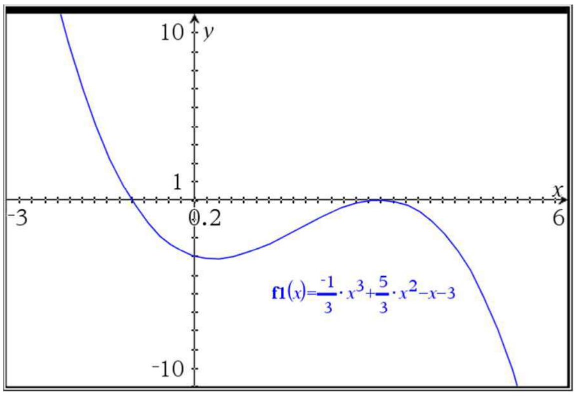
(h) $0.737 < x < 5$

15. **(a)** $\dfrac{dy}{dx} = 4x - 5$

(b) $L: 6x + 2y = -1 \;\Rightarrow\; y = -3x - \dfrac{1}{2} \;\Rightarrow\;$ Gradient of L is $m = -3$.

$\therefore \dfrac{dy}{dx} = 4x - 5 = -3 \;\Rightarrow\; x = \dfrac{1}{2}$.

16. **(a)**



(b) $f(-1)=-\frac{1}{3}(-1)^3+\frac{5}{3}(-1)^2-(-1)-3=0$

(c) $(0,-3)$

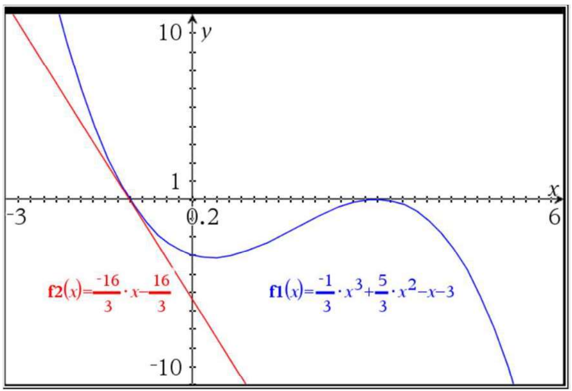
(d) $f'(x)=-x^2+\frac{10}{3}x-1$

(e) $f'(-1)=-(-1)^2+\frac{10}{3}(-1)-1=-\frac{16}{3}$

(f) $f'(-1)$ gives the gradient of the tangent to the curve at the point where $x=-1$.

(g) $y-0=-\frac{16}{3}(x+1)\ \Rightarrow\ y=-\frac{16}{3}x-\frac{16}{3}$

(h)



(i) **(i)** $a=\frac{1}{3}$ **(ii)** $b=3$

(j) $f(x)$ is increasing on the interval $a<x<b$

17. **(a)** $4(2x)+4(x)+4(y)=48 \Rightarrow 3x+y=12 \Rightarrow y=12-3x$

(b) $V=lwh=(x)(2x)(y)=2x^2y=2x^2(12-3x)=24x^2-6x^3$

(c) $\frac{dV}{dx}=48x-18x^2$

(d) $\frac{dV}{dx}=48x-18x^2=0 \Rightarrow x=0,\frac{8}{3}$. $x=0$ is nonsensical, so maximum volume occurs at $x=\frac{8}{3}=2.67$ m.

(e) $V=24\left(\frac{8}{3}\right)^2-6\left(\frac{8}{3}\right)^3=56.9$ m^3.

(f) length $=2x=5.33$ m, height $=y=12-3\left(\frac{8}{3}\right)=4$ m

(g) SA $=2\times\frac{16}{3}\times4+2\times\frac{8}{3}\times4+2\times\frac{16}{3}\times\frac{8}{3}=92.4$ m^2.

Therefore $\frac{92.4}{15\times4}=1.54 \Rightarrow 2$ tins are required.

18. **(a)** $\frac{dy}{dx}=\frac{-1}{2}\left(\frac{1}{2}-\frac{1}{4}x^2\right)^{\frac{-1}{2}}\cdot\left(\frac{-1}{2}x\right)$

when $x=1$, $\frac{dy}{dx}=\frac{-1}{2}\left(\frac{1}{2}-\frac{1}{4}(1)^2\right)^{\frac{-1}{2}}\cdot\left(\frac{-1}{2}(1)\right)=0.5$ so $y=0.5x+c$

when $x=1$, $y=-0.5$ so $-0.5=0.5(1)+c \Rightarrow c=-1$ and $y=0.5x-1$

(b) Equation is of the form $y=mx$ since it passes through the origin.

Angle with x-axis is $\frac{34.92°}{2}=17.46°$. Gradient $m=\tan(17.46°)=0.315$;

therefore equation is $y=0.315x$.

(c) Solving simultaneously $y=0.315x$ and $y=0.5x-1$ using the GDC leads to (5.39, 1.70)

(Be careful to use the value $\tan(17.46°)$ and not the rounded value 0.315 which leads to an x-value of 5.41)

(d) The throw is not valid, since the point Q is 5.65 m from the centre of the throwing circle, which implies that the discus will land outside the sector.

(e) **(i)** $R(0.795, -0.585)$

(ii) $y=0.340x-0.855$

19. The rate of increase of the diameter is twice $\frac{dr}{dt}$ where r is the radius (since it is growing from both sides of the cone).

We are given $\frac{dV}{dt} = 0.5$

Model involving V and r: $V = \frac{1}{3}\pi r^2 h$

From the right-angled triangle with sides r and h and angle 34°, we obtain

$\tan(34) = \frac{h}{r}$ so $h = r\tan(34)$ and $V = \frac{1}{3}\pi r^3 \tan(34)$ --- (1)

Differentiating with respect to time: $\frac{dV}{dt} = \frac{1}{3}\pi \tan(34)\cdot 3r^2 \frac{dr}{dt}$ ---(2)

If the pile was started 4 hours ago, its volume is now $0.5 \times 4 = 2$ m^3 and, substituting into (1): $2 = \frac{1}{3}\pi r^3 \tan(34)$ gives $r \approx 1.41$ m

Substituting these values into (2): $0.5 = \frac{1}{3}\pi \tan(34)\cdot 3(1.41)^2 \frac{dr}{dt}$

which leads to $\frac{dr}{dt} = 0.118$ m h^{-1} so a diameter increase of 0.236 m h^{-1}

20. Let x be the distance between the foot of the wall and the base of the ladder so $\frac{dx}{dt} = 0.5$

Let L be the length of the ladder, so $\frac{dL}{dt} = -1$ (negative since the length is decreasing).

Let y be the height of the top of the ladder up the wall We are asked to find $\frac{dy}{dt}$ which should be a negative value since the ladder is falling.

Using the right-angled triangle with sides x, y and L: $L^2 = x^2 + y^2$

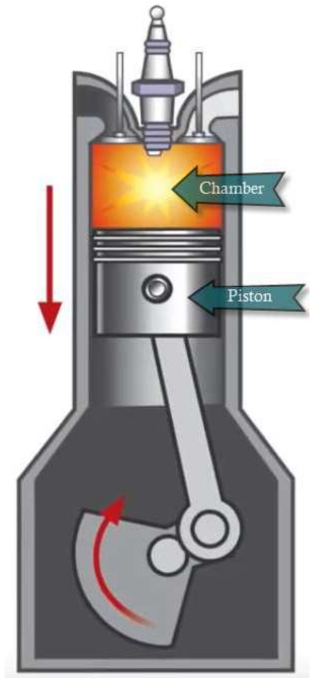
Differentiating with respect to time: $2L\frac{dL}{dt} = 2x\cdot\frac{dx}{dt} + 2y\cdot\frac{dy}{dt}$

When $x = 1.2$ and $L = 3.7$, $y = \sqrt{3.7^2 - 1.2^2} = 3.5$

Substituting the given values, we have $2(3.7)(-1) = 2(1.2)\cdot(0.5) + 2(3.5)\cdot\frac{dy}{dt}$

leading to $\frac{dy}{dt} = -1.23$ m s^{-1}

21. Boyle's law: $V = \frac{k}{P}$



Find $\frac{dP}{dt}$ given that $\frac{dh}{dt} = -4$

$V = \pi(4.5)^2(9) \approx 573$ so $573 = \frac{k}{80} \Rightarrow k = 45800$ and

$V = \frac{45800}{P}$

Differentiating with respect to time:

$\frac{dV}{dt} = 45800 \cdot (-1)P^{-2}\frac{dP}{dt}$

First we need to find $\frac{dV}{dt}$ knowing $\frac{dh}{dt}$.

For this we use $V = \pi(4.5)^2 \cdot h$

Differentiating with respect to time: $\frac{dV}{dt} = \pi(4.5)^2 \cdot \frac{dh}{dt}$

so $\frac{dV}{dt} = \pi(4.5)^2 \cdot (-4) = -81\pi$ and, when $h = 2$, $V = \pi(4.5)^2 \cdot 2 = 40.5\pi$ and

$40.5\pi = \frac{45800}{P} \Rightarrow P = \frac{45800}{40.5\pi}$

So $-81\pi = 45800 \cdot (-1)\left(\frac{45800}{40.5\pi}\right)^{-2}\frac{dP}{dt}$ leading to $\frac{dP}{dt} = 720$ kPa s^{-1}

22. **(a)** Consider the right-angled triangle with sides 2 km and 6 km. The distance is the hypotenuse. Using Pythagoras: $\sqrt{2^2 + 6^2} = \sqrt{40} = 2\sqrt{10}$ km

(b) Consider the right-angled triangle with sides 2 km, $(6 - w)$ km and hypotenuse r km. Using Pythagoras: $r^2 = (6-w)^2 + 2^2$ so $r = \sqrt{(6-w)^2 + 4}$

(c) $\text{Time} = \frac{\text{distance}}{\text{speed}}$.

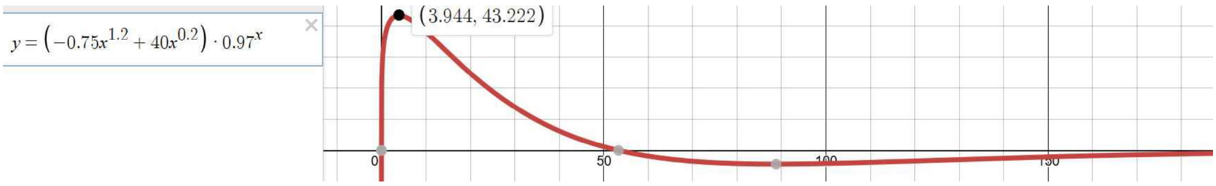
Time rowing is $T_R = \frac{\sqrt{(6-w)^2 + 4}}{3}$ while the time walking is $T_W = \frac{w}{5}$

Therefore, the total time is $T = \frac{\sqrt{(6-w)^2 + 4}}{3} + \frac{w}{5}$

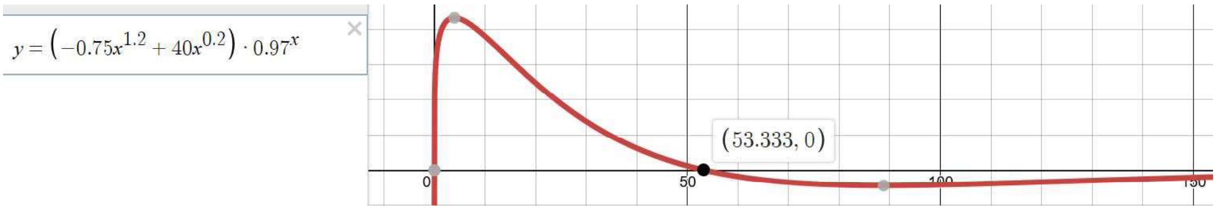
(d) Using a GDC leads to $w = 4.50$ km and so $r = \sqrt{(6 - 4.50)^2 + 4} = 2.50$ km

(e) Substituting, we find $T = 1.73$ hours

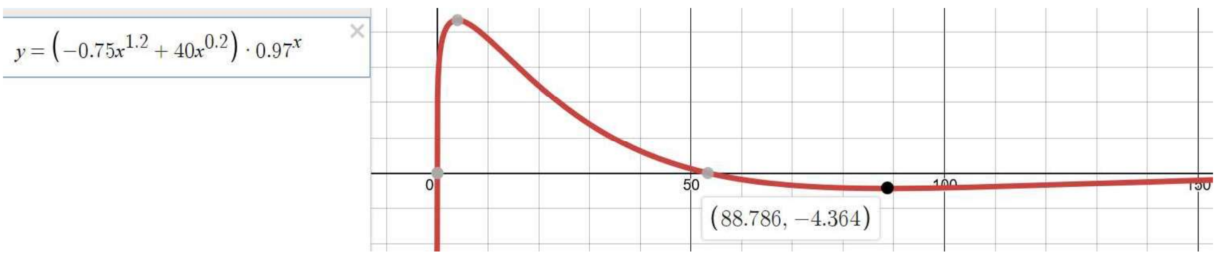
23. **(a)** Using a GDC or graphing package, we find :
3.94 min, 43.2 nanograms ml^{-1} min^{-1}



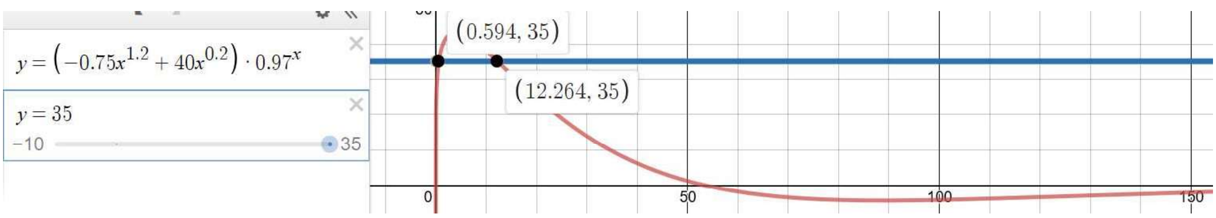
(b) Using a GDC or graphing package, we are looking for a zero of the derivative function, going from positive to negative. This point has t-value 53.3 min.



(c) Using a GDC or graphing package, we are looking for the minimum point on the derivative function, so t = 88.8 min and $\frac{dC}{dt}$ = −4.36 nanograms l^{-1} min^{-1}

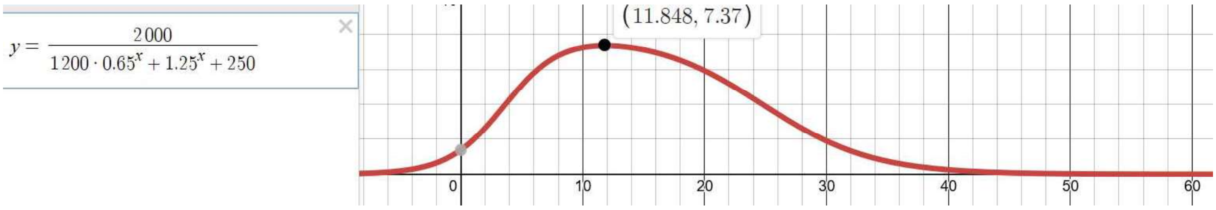


(d) Using a GDC or graphing package, we find that this is the case from 0.594 min to 12.3 min.



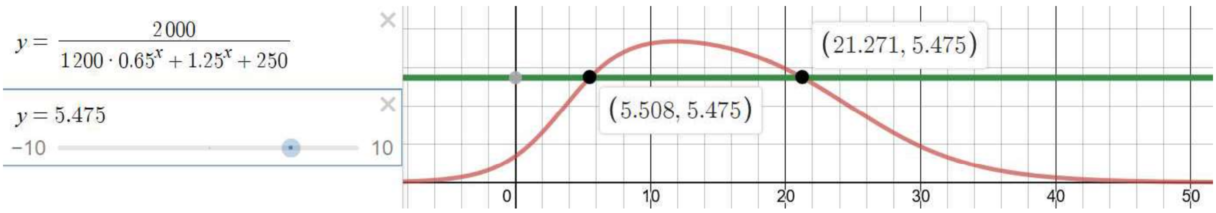
24. **(a)** Thousands of new subscribers per year.

(b) Using a GDC or graphing package, we find that it is when $t = 11.8 \Rightarrow$ late in 2011 at a rate of 7370 new subscribers per year.



(c) 15 subscribers per day corresponds to $15 \times 365 = 5475$ subscribers per year, or $S'(t) = 5.475$ since it is expressed in thousands of new subscribers per year.

Using a GDC or graphing package, we find that it is for $5.51 < t < 21.3 \Rightarrow$ mid-2005 to early 2021.



(d) We can see from the graph that $S'(t)$ remains positive for all $t \in \mathbb{R}$.

25. **(a)** We want to maximise the volume of a cylinder $V = \pi r^2 h$

40 rectangles of 5 cm by 100 cm covers a total (flat) surface of $40 \times 5 \times 100 = 20000$ cm^2

The total surface area of a cylinder is $A = 2\pi rh + 2\pi r^2$ so $20000 = 2\pi rh + 2\pi r^2$

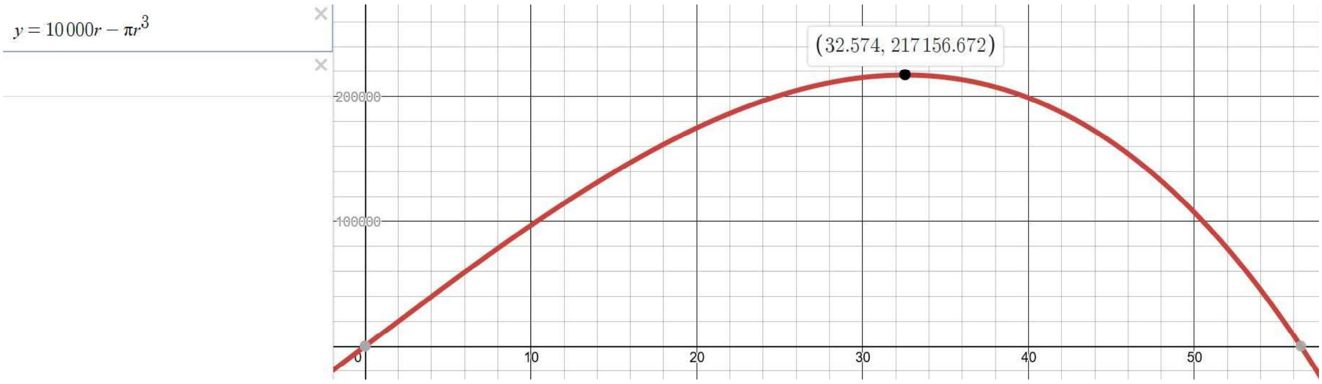
From this, we can isolate *h*: $h = \frac{20000 - 2\pi r^2}{2\pi r} = \frac{10000 - \pi r^2}{\pi r}$

And substitute in the formula for the volume:

$$V = \pi r^2 \cdot \frac{10000 - \pi r^2}{\pi r} = r \cdot \left(10000 - \pi r^2\right) = 10000r - \pi r^3$$

Using a GDC, we find that the radius = 32.6 cm

and, therefore, the height is $h = \frac{10000 - \pi (32.6)^2}{\pi (32.6)} \approx 65.1$ cm



(b) From the graph above, we see that the maximum volume is 217 litres.

Exercise 15.1

1. **(a)** continuous

(b) discrete

(c) continuous

(d) discrete

(e) continuous

(f) continuous

2. for example, the even numbers, the number of times you can play the lottery and lose

3. mean $\mu = \frac{1}{6} \times (1+2+3+4+5+6) = \frac{21}{6} = 3.5$

Variance $\sigma^2 = \frac{1}{6} \times (1^2+2^2+3^2+4^2+5^2+6^2) - 3.5^2 = \frac{35}{12} = 2.92$

4. mean $\mu = \frac{1}{6} \times (1+2+3+7+8+9) = 5$

Variance $\sigma^2 = \frac{1}{6} \times (1^2+2^2+3^2+7^2+8^2+9^2) - 5^2 = \frac{58}{6} = 9.67$

5. mean $\mu = \frac{1}{6} \times (1+1+2+3+4+5) = \frac{8}{3} = 2.67$

Variance $\sigma^2 = \frac{1}{6} \times (1^2+1^2+2^2+3^2+4^2+5^2) - \left(\frac{8}{3}\right)^2 = 2.06$

6. mean $\mu = \frac{1}{6} \times (1+2+3+4+5+6+7+8+9+10+11+12)$

use $\sum_{i=1}^{12} i = \frac{12}{2}(1+12)$

So mean $\mu = 13$

Variance $\sigma^2 = \frac{1}{6} \times (1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2+9^2+10^2+11^2+12^2) - 13^2$

use $\sum_{i=1}^{n} i^2 = \frac{1}{6} n(n+1)(2n+1) \Rightarrow \sum_{i=1}^{12} i^2 = \frac{1}{6} 12 \times 13 \times 25 = 1950$

So Variance $\sigma^2 = \frac{1}{6} \times 1950 - 13^2 = 156$

7. $E(x) = 1 \times 0.4 + 2a + 3b = 1 \quad \Rightarrow \quad 2a + 3b = 0.6$

8. $E(X) = 2\times0.4+4\times0.3+6\times0.2+8\times0.1 = 4$

$\text{Var}(X) = \left(2^2\times0.4+4^2\times0.3+6^2\times0.2+8^2\times0.1\right)-4^2 = 4$

9. probability of each score $= \dfrac{1}{5}$

$\text{E}(X) = \dfrac{1}{5}\times(1+2+3+4+5) = \dfrac{15}{5} = 3$

Variance $\text{Var}(X) = \dfrac{1}{5}\times\left(1^2+2^2+3^2+4^2+5^2\right)-3^2 = 2$

10. $E(X) = 1\times0.25+2\times0.375+3\times0.25+4\times0.0625 = 2$

$Var(X) = (1^2\times0.25+2^2\times0.375+3^2\times0.25+4^2\times0.0625)-2^2 = 1$

11. (a) $\text{E}(X) = \int_1^2 x\times\dfrac{2}{3}x\,\text{d}x = \int_1^2 \dfrac{2}{3}x^2\,\text{d}x = \left[\dfrac{2}{9}x^3\right]_1^2 = \dfrac{14}{9}$

$\text{Var}(X) = \int_1^2 x^2\times\dfrac{2}{3}x\,\text{d}x - \left(\dfrac{14}{9}\right)^2 = \int_1^2 \dfrac{2}{3}x^3\,\text{d}x = \left[\dfrac{1}{6}x^4\right]_1^2 = 2.5$

(b) $\text{P}(1.5\le X\le 2) = \int_{1.5}^2 \dfrac{2}{3}x\,\text{d}x = \int_{1.5}^2 \dfrac{2}{3}x\,\text{d}x = \left[\dfrac{1}{3}x^2\right]_{1.5}^2 = \dfrac{4}{3}-\dfrac{3}{4} = \dfrac{7}{12}$

12. (a) $\int_0^3 \dfrac{1}{9}x^2\,\text{d}x = \left[\dfrac{1}{27}x^3\right]_0^3 = 1$, i.e. area under pdf = 1 as required

(b) (i) $\text{E}(X) = \int_0^3 \dfrac{1}{9}x^2\times x\,\text{d}x \quad = \int_0^3 \dfrac{x^3}{9}\,\text{d}x = \left[\dfrac{1}{36}x^4\right]_0^3 = \dfrac{81}{36} = \dfrac{9}{4}$

$$\begin{aligned}\text{Var}(X) &= \int_0^3 \frac{1}{9}x^2\times x^2\,\text{d}x - \left(\frac{9}{4}\right)^2\\ &= \int_0^3 \frac{x^4}{9}\,\text{d}x - \left(\frac{9}{4}\right)^2\\ &= \left[\frac{1}{45}x^5\right]_0^3 - \left(\frac{9}{4}\right)^2\\ &= 0.3375\end{aligned}$$

(ii) $\text{P}(1\le X\le 2) = \int_1^2 \dfrac{1}{9}x^2\,\text{d}x = \left[\dfrac{1}{27}x^3\right]_1^2 = \dfrac{7}{27}$

13. area under pdf = 1 requires $\int_1^3 kx^2\,\text{d}x = 1 \;\Rightarrow\; \left[\dfrac{k}{3}x^3\right]_1^3 = 1 \;\Rightarrow\; 9k-\dfrac{k}{3} = 1 \;\Rightarrow\; k = \dfrac{3}{26}$

14. **(a)** Use GDC to evaluate $\int_{-3}^{3} e^{-x^2}\, dx$ (graph the function, then e.g. math function 9: FnInt(…))

$$\int_{-3}^{3} e^{-x^2}\, dx = 1.77$$

$$\int_{-3}^{3} k e^{-x^2}\, dx = k \times \int_{-3}^{3} e^{-x^2}\, dx = 1.77k$$

So $\int_{-3}^{3} k e^{-x^2}\, dx = 1 \Rightarrow 1.77k = 1 \Rightarrow k = 0.564\,20$

(b) **(i)** repeat calculation with limits -4, 4: k= 0.56419

(ii) repeat calculation with limits -100, 100: k= 0.56419 (same result to 5 d.p.)

(c) bell (normal) curve

Exercise 15.2

1. **(a)** binomial distribution with $n = 20, p = 0.5$

$$\mathrm{E}(X) = \mu = np = 20 \times \frac{1}{2} = 10$$

$$\mathrm{Var}(X) = np(1-p) = 20 \times \frac{1}{2} \times \left(1 - \frac{1}{2}\right) = 5$$

(b) $\mathrm{P}(X = 10) = \frac{20!}{10!10!} \times \frac{1}{2}^{10} \times \frac{1}{2}^{10} = 184756 \times 9.53674 \times 10^{-7} = 0.176$

or use GDC binomialpdf$(20, 0.5, 10)$ etc

(c) $\mathrm{P}(X = 0) = 9.54 \times 10^{-7}$

$\mathrm{P}(X = 4) = 4.62 \times 10^{-3}$

$\mathrm{P}(X = 8) = 0.120$

$\mathrm{P}(X = 12) = 0.120$

$\mathrm{P}(X = 16) = 4.62 \times 10^{-3}$

$\mathrm{P}(X = 20) = 9.54 \times 10^{-7}$

2. **(a)** binomial distribution with $n = 50, p = 0.5$

$$\mathrm{E}(X) = \mu = np = 50 \times \frac{1}{2} = 25$$

$$\mathrm{Var}(X) = np(1-p) = 50 \times \frac{1}{2} \times \left(1 - \frac{1}{2}\right) = 12.5$$

(b) $\mathrm{P}(X = 25) = \frac{50!}{25!25!} \times \frac{1}{2}^{25} \times \frac{1}{2}^{25} = 0.112$

3. **(a)** binomial distribution with $n = 10, p = 0.8$

$$\mathrm{E}(X) = \mu = np = 10 \times 0.8 = 8$$

$$\mathrm{Var}(X) = np(1-p) = 10 \times 0.8 \times (1 - 0.8) = 1.6$$

(b) $\mathrm{P}(X = 8) = \frac{10!}{8!2!} \times 0.8^8 \times 0.2^2 = 0.302$

(c) $\mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0) = 1 - 0.2^{10} = 0.999999897\,(9\mathrm{dp})$

4. **(a)** binomial distribution with $n = 60, p = \frac{1}{6}$

(i) $\mathrm{E}(X) = \mu = np = 60 \times \frac{1}{6} = 10$

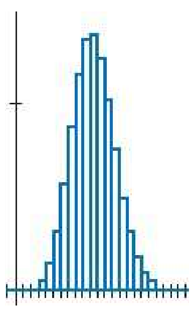
$$\mathrm{Var}(X) = np(1-p) = 60 \times \frac{1}{6} \times \left(1 - \frac{1}{6}\right) = \frac{50}{6}$$

(ii) $\mathrm{P}(X = 10) = \frac{60!}{10!50!} \times \frac{1}{6}^{10} \times \frac{5}{6}^{50} = 0.137$

(iii) $\mathrm{P}(X = 0) = \left(\frac{5}{6}\right)^{60} = 1.77 \times 10^{-5}$

(iv) $\mathrm{P}(X = 60) = \frac{1}{6}^{60} = 2.05 \times 10^{-47}$

(b)



5. **(a)** binomial distribution with $n = 30, p = 0.8$

$\mathrm{E}(X) = \mu = np = 30 \times 0.8 = 24$

$\mathrm{Var}(X) = np(1-p) = 30 \times 0.8 \times (1-0.8) = 4.8$

(b) $\mathrm{P}(X = 24) = \dfrac{30!}{24!6!} \times 0.8^{24} \times 0.2^{6} = 0.179$

6. **(a)** binomial distribution with $n = 6, p = 0.3$

$\mathrm{E}(X) = \mu = np = 6 \times 0.3 = 1.8$

$\mathrm{Var}(X) = np(1-p) = 6 \times 0.3 \times (1-0.3) = 1.26$

(b) $\mathrm{P}(X = 1) = \dfrac{6!}{1!5!} \times 0.3^{1} \times 0.7^{5} = 0.303$

(c) $\mathrm{P}(X < 6) = 1 - \mathrm{P}(X = 6) = 1 - 0.3^{6} = 0.9993$

7. **(a)** $\mathrm{P}(X = 10) = \dfrac{60!}{10!50!} \times 0.3^{10} \times 0.7^{50} = 0.0080$, so not the same

(b) $\mathrm{E}(X) = \mu = np = 60 \times 0.3 = 18$

$\mathrm{Var}(X) = np(1-p) = 60 \times 0.3 \times (1-0.3) = 12.6$

(c) variance is factor 10, so standard deviation is factor $\sqrt{10}$ greater

8. **(a)** binomial distribution with $n = 20, p = 0.1$

$\mathrm{E}(X) = \mu = np = 20 \times 0.1 = 2$

$\mathrm{Var}(X) = np(1-p) = 20 \times 0.1 \times (1-0.1) = 1.8$

(b) $\mathrm{P}(X = 8) = \dfrac{20!}{8!12!} \times 0.1^{8} \times 0.9^{12} = 0.000356$

(c) GDC $\mathrm{P}(X \geq 8) = 1 - \mathit{binomialcdf}(20, 0.1, 7) = 0.000416$

9. binomial distribution with $n = 10, p = 0.9$

10. binomial distribution with $n = 4, p = 0.2$

$\mathrm{P}(X > 1) = 1 - \mathrm{P}(X = 0) - \mathrm{P}(X = 1) = 1 - 0.8^{4} - 4 \times 0.2^{1} \times 0.8^{3} = 0.181$

11. binomial distribution with $n = 10, p = 0.8$

(a) $\mathrm{P}(X \geq 8) = \mathrm{P}(X = 9) + \mathrm{P}(X = 10) = \dfrac{10!}{9!1!} \times 0.8^{9} \times 0.2^{1} + 0.8^{10} = 0.678$

(b) $\mathrm{P}(X = 10) = 0.8^{10} = 0.107$

Exercise 15.3

1. model with poisson distribution with mean $m = 0.01 \times 50 = 0.5$

(a) $P(X=2) = \dfrac{0.5^2 \times e^{-0.5}}{2!} = \dfrac{1}{8\sqrt{e}} = 0.0758$

(b) $P(X=3) = \dfrac{0.5^3 \times e^{-0.5}}{3!} = \dfrac{1}{48\sqrt{e}} = 0.0126$

(c) $P(X=0) = \dfrac{0.5^0 \times e^{-0.5}}{0!} = \dfrac{e^{-0.5}}{1} = \dfrac{1}{\sqrt{e}} = 0.607$

2. model with poisson distribution with mean $m = 0.006 \times 1000 = 6$

$P(X=5) = \dfrac{6^5 \times e^{-6}}{5!} = 0.161$

3. model with poisson distribution with mean $m = 0.001 \times 1000 = 1$

$P(X=2) = \dfrac{1^2 \times e^{-1}}{2!} = \dfrac{1}{2e} = 0.184$

4. model with poisson distribution with mean $m = 0.0001 \times 100 = 0.01$

$P(X=2) = \dfrac{0.01^2 \times e^{-0.01}}{2!} = \dfrac{10^{-4} \times e^{-0.01}}{2} = 4.95 \times 10^{-5}$

5. model with poisson distribution with mean $m = \dfrac{1}{44} \times 100 = \dfrac{25}{11}$

$P(X=3) = \dfrac{\left(\dfrac{25}{11}\right)^3 \times e^{-\frac{25}{11}}}{3!} = 0.202$

6. model with poisson distribution with mean $m = \dfrac{1}{2} \times 20 = 10$

$P(X=10) = \dfrac{10^{10} \times e^{-10}}{10!} = 0.125$

7. model with poisson distribution with mean $m = \dfrac{1500}{365}$ earthquakes per day

(a) $P(X=3) = \dfrac{\left(\dfrac{1500}{365}\right)^3 \times e^{-\frac{1500}{365}}}{3!} = 0.190$

(b) $P(X=0) = \dfrac{1 \times e^{-\frac{1500}{365}}}{0!} = e^{-\frac{1500}{365}} = 0.0164$

(c) $P(X>0) = 1 - P(X=0) = 1 - 0.0164 = 0.984$

8. model with poisson distribution with mean $m = 1$ absence per 100 days

$$P(X=2) = \frac{1^2 \times e^{-1}}{2!} = \frac{1}{2e} = 0.184$$

9. model with poisson distribution with mean $m = \frac{1}{2}$ delays per 50 days

$$P(X>1) = 1 - P(X=0) - P(X=1) = 1 - \frac{0.5^0 \times e^{-0.5}}{0!} - \frac{0.5^1 \times e^{-0.5}}{1!}$$

$$= 1 - \frac{1}{\sqrt{e}} - \frac{1}{2\sqrt{e}} \qquad = 1 - \frac{3}{2\sqrt{e}} = 0.0902$$

10. model with poisson distribution with mean $m = \frac{100}{240} = \frac{5}{12}$ accidents per 100 days

$$P(X=2) = \frac{\left(\frac{5}{12}\right)^2 \times e^{-\frac{5}{12}}}{2!} = 0.0572$$

Exercise 15.4

1. $\mu = 181, \sigma = 8$

So between $181-8$ and $181+8$, 68% of population: 173 cm to 189 cm

between $181-16$ and $181+16$, 68% of population: 165 cm to 197 cm

between $181-24$ and $181+24$, 68% of population: 157 cm to 205 cm

2. $X \sim N(67, 6.4^2)$ Use GDC normalcdf$(63, 73, 67, 4)$

$P(63 \le X \le 73) = 0.560$ so 56.0%

3. $X \sim N(60, 6.2^2)$ Use GDC normalcdf$(50, 65, 60, 6.2)$

$P(50 \le X \le 65) = 0.737$ so 73.7%

4. $X \sim N(165, 9.3^2)$

$P(X > 180) = 1 - P(X \le 180) = 1 - 0.947 = 0.053$ so 5.3%

5. $X \sim N(43, 0.3^2)$

(a) $P(X) = 0.05 \Rightarrow X = 42.5$ so 42.5 km

(b) $P(X) = 0.95 \Rightarrow X = 43.5$ so 43.5 km

6. $X \sim N\left(30, 6^2\right)$

$\mathrm{P}\left(24 \le X \le 36\right) = 0.68$ so 68%

7. $X \sim N\left(160, 7.1^2\right)$

$\mathrm{P}\left(X\right) = 0.90 \Rightarrow X = 169$ so ≥ 169cm

8. $X \sim N\left(0.04, 0.1^2\right)$

$\mathrm{P}\left(-0.1 \le X \le 0.1\right) = 0.645$ so 64.5%

9. $X \sim N\left(100, 0.3^2\right)$

$\mathrm{P}\left(X > 3\right) = 1 - \mathrm{P}\left(X \le 3\right) = 1 - 0.748 = 0.252$

10. $X \sim N\left(100, \sigma^2\right)$ and $\mathrm{P}\left(X > 140\right) = 0.31$

Use normalcdf $\left(100,140,100,\sigma\right) = 0.50 - 0.31 = 0.19$, to get $\sigma = 80.7$

So variance $= 80.7^2 = 6500$ mm^{2}Hg

Exercise 15.5

1. $\mathrm{E}\left(X\right) = \dfrac{2+4+6+8}{4} = 5$, $\mathrm{Var}\left(X\right) = \dfrac{(2-5)^2 + (4-5)^2 + (6-5)^2 + (8-5)^2}{4} = 5$

(a) $\mathrm{E}\left(X+5\right) = \mathrm{E}\left(X\right) + 5 = 5 + 5 = 10$, $\mathrm{Var}\left(X+5\right) = \mathrm{Var}\left(X\right) = 5$

(b) $\mathrm{E}\left(3X\right) = 3\mathrm{E}\left(X\right) = 3 \times 5 = 15$, $\mathrm{Var}\left(3X\right) = 3^2 \times \mathrm{Var}\left(X\right) = 9 \times 5 = 45$

2. **(a)** $\mathrm{E}\left(X+5\right) = \mathrm{E}\left(X\right) + 5 = 100 + 5 = 105$, $\mathrm{Var}\left(X+5\right) = \mathrm{Var}\left(X\right) = 4^2 = 16$

(b) $\mathrm{E}\left(3X\right) = 3\mathrm{E}\left(X\right) = 3 \times 100 = 300$, $\mathrm{Var}\left(3X\right) = 3^2 \times \mathrm{Var}\left(X\right) = 9 \times 16 = 144$

3. **(a)** $\mathrm{E}\left(X_1 + X_2\right) = \mathrm{E}\left(X_1\right) + \mathrm{E}\left(X_2\right) = 67 + 33 = 100$,

$\mathrm{Var}\left(X_1 + X_2\right) = \mathrm{Var}\left(X_1\right) + \mathrm{Var}\left(X_2\right) = 12^2 + 5^2 = 13^2 = 169$

(b) $\mathrm{E}\left(2X_1 - X_2\right) = 2 \times \mathrm{E}\left(X_1\right) - \mathrm{E}\left(X_2\right) = 2 \times 67 - 33 = 101$,

$\mathrm{Var}\left(2X_1 - X_2\right) = 2^2 \times \mathrm{Var}\left(X_1\right) + \mathrm{Var}\left(X_2\right) = 4 \times 12^2 + 5^2 = 601$

4. difference Y between two 2 kg and one 4 kg is distributed as $Y = 2X_1 - X_2$ with

$\mathrm{E}(Y) = \mathrm{E}(2X_1 - X_2) = 2\times\mathrm{E}(X_1) - \mathrm{E}(X_2) = 2\times 2.3 - 4.4 = 0.2$,

$\mathrm{Var}(Y) = \mathrm{Var}(2X_1 - X_2) = 2^2\times\mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 4\times 0.2^2 + 0.2^2 = 0.2$

i.e. $Y \sim N(0.2, 0.2^2)$

$\mathrm{P}(Y > 0) = \mathrm{P}(Y \le 0.2 + 0.2)(\text{symmetry about mean}) = \mathrm{P}(Y \le 0.4) = 0.841$

5. $\mathrm{E}(X) = \mathrm{E}(2X_1 + X_2) = 2\times\mathrm{E}(X_1) + \mathrm{E}(X_2) = 2\times 13 + 12 = 38$,

$\mathrm{Var}(Y) = \mathrm{Var}(2X_1 + X_2) = 2^2\times\mathrm{Var}(X_1) + \mathrm{Var}(X_2) = 4\times 0.4^2 + 0.3^2 = 0.730$

$\sigma = \sqrt{0.730} = 0.854$

i.e. $X \sim N(38, 0.854^2)$

$\mathrm{P}(X \le 40) = 0.990$

6. $\mathrm{E}(X) = \mathrm{E}(X_1 + X_2) = \mathrm{E}(X_1) + \mathrm{E}(X_2) = 3 + 4 = 7$

$\mathrm{P}(X = 5) = \dfrac{7^5 \mathrm{e}^{-7}}{5!} = 0.128$

7. **(a)** travel agency: mean $= \dfrac{40}{8} = 5$ calls per hour, $\mathrm{P}o(5)$

Used car: mean $= \dfrac{20}{8} = 2.5$ calls per hour, $\mathrm{Po}(2.5)$

(b) **(i)** $\mathrm{E}(X_{TA}) = 5$ calls per hour

(ii) $\mathrm{E}(X_{UC}) = 2.5$ calls per hour

(iii) $\mathrm{E}(X_{\text{total}}) = \mathrm{E}(X_{TA}) + \mathrm{E}(X_{UC}) = 5 + 2.5 = 7.5$ calls per hour

(iv) $\mathrm{P}(X_{\text{total}} = 10) = \dfrac{7^{10}\mathrm{e}^{-7.5}}{10!} = 0.0858$

8. hummingbird feeder X_1: mean $=\frac{6}{6}=1$ visit per 10 minutes bird feeder X_2: mean $=\frac{12}{6}=2$ visits per 10 minutes

(a) $\mathrm{P}(X_1=2)=\frac{1^2\mathrm{e}^{-1}}{2!}=\frac{1}{2\mathrm{e}}=0.184$

(b) $\mathrm{P}(X_2=2)=\frac{2^2\mathrm{e}^{-2}}{2!}=\frac{2}{\mathrm{e}^2}=0.271$

(c) sum (X_1+X_2) is distributed as $Po(1+2)=Po(3)$.

$\mathrm{P}(X_1+X_2=4)=\frac{3^4\mathrm{e}^{-3}}{4!}=\frac{27}{8\mathrm{e}^3}=0.168$

Note $\mathrm{P}(X_1+X_2=4)\neq\mathrm{P}(X_1=2)+\mathrm{P}(X_2=2)$

(d) at most 4 times ⇒ $(X_1+X_2)=0\,or\,1\,or\,2\,or\,3\,or\,4$.

$$\mathrm{P}(X_1+X_2=0,1,2,3,4)=\frac{3^0\mathrm{e}^{-3}}{0!}+\frac{3^1\mathrm{e}^{-3}}{1!}+\frac{3^2\mathrm{e}^{-3}}{2!}+\frac{3^3\mathrm{e}^{-3}}{3!}+\frac{3^4\mathrm{e}^{-3}}{4!}$$

$$=\left(1+3+\frac{9}{2}+\frac{27}{6}+\frac{81}{24}\right)\times\frac{1}{\mathrm{e}^3}=0.815$$

Exercise 15.6

1. (a) (i)

$$x_1=\begin{pmatrix}0.7 & 0.2 & 0.1\end{pmatrix}\times\begin{pmatrix}0.8 & 0.1 & 0.1\\ 0.4 & 0.4 & 0.2\\ 0.4 & 0.3 & 0.3\end{pmatrix}$$

$$=\begin{pmatrix}0.68 & 0.18 & 0.14\end{pmatrix}$$

(ii)

$$x_2=\begin{pmatrix}0.68 & 0.18 & 0.14\end{pmatrix}\times\begin{pmatrix}0.8 & 0.1 & 0.1\\ 0.4 & 0.4 & 0.2\\ 0.4 & 0.3 & 0.3\end{pmatrix}$$

$$=\begin{pmatrix}0.672 & 0.182 & 0.146\end{pmatrix}$$

$$x_3=\begin{pmatrix}0.672 & 0.182 & 0.146\end{pmatrix}\times\begin{pmatrix}0.8 & 0.1 & 0.1\\ 0.4 & 0.4 & 0.2\\ 0.4 & 0.3 & 0.3\end{pmatrix}$$

$$=\begin{pmatrix}0.6688 & 0.1838 & 0.1474\end{pmatrix}$$

(iii) $$x_{10} = \begin{pmatrix} 0.7 & 0.2 & 0.1 \end{pmatrix} \times \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}^{10}$$

$$= \begin{pmatrix} 0.66667 & 0.18518 & 0.14815 \end{pmatrix}$$

(b) e,g. $$\begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}^{100} = \begin{pmatrix} 0.666667 & 0.185185 & 0.148148 \\ 0.666667 & 0.185185 & 0.148148 \\ 0.666667 & 0.185185 & 0.148148 \end{pmatrix}$$ (6 decimal places)

$$x_{100} = \begin{pmatrix} 0.7 & 0.2 & 0.1 \end{pmatrix} \times \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}^{100}$$

$$= \begin{pmatrix} 0.7 & 0.2 & 0.1 \end{pmatrix} \times \begin{pmatrix} 0.666667 & 0.185185 & 0.148148 \\ 0.666667 & 0.185185 & 0.148148 \\ 0.666667 & 0.185185 & 0.148148 \end{pmatrix}$$

$$= \begin{pmatrix} 0.666667 & 0.185183 & 0.148147 \end{pmatrix}$$ (6 decimal places)

2. **(a)** **(i)** $$x_1 = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix} \times \begin{pmatrix} 0.1 & 0.2 & 0.5 & 0.2 \\ 0.6 & 0 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.6 & 0.3 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3 & 0.25 & 0.3 & 0.15 \end{pmatrix}$$

(ii) $$x_2 = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix} \times \begin{pmatrix} 0.1 & 0.2 & 0.5 & 0.2 \\ 0.6 & 0 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.6 & 0.3 & 0 \end{pmatrix}^2$$

$$= \begin{pmatrix} 0.315 & 0.21 & 0.3 & 0.175 \end{pmatrix}$$

$$x_3 = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix} \times \begin{pmatrix} 0.1 & 0.2 & 0.5 & 0.2 \\ 0.6 & 0 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.6 & 0.3 & 0 \end{pmatrix}^3$$

$$= \begin{pmatrix} 0.295 & 0.228 & 0.303 & 0.174 \end{pmatrix}$$

Trend: values appear to oscillate with decreasing amplitudes.

(iii) $x_{10} = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix} \times \begin{pmatrix} 0.1 & 0.2 & 0.5 & 0.2 \\ 0.6 & 0 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.6 & 0.3 & 0 \end{pmatrix}^{10}$

$= \begin{pmatrix} 0.30225\ 0.22433\ 0.30037\ 0.17301 \end{pmatrix}$

Trend: values only changed in 3rd decimal place – approaching a steady state

(b) e,g. to 6 dp, $x_{100} = \begin{pmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix} \times \begin{pmatrix} 0.1 & 0.2 & 0.5 & 0.2 \\ 0.6 & 0 & 0.3 & 0.1 \\ 0.4 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.6 & 0.3 & 0 \end{pmatrix}^{100}$

$= \begin{pmatrix} 0.302281\ 0.224335\ 0.300380\ 0.173004 \end{pmatrix}$

3. **(a)** **(i)** $x_1 = \begin{pmatrix} 0.8 & 0.15 & 0.05 \end{pmatrix} \times \begin{pmatrix} 0.9 & 0 & 0.1 \\ 0.8 & 0.1 & 0.1 \\ 0.8 & 0.2 & 0 \end{pmatrix} = \begin{pmatrix} 0.88 & 0.025 & 0.095 \end{pmatrix}$

(ii) $x_2 = \begin{pmatrix} 0.88 & 0.025 & 0.095 \end{pmatrix} \times \begin{pmatrix} 0.9 & 0 & 0.1 \\ 0.8 & 0.1 & 0.1 \\ 0.8 & 0.2 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0.888 & 0.0215 & 0.0905 \end{pmatrix}$

$x_3 = \begin{pmatrix} 0.888 & 0.0215 & 0.0905 \end{pmatrix} \times \begin{pmatrix} 0.9 & 0 & 0.1 \\ 0.8 & 0.1 & 0.1 \\ 0.8 & 0.2 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0.8888 & 0.02025 & 0.09095 \end{pmatrix}$

(iii) $x_{10} = \begin{pmatrix} 0.8 & 0.15 & 0.05 \end{pmatrix} \times \begin{pmatrix} 0.9 & 0 & 0.1 \\ 0.8 & 0.1 & 0.1 \\ 0.8 & 0.2 & 0 \end{pmatrix}^{10}$

$= \begin{pmatrix} 0.888889 & 0.020202 & 0.090909 \end{pmatrix}$ (6 decimal places)

(b) e.g. $\begin{pmatrix} 0.9 & 0 & 0.1 \\ 0.8 & 0.1 & 0.1 \\ 0.8 & 0.2 & 0 \end{pmatrix}^{100} = \begin{pmatrix} 0.888889 & 0.020202 & 0.090909 \\ 0.888889 & 0.020202 & 0.090909 \\ 0.888889 & 0.020202 & 0.090909 \end{pmatrix}$ (6 decimal places)

$$x_{100} = \begin{pmatrix} 0.8 & 0.15 & 0.05 \end{pmatrix} \times \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.4 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.3 \end{pmatrix}^{100}$$

$$= \begin{pmatrix} 0.8 & 0.15 & 0.05 \end{pmatrix} \times \begin{pmatrix} 0.888889 & 0.020202 & 0.090909 \\ 0.888889 & 0.020202 & 0.090909 \\ 0.888889 & 0.020202 & 0.090909 \end{pmatrix}$$

$= \begin{pmatrix} 0.888889 & 0.020202 & 0.090909 \end{pmatrix}$ (same as x_{10} to 6 decimal places)

4. **(a)** T^n where n is large, determines the steady state.

(b) sum of columns: $\text{col1} = 0.9,\ \text{col2} = 1.1,\ \text{col3} = 1.0$

so brand B since column 2 has largest sum

(c) $\begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \times \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.5 & 0.4 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.35 & 0.35 \end{pmatrix}$

multiply both sides of equation by inverse of 3x3 matirx:

$$\Rightarrow \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} 0.3 & 0.35 & 0.35 \end{pmatrix} \times \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.5 & 0.4 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 0.3 & 0.35 & 0.35 \end{pmatrix} \times \begin{pmatrix} 2 & 0.5 & -1.5 \\ 2 & -4.5 & 3.5 \\ -3 & 5.5 & -1.5 \end{pmatrix}$$

$$= \begin{pmatrix} 0.25 & 0.5 & 0.25 \end{pmatrix}$$

(d) e.g. $x_{100+n} = \begin{pmatrix} 0.3 & 0.35 & 0.35 \end{pmatrix} \times \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.3 & 0.3 & 0.4 \\ 0.1 & 0.5 & 0.4 \end{pmatrix}^{100}$

$$= \begin{pmatrix} 0.3 & 0.35 & 0.35 \end{pmatrix} \times \begin{pmatrix} 0.289473 & 0.368421 & 0.342105 \\ 0.289473 & 0.368421 & 0.342105 \\ 0.289473 & 0.368421 & 0.342105 \end{pmatrix}$$

$$= \begin{pmatrix} 0.289473 & 0.368421 & 0.342105 \end{pmatrix}$$

So brand B most popular in steady state.

5. **(a)** $\begin{pmatrix} 0.80 & 0.05 & 0.15 \\ 0.05 & 0.90 & 0.05 \\ 0.10 & 0.15 & 0.75 \end{pmatrix}$

since in the multiplication $\begin{pmatrix} A & B & C \end{pmatrix} \times \begin{pmatrix} 0.80 & 0.05 & 0.15 \\ 0.05 & 0.90 & 0.05 \\ 0.10 & 0.15 & 0.75 \end{pmatrix}$

e.g. the 1st figure is $0.80 \times A + 0.05 \times B + 0.10 \times C$

i,e. A retains 80% of own customers, gains 5% of B's and 10% of C's.

(b) $\begin{pmatrix} 0.4 & 0.3 & 0.3 \end{pmatrix} \times \begin{pmatrix} 0.80 & 0.05 & 0.15 \\ 0.05 & 0.90 & 0.05 \\ 0.10 & 0.15 & 0.75 \end{pmatrix} = \begin{pmatrix} 0.365 & 0.335 & 0.3 \end{pmatrix}$

(c) $\begin{pmatrix} 0.365 & 0.335 & 0.3 \end{pmatrix} \times \begin{pmatrix} 0.80 & 0.05 & 0.15 \\ 0.05 & 0.90 & 0.05 \\ 0.10 & 0.15 & 0.75 \end{pmatrix} = \begin{pmatrix} 0.33875 & 0.36475 & 0.2965 \end{pmatrix}$

6. transition matrix is $\begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$ since

$\begin{pmatrix} C & L \end{pmatrix} \times \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.7C + 0.2L & 0.3C + 0.8L \end{pmatrix}$

(a) $\begin{pmatrix} 0.6 & 0.4 \end{pmatrix} \times \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix}$ so election is tied

(b) after another 3 month period, $\begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \times \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} = \begin{pmatrix} 0.45 & 0.55 \end{pmatrix}$,

i,e. liberal wins

7. transition matrix is $\begin{pmatrix} 0.75 & 0.15 & 0.10 \\ 0.05 & 0.90 & 0.05 \\ 0.05 & 0.10 & 0.85 \end{pmatrix}$

(a) $\begin{pmatrix} 0.6 & 0.3 & 0.1 \end{pmatrix} \times \begin{pmatrix} 0.75 & 0.15 & 0.10 \\ 0.05 & 0.90 & 0.05 \\ 0.05 & 0.10 & 0.85 \end{pmatrix}^2 = \begin{pmatrix} 0.3790 & 0.4195 & 0.2015 \end{pmatrix}$

(b) $\begin{pmatrix} 0.6 & 0.3 & 0.1 \end{pmatrix} \times \begin{pmatrix} 0.75 & 0.15 & 0.10 \\ 0.05 & 0.90 & 0.05 \\ 0.05 & 0.10 & 0.85 \end{pmatrix}^5 = \begin{pmatrix} 0.239497 & 0.49706 & 0.236444 \end{pmatrix}$

Chapter 15 Practice questions

1. $\mathrm{E}(X)=\dfrac{\sum_1^8 n}{8}=\dfrac{36}{8}=4.5$

2. **(a)** $\mathrm{P}(X=15)=1-\mathrm{P}(X\neq 15)=1-(0.14+0.11+0.26+0.23)=1-0.74=0.26$

(b) $\mathrm{P}(X=12)+\mathrm{P}(X=20)=0.14+0.23=0.37$

(c) $\mathrm{P}(X\leq 18)=0.14+0.11+0.26+0.26=0.77$

(d) $\mathrm{E}(X)=12\times 0.14+13\times 0.11+15\times 0.26+18\times 0.26+20\times 0.23=16.29$

(e) $\mathrm{Var}(X)=\sum X^2.\mathrm{P}(X)-16.29^2$

$=(12^2\times 0.14+13^2\times 0.11+15^2\times 0.26+18^2\times 0.26+20^2\times 0.23)-16.29^2$

$=8.126$

3. sum of probabilities $=1 \quad\Rightarrow\quad 5a+0.5=1 \;\Rightarrow\quad a=0.1$

$\mathrm{E}(X)=1\times 0.1+2\times 0.15+3\times 0.2+4\times 0.25+5\times 0.3=3.5$

$\mathrm{Var}(X)=(1^2\times 0.1+2^2\times 0.15+3^2\times 0.2+4^2\times 0.25+5^2\times 0.3)-3.5^2=1.75$

4. $\mathrm{E}(X)=40\times 0.05+20\times 0.1+10\times 0.25+5\times 0.6=9.5$

$\mathrm{Var}(X)=(40^2\times 0.05+20^2\times 0.1+10^2\times 0.25+5^2\times 0.6)-9.5^2=69.75$

5. **(a)** sum of probabilities $=1 \quad\Rightarrow\quad \dfrac{3}{20}+\dfrac{7}{30}+k+\dfrac{3}{10}+\dfrac{13}{60}=1 \;\Rightarrow\quad k=1-\dfrac{54}{60}=\dfrac{1}{10}$

(b) $\mathrm{P}(X>10)=1-\mathrm{P}(X=5)-\mathrm{P}(X=10)=1-\dfrac{3}{20}-\dfrac{7}{30}=\dfrac{37}{60}$

(c) $\mathrm{P}(5<X\leq 20)=\mathrm{P}(X=10)+\mathrm{P}(X=15)+\mathrm{P}(X=20)=\dfrac{7}{30}+\dfrac{1}{10}+\dfrac{3}{10}=\dfrac{19}{30}$

(d) $\mathrm{E}(X)=5\times\dfrac{3}{20}+10\times\dfrac{7}{30}+15\times\dfrac{1}{10}+20\times\dfrac{3}{10}+25\times\dfrac{13}{60}=16$

$\mathrm{Var}(X)=\left(5^2\times\dfrac{3}{20}+10^2\times\dfrac{7}{30}+15^2\times\dfrac{1}{10}+20^2\times\dfrac{3}{10}+25^2\times\dfrac{13}{60}\right)-16^2$

$=49$

$\Rightarrow\quad \sigma=7$

6. $\displaystyle\int_2^3 kx^3\,\mathrm{d}x=1 \quad\Rightarrow\quad \left[\frac{k}{4}x^4\right]_2^3=1\Rightarrow\quad \frac{k}{4}\times 3^4-\frac{k}{4}\times 2^4=1 \quad\Rightarrow\quad k=\frac{4}{3^4-2^4}=\frac{4}{65}$

7. $\int_0^2 \left(2kx^2 - kx^3\right)\mathrm{d}x = 1 \quad \Rightarrow \left[\frac{2k}{3}.x^3 - \frac{k}{4}x^4\right]_0^2 = 1 \Rightarrow \frac{2k}{3}\times 2^3 - \frac{k}{4}\times 2^4 = 1$

$$\Rightarrow k\left(\frac{16}{3} - 4\right) = 1 \Rightarrow k = \frac{3}{4}$$

8. **(a)** $k\int_0^5 \left(-x^2 + 2x + 15\right)\mathrm{d}x = 1 \quad \Rightarrow k.\left[-\frac{1}{3}x^3 + x^2 + 15x\right]_0^5 = 1 \Rightarrow -\frac{125}{3} + 25 + 75 = \frac{1}{k}$

$$\Rightarrow \frac{175}{3} = \frac{1}{k} \quad \Rightarrow k = \frac{3}{175}$$

(b) mean $\mu = \int_0^5 kx\left(-x^2 + 2x + 15\right)\mathrm{d}x = \int_0^5 k\left(-x^3 + 2x^2 + 15x\right)\mathrm{d}x$

$$= k.\left[-\frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{15}{2}x^2\right]_0^5$$

$$= \frac{3}{175}\times\left(-\frac{1}{4}\times 5^4 + \frac{2}{3}\times 5^3 + \frac{15}{2}\times 5^2\right)$$

$$= \frac{3}{175}\times\left(-\frac{625}{4} + \frac{250}{3} + \frac{375}{2}\right)$$

$$= \frac{3}{175}\times\frac{1375}{12} = \frac{55}{28}$$

Mode $=$ x coordinate of maximum of function

$\Rightarrow f'(x) = 0$ when $x = 1$, ie mode $= 1$

(c) $\int_0^m k\left(-x^2 + 2x + 15\right)\mathrm{d}x = 0.5$

$$\Rightarrow k.\left[-\frac{1}{3}x^3 + x^2 + 15x\right]_0^m = 0.5$$

$$\Rightarrow \frac{3}{175}\times\left(-\frac{1}{3}m^3 + m^2 + 15m\right) = 0.5$$

$$\Rightarrow -\frac{1}{3}m^3 + m^2 + 15m = \frac{175}{6}$$

$$\Rightarrow -m^3 + 3m^2 + 45m - 87.5 = 0$$

Use GDC to solve: $m = 1.857$

9. $E(X)=\int xf(x)\,dx=\int_0^2 x\left(\frac{3}{2}x^2-\frac{3}{4}x\right)dx=\left[\frac{3}{8}x^4-\frac{3}{20}x^5\right]_0^2=6-\frac{24}{5}=\frac{6}{5}=1.2$

$\text{Var}(X)=\int x^2 f(x)\,dx-\mu^2$

$=\int_0^2 x^2\left(\frac{3}{2}x^2-\frac{3}{4}x\right)dx-1.2^2$

$=\int_0^2 \left(\frac{3}{2}x^4-\frac{3}{4}x^5\right)dx-1.2^2$

$=\left[\frac{3}{10}x^5-\frac{1}{8}x^6\right]_0^2-1.2^2$

$=\left(\frac{48}{5}-8\right)-1.2^2$

$=\frac{8}{5}-\frac{36}{25}=\frac{4}{25}$

i.e. $\sigma^2=\frac{4}{25}\Rightarrow\ \sigma=\frac{2}{5}\ =0.4$

10. binomial distribution: $n=60, p=0.4$

(a) **(i)** $P(X=24)=60C_{24}\times0.4^{24}\times0.6^{36}=\frac{60!}{24!36!}\times0.4^{24}\times0.6^{36}=0.105$

(or use GDC binompdf function)

(ii) Use GDC: $\text{binompdf}(60,0.4,22)=0.0925$,
$\text{binompdf}(60,0.4,26)=0.0902$

(iii) Use GDC: *binompdf* $(60,0.4,20)=0.0616$,
$\text{binompdf}(60,0.4,28)=0.0595$

(b) $E(X)=np=60\times0.4=24$

So peak at expected value and decreasing on either side: bell-shape

11. binomial distribution: $n=20, p=0.25$

use GDC binomcdf function

probability of less than ten correct $=\text{binompdf}(20,0.25,9)$

So $P(X\geq10)=1-\text{binomcdf}(20,0.25,9)=0.0139$

12. binomial distribution: $n = 10, p = 0.98$

(a) $\mathrm{P}(X = 10) = 0.98^{10} = 0.817$

(b) use GDC binomcdf function

probability of 5 or more $= 1 - \mathrm{P}(X \le 4) = 1 - \text{binomcdf}(10, 0.98, 4)$

$\text{binomcdf}(10, 0.98, 4) < 10^{-10}$ and so probability of 5 or more is very close to 1.

(c) probability of 8 or less $= \mathrm{P}(X \le 8) = \text{binomcdf}(10, 0.98, 8) = 0.0162$

13. binomial distribution: $n = 15, p = 0.40$

(a) $\mathrm{P}(X \ge 10) = 1 - \mathrm{P}(X \le 9) = 1 - \text{binomcdf}(15, 0.4, 9) = 0.0338$

(b) $\mathrm{P}(X = 10) = \textit{binompdf}(15, 0.4, 10) = 0.0245$

(c) at most ten are *younger* than 30 yrs = at least 5 are *older* than 30yrs

$= \mathrm{P}(X \ge 5) = 1 - \mathrm{P}(X \le 4) = 1 - \text{binomcdf}(15, 0.4, 4) = 0.783$

14. binomial distribution: $n = 100, p = 0.03$

(a) $\mathrm{E}(X) = np = 100 \times 0.03 = 3$

(b) $\mathrm{P}(X = 5) = \text{binompdf}(100, 0.03, 5) = 0.101$

(c) $\mathrm{P}(X > 10) = 1 - \mathrm{P}(X \le 10) = 1 - \textit{binomcdf}(100, 0.03, 10) = 0.000215$

15. Poisson distribution with mean $m = 6$ calls per minute

(a) $\mathrm{P}(X = 0) = \dfrac{6^0 \times \mathrm{e}^{-6}}{0!} = \mathrm{e}^{-6} = 0.00248$

(b) $\mathrm{P}(X \ge 2) = 1 - \mathrm{P}(X \le 1) = 1 - \mathrm{P}((X = 0) - \mathrm{P}(X = 1)$

$$= 1 - \frac{6^0 \times \mathrm{e}^{-6}}{0!} - \frac{6^1 \times \mathrm{e}^{-6}}{1!} = 1 - 7\mathrm{e}^{-6} = 0.9826$$

(c) Poisson distribution with mean $m = 12$ calls per 2-minute period

$\mathrm{P}(X \ge 2) = 1 - \mathrm{P}(X \le 1) = 1 - \mathrm{P}((X = 0) - \mathrm{P}(X = 1)$

$$= 1 - \frac{12^0 \times \mathrm{e}^{-12}}{0!} - \frac{12^1 \times \mathrm{e}^{-12}}{1!} = 1 - 13\mathrm{e}^{-12} = 0.9999$$

16. Poisson distribution with mean $m = 8$ passengers per 10 minutes

(a) $\mathrm{P}(X=8)=\dfrac{8^8\times \mathrm{e}^{-8}}{8!}=0.1396$

(b) $\mathrm{P}(X\le 5)=\text{poissoncdf}(8,5)=0.1912$

(c) $\mathrm{P}(X\ge 4)=1-\mathrm{P}(X\le 3)=1-\textit{poissoncdf}(8,3)=0.9576$

17. Poisson distribution with mean $m = 0.2$ hits per second

(a) $\mathrm{P}(X=0)=\mathrm{e}^{-0.2}=0.8187$

(b) $\mathrm{P}(X=0)$ in 1st and 2nd and 3rd second

$=\mathrm{P}(X=0)\times\mathrm{P}(X=0)\times\mathrm{P}(X=0)=\mathrm{P}(X=0)^3$

$=\mathrm{e}^{-0.2\times 3}=\mathrm{e}^{-0.6}=0.5488$

18. Poisson distribution with mean $m = 4.4$ faults per m^2

(a) $\mathrm{P}(X\ge 1)=1-\mathrm{P}(X=0)=1-\mathrm{e}^{-4.4}=0.9877$

(b) prob of at least 1 fault in 3 m^2 = $1-$ prob of no faults in 3 m^2

prob of no faults in 3 m^2

$=\mathrm{P}(X=0)\times\mathrm{P}(X=0)\times\mathrm{P}(X=0)=\mathrm{P}(X=0)^3=\mathrm{e}^{-4.4\times 3}=\mathrm{e}^{-13.2}$

So prob of at least 1 fault in 3 $\mathrm{m}^2=1-\mathrm{e}^{-13.2}=0.999998$

(c) prob of 3 pieces contain exactly one fault between them

$=3\times\mathrm{P}(X=1)\times\mathrm{P}(X=0)\times\mathrm{P}(X=0)$

$=3\times\mathrm{P}(X=1)\times\mathrm{P}(X=0)^2$

$=3\times\dfrac{4.4^1\times\mathrm{e}^{-4.4}}{1!}\times\mathrm{e}^{-4.4\times 2}=3\times 4.4\times\mathrm{e}^{-13.2}=0.0000244$

19. **(a)** use GDC normal cdf function: normalcdf(700, 800, 510, 106)

$\mathrm{P}(700\le X\le 800)=0.0334$

(b) use inverse normal function: invNorm(0.9, 510, 106)

$\mathrm{P}(X\le x)=0.90 \Rightarrow x=646$

(c) $\mathrm{P}(X\le 720)=0.978 \Rightarrow$ score 720 is in 98th percentile

20. use GDC normal cdf function: normalcdf(7, 1EXP1000, 6.8, 0.25)

$\mathrm{P}(X\ge 7)=0.212$

21. **(a)** use GDC normal cdf function: $normalcdf\left(1010, 1\text{EXP}1000, 1012, 5\right)$

$\text{P}\left(X \geq 1010\right) = 0.655$

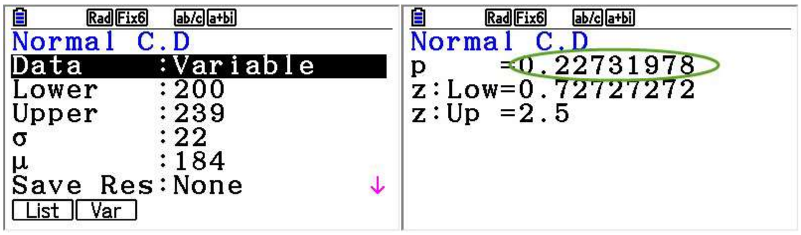
(b) use GDC normal cdf function: $normalcdf\left(-1\text{EXP}1000, 1000, 1012, 5\right)$

$\text{P}\left(X \leq 1000\right) = 0.0082$

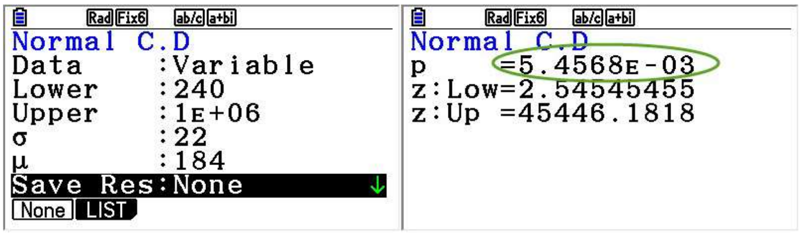
(c) expected value $= 10000 \times 0.0082 = 82$ bottles

22. (a) $\text{P}\left(200 < X < 239\right) \approx 0.227$

Sample GDC results

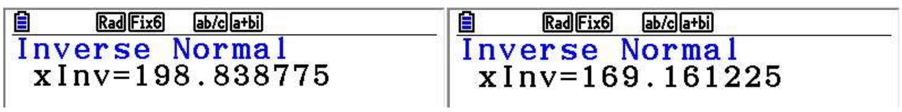


(b) $P\left(X > 249\right) \approx 0.0055 = 0.55\%$



(c) This is an inverse normal calculation; we need to find **Q1** and **Q3**.

IQR = 29.7



(d) Also an inverse normal calculation. We find the number above which we have 2% of the population. Alternatively, we find the number below which there 98% of the population: 2% of the population have a level above 229.

23. **(a)** use GDC normal cdf function: $\text{normalcdf}\left(64000, 1\text{EXP}1000, 52000, 4000\right)$

$\text{P}\left(X \geq 64000\right) = 0.0013$ so not likely

(b) $\text{normalcdf}\left(-1\text{EXP}1000, 48000, 52000, 4000\right)$

$\text{P}\left(X \leq 48000\right) = 0.159$ i.e. 15.9%

(c) $\text{normalcdf}(48000, 56000, 52000, 4000)$

$\text{P}\left(48000 \leq X \leq 56000\right) = 0.683$

(d) for lower quartile use GDC $invNorm(0.25, 52000, 4000)$

lower quartile $= 49300$, so inter quartile range $= 2\times(52000-49300) = 5400\,\text{km}$

(e) GDC $\text{invNorm}(0.02, 52000, 4000)$

$\text{P}(X \le x) = 0.02 \Rightarrow x = 43785$ km

24. Model book-shoppers and souvenir –shoppers per hour with poisson distributions:

$X_B\left(\text{mean} = \dfrac{200}{8} = 25\right)$ and $X_S\left(\text{mean} = \dfrac{80}{8} = 10\right)$

Then sum $X_B + X_S$ is $Po(25+10)$, i.e. poisson with mean 35 visitors per hour

$$\text{P}(X_B + X_S = 40) = \frac{35^{40}\text{e}^{-35}}{40!} = 0.04475$$

25. (a) (i)

$$x_1 = \begin{pmatrix}0.35 & 0.35 & 0.3\end{pmatrix}\times\begin{pmatrix}0.6 & 0.2 & 0.2\\0.4 & 0.3 & 0.3\\0.2 & 0.3 & 0.5\end{pmatrix} = \begin{pmatrix}0.41 & 0.265 & 0.325\end{pmatrix}$$

(ii)

$$x_2 = \begin{pmatrix}0.41 & 0.265 & 0.325\end{pmatrix}\times\begin{pmatrix}0.6 & 0.2 & 0.2\\0.4 & 0.3 & 0.3\\0.2 & 0.3 & 0.5\end{pmatrix}$$

$$= \begin{pmatrix}0.417 & 0.259 & 0.324\end{pmatrix}$$

$$x_3 = \begin{pmatrix}0.417 & 0.259 & 0.324\end{pmatrix}\times\begin{pmatrix}0.6 & 0.2 & 0.2\\0.4 & 0.3 & 0.3\\0.2 & 0.3 & 0.5\end{pmatrix}$$

$$= \begin{pmatrix}0.4186 & 0.2583 & 0.3231\end{pmatrix}$$

(iii)

$$x_7 = \begin{pmatrix}0.35 & 0.35 & 0.3\end{pmatrix}\times\begin{pmatrix}0.6 & 0.2 & 0.2\\0.4 & 0.3 & 0.3\\0.2 & 0.3 & 0.5\end{pmatrix}^7$$

$$= \begin{pmatrix}0.419345 & 0.258067 & 0.322588\end{pmatrix}$$

(b) e.g .

$$x_{100} = \begin{pmatrix}0.35 & 0.35 & 0.3\end{pmatrix}\times\begin{pmatrix}0.6 & 0.2 & 0.2\\0.4 & 0.3 & 0.3\\0.2 & 0.3 & 0.5\end{pmatrix}^{100}$$

$$= \begin{pmatrix}0.419355 & 0.258065 & 0.322581\end{pmatrix}$$

26. $T^{100} = \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}^{100} = \begin{pmatrix} 0.666667 & 0.333333 \\ 0.666667 & 0.333333 \end{pmatrix}$ 6dps, so steady state is $\begin{pmatrix} \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

(a) $$\begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \times \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}^{100} = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \times \begin{pmatrix} 0.666667 & 0.333333 \\ 0.666667 & 0.333333 \end{pmatrix}$$
$$= \begin{pmatrix} 0.666667 & 0.333333 \end{pmatrix}$$

(b) $$\begin{pmatrix} 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}^{100} = \begin{pmatrix} 1 & 0 \end{pmatrix} \times \begin{pmatrix} 0.666667 & 0.333333 \\ 0.666667 & 0.333333 \end{pmatrix}$$
$$= \begin{pmatrix} 0.666667 & 0.333333 \end{pmatrix}$$

(c) $$\begin{pmatrix} 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0.75 & 0.25 \\ 0.5 & 0.5 \end{pmatrix}^{100} = \begin{pmatrix} 0 & 1 \end{pmatrix} \times \begin{pmatrix} 0.666667 & 0.333333 \\ 0.666667 & 0.333333 \end{pmatrix}$$
$$= \begin{pmatrix} 0.666667 & 0.333333 \end{pmatrix}$$

27. $T^{100} = \begin{pmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{pmatrix}^{100} = \begin{pmatrix} 0.333333 & 0.666667 \\ 0.333333 & 0.666667 \end{pmatrix}$

6 decimal places, so steady state is $\begin{pmatrix} \frac{1}{3} & \frac{2}{3} \end{pmatrix}$, i.e. $\frac{1}{3}$ short, $\frac{2}{3}$ tall

28. **(a)** **(i)** normalcdf$(60,1\text{EXP}1000,53,8)$

$\text{P}(X \geq 60) = 0.1908$

(ii) normalcdf$(70,1\text{EXP}1000,53,8)$

$\text{P}(X \geq 70) = 0.0168$

$$\text{P}(X \geq 70 \text{ given that } X \geq 60) = \frac{\text{P}(X \geq 70 \, and X \geq 60)}{\text{P}(X \geq 60)}$$
$$= \frac{\text{P}(X \geq 70)}{\text{P}(X \geq 60)}$$
$$= \frac{0.0168}{0.1908}$$
$$= 0.0881$$

(b) $\text{P}(X \geq 60) \times \text{P}(X \geq 60) = \text{P}(X \geq 60)^2 = 0.1908^2 = 0.0364$

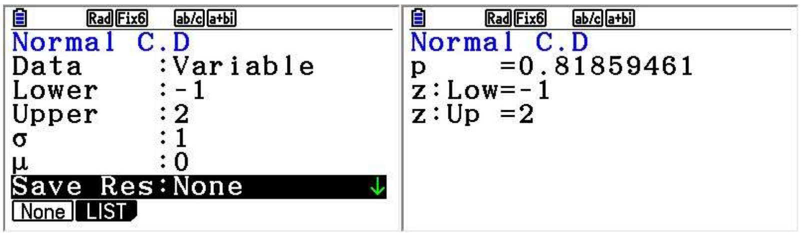
(c) **(i)** $100 \times \text{P}(\text{tall}) = 100 \times 0.191 = 19.1 \text{ trees}$

(ii) binomial distribution $n = 100, p = 0.191$

$1 - \text{binomialcdf}(100, 0.191, 25)$, $\text{P}(X \geq 25) = 0.0869$

29. **(a)** Use the standard normal distribution. $50-\sigma$ corresponds to $z=-1$; and $50+2\sigma$ is $z=+2$

$$\mathrm{P}(50-\sigma<L<50+2\sigma)=\mathrm{P}(-1<Z<2)\approx 0.819$$



(b) Again, we use the standard normal distribution.

Z that corresponds to 0.975 is 1.9599.

$$1.9599=\frac{53.92-50}{\sigma}\Rightarrow\sigma=\frac{3.92}{1.9599}\approx 2$$

(c) $\mathrm{P}(L\geq t)=0.75\Rightarrow\mathrm{P}(L\leq t)=0.25$

invNorm$(0.25,\ 50,\ 1.96)=48.678$ so 48.68 mm (2 d.p.)

(d) **(i)** prob of selecting from all nails

$\mathrm{P}(48.68\leq X<50.1)=\mathit{normalcdf}(48.68,50.1,50,1.96)=0.27$

prob of selecting from large nails:

$$\mathrm{P}(X<50.1\text{ given that }X\geq 48.68)=\frac{\mathrm{P}(X<50.1\text{ and }X\geq 48.68)}{\mathrm{P}(X\geq 48.68)}$$

$$=\frac{\mathrm{P}(48.68\leq X<50.1)}{\mathrm{P}(X\geq 48.68)}$$

$$=\frac{0.270}{0.750}$$

$$=0.360$$

(ii) binomial distribution $n=10$, $p=0.36$

$$\mathrm{P}(X\geq 2)=1-\mathrm{P}(X\leq 1)=1-\left(0.64^{10}+10\times 0.64^{9}\times 0.36\right)=0.9236$$

30. **(a)** poisson with mean 7×5.84 birds per week $= 40.88$

$\mathrm{P}(X > 40) = 1 - \mathrm{P}(X \leq 40) = 1 - \text{poissoncdf}(40.88, 40) = 1 - 0.4867 = 0.5133$

(b) $\dfrac{\text{probability there were more than 10 on Monday AND more than 40 over the week}}{\text{probability there were more than 10 on Monday}}$

possibilities for the numerator are:

there were more than 40 birds on the power line on Monday

11 on Monday and more than 29 over the course of the next 6 days

12 on Monday and more than 28 over the course of the next 6 days … until

40 on Monday and more than 0 over the course of the next 6 days

hence if X is the number on the power line on Monday and Y, the number on the power line Tuesday – Sunday then the numerator is

$$\mathrm{P}(X > 40) + \mathrm{P}(X = 11) \times \mathrm{P}(Y > 29) + \mathrm{P}(X = 12) \times \mathrm{P}(Y > 28) + \cdots + \mathrm{P}(X = 40) \times \mathrm{P}(Y > 0)$$

$$= \mathrm{P}(X > 40) + \sum_{r=11}^{40} \mathrm{P}(X = r)\mathrm{P}(Y > 40 - r)$$

Hence the solution is

$$\frac{\mathrm{P}(X > 40) + \sum_{r=11}^{40} \mathrm{P}(X = r)\mathrm{P}(Y > 40 - r)}{\mathrm{P}(X > 10)}$$

31. $\mathrm{P}(3 \text{ in 1st hour given that 5 in 1st to 4th hour})$

$$= \frac{\mathrm{P}(3 \text{ in 1st hour and 5 in 1st to 4th hour})}{\mathrm{P}(5 \text{ in 1st to 4th hour})}$$

$$= \frac{\mathrm{P}(3 \text{ in 1st hour and 2 in 2nd to 4th hour})}{\mathrm{P}(5 \text{ in 1st to 4th hour})} \quad \text{(equivalent description)}$$

$$= \frac{\mathrm{P}(3 \text{ in 1st hour}) \times \mathrm{P}(2 \text{ in 2nd to 4th hour})}{\mathrm{P}(5 \text{ in 1st to 4th hour})} \quad \text{(independent events)}$$

(1^{st} hour: Po mean λ, 2^{nd} to 4^{th} hour: Po mean 3λ, 1^{st} to 4^{th} hour: Po mean 4λ)

$$= \frac{\dfrac{\lambda^3 e^{-\lambda}}{3!} \times \dfrac{(3\lambda)^2 e^{-3\lambda}}{3!}}{\dfrac{(4\lambda)^5 e^{-4\lambda}}{5!}} = \frac{\dfrac{1}{3!} \times \dfrac{3^2}{3!}}{\dfrac{4^5}{5!}} = \frac{3^2 \times 5}{2^9} = 0.0879$$

Note: The form of answers to integration exercises may differ from one user to the other. The answers usually differ by a constant. In all integration results, the constant term, we will mostly denote it by *c*, is a real number.

Exercise 16.1

1. **(a)** $\int(x+2)\,\mathrm{d}x=\frac{1}{2}x^2+2x+c,\ c\in\mathbb{R}$

As mentioned at the beginning, if we substitute *u* for $x+2$, then d*u* = d*x* and the integration will result with $\int(x+2)\,\mathrm{d}x=\frac{1}{2}(x+2)^2+c_1,\ c_1\in\mathbb{R}$.

The two answers can be consolidated by observing that
$\frac{1}{2}(x+2)^2+c_1=\frac{1}{2}x^2+2x+2+c_1$.

By comparison you can see that $c=2+c_1$, which are constants.

(b) Direct application of the power rule:
$$\int\left(3t^2-2t+1\right)\mathrm{d}x=3\cdot\frac{1}{3}t^3-2\cdot\frac{1}{2}t^2+t+c=t^3-t^2+t+c,\ c\in\mathbb{R}$$

(c) $$\int\left(\frac{1}{3}-\frac{2}{7}x^3\right)\mathrm{d}x=\frac{1}{3}x-\frac{2}{7}\cdot\frac{1}{4}x^4+c=\frac{1}{3}x-\frac{1}{14}x^4+c,\ c\in\mathbb{R}$$

(d) Simplify the integrand first and then apply power rule:
$$\int(t-1)(2t+3)\,\mathrm{d}t=\int\left(2t^2+t-3\right)\mathrm{d}t$$
$$=2\cdot\frac{1}{3}t^3+\frac{1}{2}t^2-3t+c=\frac{2}{3}t^3+\frac{1}{2}t^2-3t+c,\ c\in\mathbb{R}$$

(e) $$\int\left(u^{\frac{2}{5}}-4u^3\right)\mathrm{d}u=\frac{1}{\frac{2}{5}+1}u^{\frac{2}{5}+1}-4\cdot\frac{1}{4}u^4+c=\frac{5}{7}u^{\frac{7}{5}}-u^4+c,\ c\in\mathbb{R}$$

(f) $$\int\left(2\sqrt{x}-\frac{3}{2\sqrt{x}}\right)\mathrm{d}x=\int\left(2x^{\frac{1}{2}}-\frac{3}{2}x^{-\frac{1}{2}}\right)\mathrm{d}x$$
$$=2\cdot\frac{1}{\frac{3}{2}}x^{\frac{3}{2}}-\frac{3}{2}\cdot\frac{1}{\frac{1}{2}}x^{\frac{1}{2}}+c=\frac{4}{3}x^{\frac{3}{2}}-3x^{\frac{1}{2}}+c,\ c\in\mathbb{R}$$

(g) $$\int(3\sin\theta+4\cos\theta)\,\mathrm{d}\theta=3(-\cos\theta)+4\sin\theta+c=-3\cos\theta+4\sin\theta+c,c\in\mathbb{R}$$

(h) $$\int\left(3t^2-2\sin t\right)\mathrm{d}t=3\cdot\frac{1}{3}t^3-2\cdot(-\cos t)+c=t^3+2\cos t+c,c\in\mathbb{R}$$

(i) $\int \sqrt{x}(2x-5)\,dx = \int\left(2x^{\frac{3}{2}} - 5x^{\frac{1}{2}}\right)dx = 2\cdot\frac{1}{\frac{5}{2}}x^{\frac{5}{2}} - 5\cdot\frac{1}{\frac{3}{2}}x^{\frac{3}{2}} + c = \frac{4}{5}x^{\frac{5}{2}} - \frac{10}{3}x^{\frac{3}{2}} + c,\ c\in\mathbb{R}$

(j) $\int\left(3\cos\theta - \frac{2}{\cos^2\theta}\right)d\theta = 3\sin\theta - 2\tan\theta + c,\ c\in\mathbb{R}$

(k) $\int e^{3t-1}\,dt = \int\frac{1}{3}e^{3t-1}\,d(3t-1) = \frac{1}{3}e^{3t-1} + c,\ c\in\mathbb{R}$

(l) $\int\frac{2}{t}\,dt = 2\int\frac{1}{t}\,dt = 2\ln|t| + c,\ c\in\mathbb{R}$

(m) $\int\frac{t}{3t^2+5}\,dt = \int\frac{1}{6}\frac{6t\,dt}{3t^2+5} = \frac{1}{6}\int\frac{d(3t^2+5)}{3t^2+5} = \frac{1}{6}\ln(3t^2+5) + c,\ c\in\mathbb{R}$

(n) $\int e^{\sin\theta}\cos\theta\,d\theta = \int e^{\sin\theta}\,d(\sin\theta) = e^{\sin\theta} + c,\ c\in\mathbb{R}$

(o) $\int(3+2x)^2\,dx = \int\frac{1}{2}(3+2x)^2\,d(3+2x)$

$$= \frac{1}{2}\times\frac{1}{3}(3+2x)^3 + c$$

$$= \frac{1}{6}(3+2x)^3 + c,\ c\in\mathbb{R}$$

2. **(a)** $f'(x) = \int(4x-15x^2)\,dx = 4\times\frac{1}{2}x^2 - 15\times\frac{1}{3}x^3 + c = 2x^2 - 5x^3 + c,\ c\in\mathbb{R}$,

Integrate again,

$$f(x) = \int(2x^2 - 5x^3 + c)\,dx = 2\times\frac{1}{3}x^3 - 5\times\frac{1}{4}x^4 + cx + k$$

$$= \frac{2}{3}x^3 - \frac{5}{4}x^4 + cx + k,\ c,k\in\mathbb{R}$$

(b) $f'(x) = \int(1+3x^2-4x^3)\,dx = x + 3\times\frac{1}{3}x^3 - 4\times\frac{1}{4}x^4 + c = x + x^3 - x^4 + c,\ c\in\mathbb{R}$

With the initial condition that $f'(0)=2$, so we can calculate the constant c:

$f'(0) = 2 \Rightarrow 0 + c = 2 \Rightarrow c = 2 \Rightarrow f(x) = 2 + x + x^3 - x^4$. Hence,

$f(x) = \int(2+x+x^3-x^4)\,dx = 2x + \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^5}{5} + k,\ k\in\mathbb{R}$.

Also, the initial condition that $f(1)=2$, so we can calculate the constant k:

$f(1) = 2 + \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + k = 2 \Rightarrow k = -\frac{11}{20} \Rightarrow f(x) = -\frac{x^5}{5} + \frac{x^4}{4} + \frac{x^2}{2} + 2x - \frac{11}{20}$

(c) $f'(t)=\int(8t-\sin t)\,\mathrm{d}t=8\times\frac{1}{2}t^2-(-\cos t)+c=4t^2+\cos t+c, c\in\mathbb{R}$

$f(t)=\int(4t^2+\cos t+c)\,\mathrm{d}t=\frac{4}{3}t^3+\sin t+ct+k; c,k\in\mathbb{R}$.

(d) $f(x)=\int(12x^3-8x+7)\,\mathrm{d}x=12\times\frac{1}{4}x^4-8\times\frac{1}{2}x^2+7x+c$

$=3x^4-4x^2+7x+c, \quad c\in\mathbb{R}$

With the initial condition that, $f(0)=3$, so we can calculate the constant c:

$f(0)=3\Rightarrow 0+c=3\Rightarrow c=3\Rightarrow f(x)=3x^4-4x^2+7x+3.$

(e) $f(\theta)=\int(2\cos\theta-\sin(2\theta))\,\mathrm{d}\theta=2\sin\theta-\int\sin(2\theta)\times\frac{1}{2}\mathrm{d}(2\theta)+c$

$=2\sin\theta-\frac{1}{2}(-\cos(2\theta))+c=2\sin\theta+\frac{1}{2}\cos(2\theta)+c, c\in\mathbb{R}$

3. **(a)** Use substitution $u=3x^2+7\Rightarrow \mathrm{d}u=6x\,\mathrm{d}x\Rightarrow x\,\mathrm{d}x=\frac{1}{6}\mathrm{d}u$

$\int x(3x^2+7)^5\,\mathrm{d}x=\int u^5\times\frac{1}{6}\mathrm{d}u=\frac{1}{6}\times\frac{u^6}{6}+c=\frac{(3x^2+7)^6}{36}+c$

(b) Use substitution $u=3x^2+5\Rightarrow \mathrm{d}u=6x\mathrm{d}x$

$\int\frac{x}{(3x^2+5)^4}\mathrm{d}x=\int u^{-4}\times\frac{1}{6}\mathrm{d}u=\frac{1}{6}\times\frac{u^{-3}}{-3}+c=-\frac{1}{18(3x^2+5)^3}+c$

(c) Use substitution $u=5x^3+2\Rightarrow \mathrm{d}u=15x^2\mathrm{d}x$

$\int 2x^2\sqrt[4]{5x^3+2}\,\mathrm{d}x=\int u^{\frac{1}{4}}\times\frac{2}{15}\mathrm{d}u=\frac{2}{15}\times\frac{u^{\frac{5}{4}}}{\frac{5}{4}}+c=\frac{8\sqrt[4]{(5x^3+2)^5}}{75}+c$

Note: in the rest of the exercises, substitutions are implied by the work and not given directly. Some "obvious" intermediate steps are also not included.

(d) $\int\frac{(3+2\sqrt{x})^5}{\sqrt{x}}\mathrm{d}x=\int u^5\mathrm{d}u=\frac{(3+2\sqrt{x})^6}{6}+c$

(e) $\int t^2\sqrt{2t^3-7}\,\mathrm{d}t=\int u^{\frac{1}{2}}\times\frac{1}{6}\mathrm{d}u=\frac{1}{6}\times\frac{(2t^3-7)^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{\sqrt{(2t^3-7)^3}}{9}+c$

(f) $\int\left(2+\frac{3}{x}\right)^5\left(\frac{1}{x^2}\right)\mathrm{d}x=\int u^5\times(-3)\,\mathrm{d}u=-\frac{\left(2+\frac{3}{x}\right)^6}{18}+c=-\frac{(2x+3)^6}{18x^6}+c$

(g) $\int\sin(7x-3)\,\mathrm{d}x=\int\sin u\times\frac{1}{7}\mathrm{d}u=\frac{1}{7}\times(-\cos(7x-3))+c=-\frac{\cos(7x-3)}{7}+c$

(h) $\int\frac{\sin(2\theta-1)}{\cos(2\theta-1)+3}\mathrm{d}\theta=\int\frac{1}{u}\times\left(-\frac{1}{2}\right)\mathrm{d}u=-\frac{1}{2}\ln(\cos(2\theta-1)+3)+c$

(i) $\int\frac{\mathrm{d}\theta}{\cos^2(5\theta-2)}=\frac{1}{5}\int\frac{1}{\cos^2 u}\mathrm{d}u=\frac{\tan(5\theta-2)}{5}+c$

(j) $\int\cos(\pi x+3)\,\mathrm{d}x=\int\cos u\times\frac{1}{\pi}\mathrm{d}u=\frac{1}{\pi}\sin(\pi x+3)+c$

(k) $\int xe^{x^2+1}\mathrm{d}x=\int e^u\times\frac{1}{2}\mathrm{d}u=\frac{1}{2}e^{x^2+1}+c$

(l) $\int\sqrt{t}e^{2t\sqrt{t}}\mathrm{d}t=\int t^{\frac{1}{2}}e^{2t^{\frac{3}{2}}}\mathrm{d}t=\int e^u\times\frac{1}{3}\mathrm{d}u=\frac{1}{3}e^{2t^{\frac{3}{2}}}+c=\frac{1}{3}e^{2t\sqrt{t}}+c$

(m) $\int\frac{2}{\theta}(\ln\theta)^2\,\mathrm{d}\theta=\int u^2\times2\mathrm{d}u=\frac{2(\ln\theta)^3}{3}+c$

(n) Let $u=\ln 2z\Rightarrow\mathrm{d}u=\frac{\mathrm{d}z}{z}\Rightarrow\int\frac{\mathrm{d}z}{z\ln 2z}=\int\frac{\mathrm{d}u}{u}=\ln|\ln 2z|+c$

4. **(a)** $\int t\sqrt{3-5t^2}\,\mathrm{d}t=\int u^{\frac{1}{2}}\times\left(-\frac{1}{10}\right)\mathrm{d}u=-\frac{1}{10}\times\frac{2}{3}u^{\frac{3}{2}}+c=-\frac{\sqrt{(3-5t^2)^3}}{15}+c$

(b) $\int\frac{\theta^2}{\cos^2\theta^3}\mathrm{d}\theta=\frac{1}{3}\int\frac{1}{\cos^2 u}\mathrm{d}u=\frac{1}{3}\tan\theta^3+c$

(c) $\int\frac{\sin\sqrt{t}}{2\sqrt{t}}\mathrm{d}t=\int\sin u\times\mathrm{d}u=-\cos\left(\sqrt{t}\right)+c$

(d) $\int\frac{\tan^5 2t}{\cos^2 2t}\mathrm{d}t=\int u^5\times\frac{1}{2}\mathrm{d}u=\frac{1}{2}\frac{u^6}{6}+c=\frac{\tan^6 2t}{12}+c$

(e) $\int\frac{\mathrm{d}x}{\sqrt{x}\left(\sqrt{x}+2\right)}=\int\frac{1}{u}\times2\mathrm{d}u=2\ln u+c=2\ln\left(\sqrt{x}+2\right)+c$

(f) $\int\frac{x+3}{x^2+6x+7}\mathrm{d}x=\int\frac{1}{u}\times\frac{1}{2}\mathrm{d}u=\frac{1}{2}\ln\left|x^2+6x+7\right|+c$

(g) $$\int \frac{k^3x^3}{\sqrt{a^2-a^4x^4}}\,dx = \int u^{-\frac{1}{2}}\times\left(-\frac{k^3}{4a^4}\right)du = -\frac{k^3}{4a^4}\times\frac{u^{\frac{1}{2}}}{\frac{1}{2}}+c = -\frac{k^3\sqrt{a^2-a^4x^4}}{2a^4}+c$$

$$= -\frac{k^3|a|\sqrt{1-a^2x^4}}{2a^4}+c = -\frac{k^3\sqrt{1-a^2x^4}}{2|a|^3}+c$$

(h) In this question we are going to use a slightly different method of substitution, since a direct one may lead to a more complex set up.

$x-1=t \Rightarrow x=t+1 \Rightarrow dx = dt$

$$\int 3x\sqrt{x-1}\,dx = \int 3(t+1)\sqrt{t}\,dt$$

$$= 3\int\left(t^{\frac{3}{2}}+t^{\frac{1}{2}}\right)dt$$

$$= 3\left(\frac{t^{\frac{5}{2}}}{\frac{5}{2}}+\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right)+c$$

$$= \frac{2}{5}t^{\frac{3}{2}}(3t+5)+c$$

$$= \frac{2}{5}\sqrt{x-1}\left(3(x-1)^2+5(x-1)\right)+c = \frac{2}{5}\left(3x^2-x-2\right)\sqrt{x-1}+c$$

(i) $$\int\sqrt{1+\cos\theta}\,\sin\theta\,d\theta = \int u^{\frac{1}{2}}\times(-du) = -\frac{(1+\cos\theta)^{\frac{3}{2}}}{\frac{3}{2}}+c = -\frac{2}{3}\sqrt{(1+\cos\theta)^3}+c$$

In questions **(j)** – **(l)** we will use 2-stage methods of substitution.

(j) Start with the substitution $1-t=u \Rightarrow 1-u=t \Rightarrow dt=-du$

$$\Rightarrow \int t^2\sqrt{1-t}\,dt = \int(1-u)^2\sqrt{u}(-du) = -\int\left(u^{\frac{1}{2}}-2u^{\frac{3}{2}}+u^{\frac{5}{2}}\right)du$$

$$= -\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+2\frac{u^{\frac{5}{2}}}{\frac{5}{2}}-\frac{u^{\frac{7}{2}}}{\frac{7}{2}}+c = \frac{2}{105}u^{\frac{1}{2}}\left(-35u+42u^2-15u^3\right)+c$$

$$= \frac{2\sqrt{1-t}}{105}\left(-35(1-t)+42(1-t)^2-15(1-t)^3\right)+c$$

$$= \frac{2\sqrt{1-t}}{105}(t-1)\left(15t^2+12t+8\right)+c$$

(k) Start with the substitution $2r-1=x \Rightarrow r=\frac{x+1}{2} \Rightarrow \mathrm{d}r=\frac{\mathrm{d}x}{2}$

$$\int \frac{r^2-1}{\sqrt{2r-1}}\,\mathrm{d}r=\int \frac{\left(\frac{x+1}{2}\right)^2-1}{\sqrt{x}}\frac{\mathrm{d}x}{2}=\frac{1}{8}\int\left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}}\right)\mathrm{d}x$$

$$=\frac{1}{8}\left(\frac{x^{\frac{5}{2}}}{\frac{5}{2}}+2\frac{x^{\frac{3}{2}}}{\frac{3}{2}}-3\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right)+c=\frac{1}{60}x^{\frac{1}{2}}\left(3x^2+10x-45\right)+c$$

$$=\frac{1}{60}\sqrt{2r-1}\left(3(2r-1)^2+10(2r-1)-45\right)+c$$

$$=\frac{\sqrt{2r-1}}{15}\left(3r^2+2r-13\right)+c$$

(l) Substitution: $e^{x^2}+e^{-x^2}=t \Rightarrow \left(2xe^{x^2}-2xe^{-x^2}\right)\mathrm{d}x=\mathrm{d}t$

$$\int \frac{\mathrm{e}^{x^2}-\mathrm{e}^{-x^2}}{\mathrm{e}^{x^2}+\mathrm{e}^{-x^2}}x\,\mathrm{d}x=\int\frac{1}{t}\frac{\mathrm{d}t}{2}=\frac{1}{2}\ln t+c=\frac{1}{2}\ln\left(\mathrm{e}^{x^2}+\mathrm{e}^{-x^2}\right)+c$$

(m) Substitute $u=t-5 \Rightarrow t=u+5$ and $\mathrm{d}t=\mathrm{d}u$

$$\int\frac{t^2+2}{\sqrt{t-5}}\mathrm{d}t=\int u^{-\frac{1}{2}}\left((u+5)^2+2\right)\mathrm{d}u=\int u^{-\frac{1}{2}}\left(u^2+10u+27\right)\mathrm{d}u$$

$$=\int\left(u^{\frac{3}{2}}+10u^{\frac{1}{2}}+27u^{-\frac{1}{2}}\right)\mathrm{d}u=\frac{2}{5}u^{\frac{5}{2}}+\frac{20}{3}u^{\frac{3}{2}}+54u^{\frac{1}{2}}+c$$

$$=\frac{2}{5}(t-5)^{\frac{5}{2}}+\frac{20}{3}(t-5)^{\frac{3}{2}}+54(t-5)^{\frac{1}{2}}+c$$

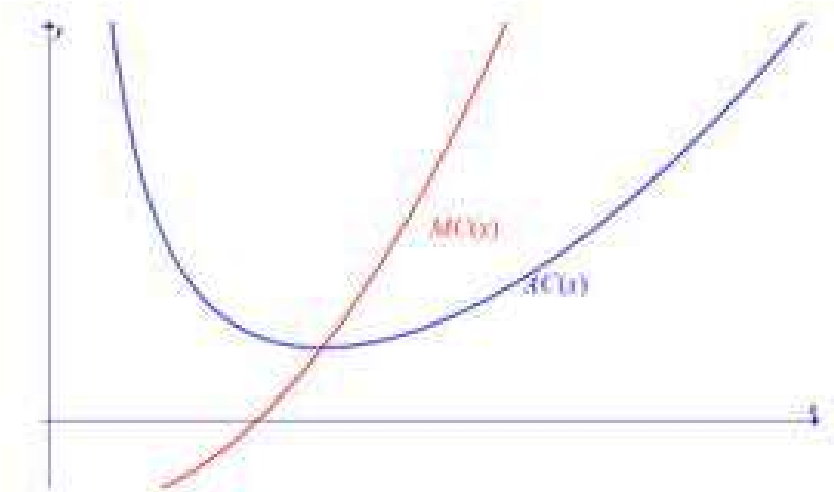
5. (a) $C(x)=\int MC(x)dx=\int\left(0.0006x^2-0.02x-10\right)dx=0.0002x^3-0.01x^2-10x+c$

Fixed costs €2500 are incurred with and without production, i.e., $x = 0$:

$C(0)=2500=0.0002\times 0-0.01\times x-10\times 0+c \Rightarrow c=2500$

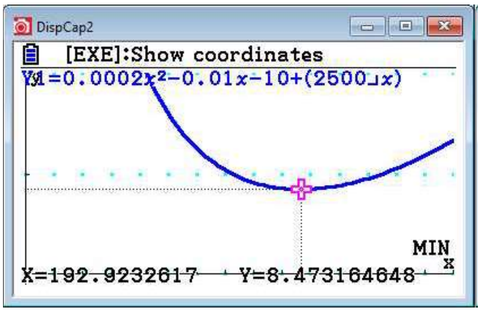
$\Rightarrow C(x)=0.0002x^3-0.01x^2-10x+2500$

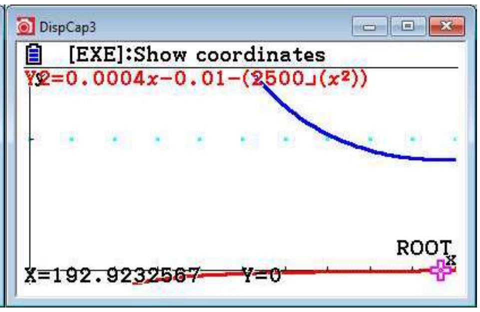
(b) $AC(x)=\frac{C(x)}{x}=0.0002x^2-0.01x-10+\frac{2500}{x}$. Graphs given below.



(c) To find the minimum, we either delegate the whole process to GDC, or we use our knowledge of function behaviour. That is, the minimum average cost happens at the point where the derivative is zero.

$$\frac{\mathrm{d}\left(AC(x)\right)}{\mathrm{d}x} = 0.0004x - 0.01 - \frac{2500}{x^2}$$





So, the minimum average cost is €8.5 and production level is 193 wallets

(d) Evaluate MC(193) $\approx$ €8.5. Same as in **(c)**. This is so because at minimum average cost, average cost and marginal cost are equal.

(e) AC(400) $\approx$ 24.25, They should sell it for 26.25.

6. **(a)** Let P be the amount of pollution added to the lake. Since the only pollution comes from the plant, then, $\frac{\mathrm{d}P}{\mathrm{d}t} = 40\sqrt{t^3}$

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 40\sqrt{t^3} \Rightarrow P(t) = \frac{80}{5}t^{\frac{5}{2}} + c = 16t^{\frac{5}{2}} + c$$

We are assuming that there was no pollution before the plant started, then $P(0) = 0$, thus, $c = 0$. Therefore a model that is appropriate to this case is $P(t) = 16t^{\frac{5}{2}}$.

(b) As described by the model, for the first 3 years, the amount of pollution is

$P(3) = 16 \times 3^{\frac{5}{2}} \approx 249$ tons.

(c) We have to find the time, t that corresponds to a pollution of 400 tons

$$400 = 16t^{\frac{5}{2}} \Rightarrow t^{\frac{5}{2}} = \frac{400}{16} = 25 \Rightarrow t = 25^{\frac{2}{5}} \approx 3.6$$

7. **(a)** let f be the number of cases at time t days after the epidemic started. Then the rate per day is given as $\frac{\mathrm{d}f}{\mathrm{d}t} = 12\mathrm{e}^{0.2t}$.

So, the number of cases at any time t is given by

$$f(t) = \int 12\mathrm{e}^{0.2t}\,\mathrm{d}t = 60\mathrm{e}^{0.2t} + c$$

At time $t = 0$, there were 4 cases, thus,

$$4 = 60\mathrm{e}^{0} + c \Rightarrow c = -56 \Rightarrow f(t) = 60\mathrm{e}^{0.2t} - 56$$

(b) $f(30) \approx 24150$.

8. Let the consumption of tin be $C(t)$ where t is the number of years after 2013.

The rate of consumption is given as $\frac{\mathrm{d}C}{\mathrm{d}t} = 0.22\mathrm{e}^{0.01t}$ million tons per year, thus,

$$C(t) = \int 0.22\mathrm{e}^{0.01t}\,\mathrm{d}t = 22\mathrm{e}^{0.01t} + c$$

With the assumption that we are starting the measurement from 2013 onwards, and so, $C(0) = 0$.

Then, $C(0) = 0 = 22\mathrm{e}^{0} + c \Rightarrow -22$ and so, $C(t) = 22\mathrm{e}^{0.01t} - 22$

For the reserves of 156 million tons to be exhausted, then we need to find t such that $22\mathrm{e}^{0.01t} - 22 = 156$. Use your GDC to solve the equation, or otherwise

$22\mathrm{e}^{0.01t} = 178 \Rightarrow t = 100\ln\left(\frac{178}{22}\right) \Rightarrow t \approx 209$ That is in year 2222.

9. Let the consumption of silver be C(t) where t is the number of years after 2017.

The rate of consumption is given as $\frac{dC}{dt} = 12e^{0.015t}$ thousand tons per year, thus,

$C(t) = \int 12e^{0.015t}dt = 800e^{0.015t} + c$

With the assumption that we are starting the measurement from 2017 onwards, and so, $C(0) = 0$.

Then, $C(0) = 0 = 800e^0 + c \Rightarrow c = -800$ and so, $C(t) = 800e^{0.015t} - 800$

For the reserves of 533 thousand tons to be exhausted, then we need to find t such that $800e^{0.015t} - 800 = 533$.

Use your GDC to solve the equation, or otherwise

$800e^{0.015t} = 1333 \Rightarrow t = \frac{1}{0.015}\ln\left(\frac{1333}{800}\right) \Rightarrow t \approx 34$, that is in year 2051.

10. $\bar{C}(x) = \int -\frac{1000}{x^2}dx = \frac{1000}{x} + c$, with $\bar{C}(100) = 25 \Rightarrow 25 = 10 + c \Rightarrow c = 15$.

So, $\bar{C}(x) = \frac{1000}{x} + 15$. Total cost is the average cost per unit multiplied by the number of units produced: $C(x) = x \times \bar{C}(x) = 15x + 1000 \Rightarrow C(0) = 1000$ SF, and that represents the fixed costs.

11. $f(t) = \int(0.42 + 0.1062t - 0.00747t^2)dt = 0.42t + 0.0531t^2 - 0.00249t^3 + c$

But $f(0) = 0.0242 = 0 + c \Rightarrow c = 0.0242$, thus,
$f(t) = 2.42 + 0.42t + 0.0531t^2 - 0.00249t^3$

In 2020 we will have $f(2020) = 12.18\%$

12. **(a)** Quantity produced = $\int\left(\frac{100}{t+1} + 5\right)dt = 100\ln(t+1) + 5t + c$, with initial production at 0, $c = 0$.

(b) The entire useable life is 20 years, and thus,

Quantity produced = $100\ln(20+1) + 5 \times 20 \approx 404$ thousand barrels

13. **(a)** **(i)** $C(x) = \int\left(12 + \frac{500}{x+1}\right)dx = 12x + 500\ln(x+1) + c$

With fixed costs of €2000,

$C(0) = 0 + c = 2000 \Rightarrow C(x) = 12x + 500\ln(x+1) + 2000$

(ii) $\bar{C}(x) = \frac{C(x)}{x} = 12 + \frac{500\ln(x+1)}{x} + \frac{2000}{x} \Rightarrow \bar{C}(1000) = 17.45$

(b) $R(x)=\int\left(40-0.02x+\frac{200}{x+1}\right)\mathrm{d}x=40x-0.01x^2+200\ln(x+1)+c$

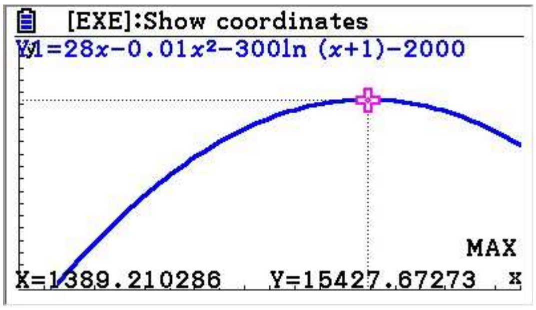
With no revenue when production is zero, then $c = 0$, and thus,

$R(x)=40x-0.01x^2+200\ln(x+1)$

(c) Since profit = revenue – cost, then

$P(x)=R(x)-C(x)=28x-0.01x^2-300\ln(x+1)-2000$

To maximise profit, we need to find the maximum of the profit function. A task that can be delegated to GDC.



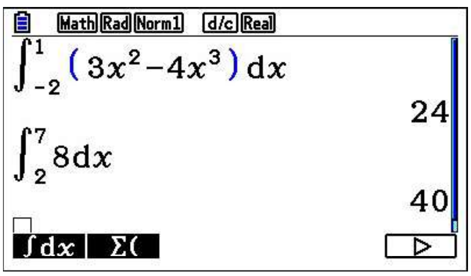
1389 pairs and 15428 Euros profit.

Exercise 16.2

All calculations of definite integrals can be delegated to GDC. We present a minimum algebraic procedures here for explanation purposes. You don't need to do the algebra on exams. You need to set up your integral properly, then give the GDC output. We will demonstrate a GDC output in few cases.

1. **(a)** $\int_{-2}^{1}\left(3x^2-4x^3\right)\mathrm{d}x=\left[x^3-x^4\right]_{-2}^{1}=\left(1^3-1^4\right)-\left((-2)^3-(-2)^4\right)=24$

(b) $\int_{2}^{7}8\,\mathrm{d}x=\left[8x\right]_{2}^{7}=56-16=40$



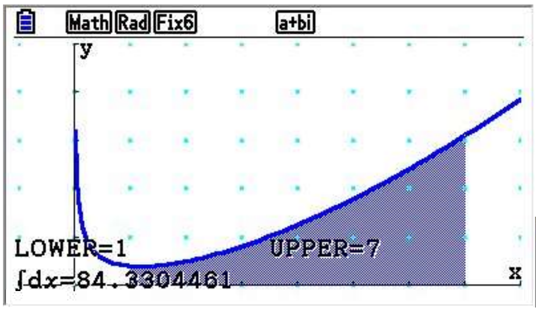
(c) $\int_{1}^{5}\frac{2}{t^3}\mathrm{d}t=\left[2\times\frac{t^{-2}}{-2}\right]_{1}^{5}=\left(-\frac{1}{25}\right)-\left(-\frac{1}{1}\right)=\frac{24}{25}$

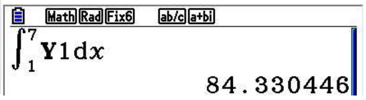
(d) $\int_{2}^{2}(\cos t-\tan t)\mathrm{d}t=0,$ since the upper and lower limits are equal.

(e) Simplify the integrand before attempting to evaluate it

$$\int_1^7 \frac{2x^2-3x+5}{\sqrt{x}}\,dx = \int_1^7 \left(2x^{\frac{3}{2}} - 3x^{\frac{1}{2}} + 5x^{-\frac{1}{2}}\right) dx = \left[2\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 5\frac{x^{\frac{1}{2}}}{\frac{1}{2}}\right]_1^7$$

$$= \frac{176\sqrt{7}-44}{5} \approx 84.33$$





(f) $\int_0^{\pi} \cos\theta\, d\theta = [\sin\theta]_0^{\pi} = \sin\pi - \sin 0 = 0$

You can get the result without integration if you notice that the graph is symmetric about the point $\left(\frac{\pi}{2}, 0\right)$ and the negative area will "cancel" the positive one!

(g) $\int_0^{\pi} \sin\theta\, d\theta = [-\cos\theta]_0^{\pi} = -\cos\pi - (-\cos 0) = -(-1) - (-1) = 2$

(h) $\int_3^1 \left(5x^4 + 3x^2\right) dx = \left[5\frac{x^5}{5} + 3\frac{x^3}{3}\right]_3^1 = \left(1^5 + 1^3\right) - \left(3^5 + 3^3\right) = -268$

(i) $\int_1^3 \frac{u^5+2}{u^2}\, du = \int_1^3 u^3 + 2u^{-2} du = \left[\frac{u^4}{4} + 2\frac{u^{-1}}{-1}\right]_1^3 = \left(\frac{3^4}{4} - \frac{2}{3}\right) - \left(\frac{1}{4} - 2\right) = \frac{64}{3}$

(j) $\int_1^e \frac{2dx}{x} = [2\ln x]_1^e = 2\ln e - 2\ln(1) = 2$

(k) $\int_1^3 \frac{2x}{x^2+2}\, dx = \int_1^3 \frac{d\left(x^2+2\right)}{x^2+2} = \left[\ln\left(x^2+2\right)\right]_1^3 = \ln 11 - \ln 3 = \ln\left(\frac{11}{3}\right)$

This integral could have been evaluated without going back to the original variable too

$$\int_1^3 \frac{2x}{x^2+2}\, dx = \int_3^{11} \frac{du}{u} = \left[\ln|u|\right]_3^{11} = \ln 11 - \ln 3 = \ln\left(\frac{11}{3}\right)$$

(l) $\int_1^3 \left(2-\sqrt{x}\right)^2 \mathrm{d}x = \int_1^3 \left(4-4\sqrt{x}+x\right)\mathrm{d}x = \left[4x-4\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{x^2}{2}\right]_1^3 = \frac{44}{3}-8\sqrt{3}$

(m) $\int_0^{\frac{\pi}{4}} 3\sec^2\theta \,\mathrm{d}\theta = \left[3\tan\theta\right]_0^{\frac{\pi}{4}} = 3\left(\tan\left(\frac{\pi}{4}\right)-\tan 0\right) = 3$

(n) $\int_0^1 \left(8x^7+\sqrt{\pi}\right)\mathrm{d}x = \left[8\frac{x^8}{8}+\sqrt{\pi}\times x\right]_0^1 = 1+\sqrt{\pi}$

(o) Use the fact that $|3x| = \begin{cases} 3x, x \ge 0 \\ -3x, x < 0 \end{cases}$

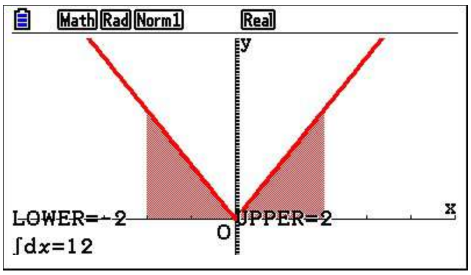
$\int_0^2 |3x|\,\mathrm{d}x = \int_0^2 3x\,\mathrm{d}x = \left[3\frac{x^2}{2}\right]_0^2 = 6$

(p) $\int_{-2}^0 |3x|\,\mathrm{d}x = \int_{-2}^0 -3x\,\mathrm{d}x = \left[-3\frac{x^2}{2}\right]_{-2}^0 = 0-(-6) = 6$

(q) In this part, we simply split the definite integral into the two previous parts:

$\int_{-2}^2 |3x|\,\mathrm{d}x = \int_{-2}^0 |3x|\,\mathrm{d}x + \int_0^2 |3x|\,\mathrm{d}x = 6+6 = 12$.

A GDC demonstration is shown below Questions (o-q)



(r) $\int_0^{\frac{\pi}{2}} \sin 2x\,\mathrm{d}x = \left(-\frac{1}{2}\cos 2x\right)\Big]_0^{\frac{\pi}{2}} = -\frac{1}{2}\left(\underbrace{\cos\pi}_{-1}-\underbrace{\cos 0}_{1}\right) = 1$

(s) $\int_1^9 \frac{1}{\sqrt{x}}\mathrm{d}x = \left(2\sqrt{x}\right)\Big]_1^9 = 2\left(\sqrt{9}-\sqrt{1}\right) = 4$

(t) $\int_{-2}^2 \left(\mathrm{e}^x-\mathrm{e}^{-x}\right)\mathrm{d}x = \left(\mathrm{e}^x+\mathrm{e}^{-x}\right)\Big]_{-2}^2 = \left(\mathrm{e}^2+\mathrm{e}^{-2}\right)-\left(\mathrm{e}^{-2}+\mathrm{e}^2\right) = 0$

If you recognise the symmetry about the origin, the result will be obvious without integration.

Note: In the integrals where we use substitution, if we change the limits of integration we do not need to go back to the original variable.

2. **(a)** Use substitution: $x^2+1=t; 2x\,\mathrm{d}x=\mathrm{d}t$

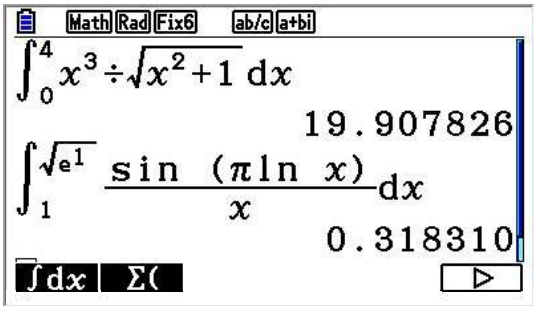
$$\int_0^4 \frac{x^3\mathrm{d}x}{\sqrt{x^2+1}}=\int_1^{17}\frac{(t-1)\frac{1}{2}\mathrm{d}t}{\sqrt{t}}=\frac{1}{2}\int_1^{17}\left(\sqrt{t}-\frac{1}{\sqrt{t}}\right)\mathrm{d}t=\frac{1}{2}\left(\frac{2}{3}t^{\frac{3}{2}}-2t^{\frac{1}{2}}\right)\Bigg]_1^{17}$$

$$=\frac{14\sqrt{17}+2}{3}$$

(b) Use substitution: $\pi\ln x=t; \frac{\pi}{x}\mathrm{d}x=\mathrm{d}t$

$$\int_1^{\sqrt{e}}\frac{\sin(\pi\ln x)\mathrm{d}x}{x}=\int_0^{\frac{\pi}{2}}\frac{\sin t\,\mathrm{d}t}{\pi}=\frac{1}{\pi}(-\cos t)\Big]_0^{\frac{\pi}{2}}=\frac{1}{\pi}$$

Note that when $x=1$, $t=0$, and when $x=\sqrt{e}, t=\pi\ln\sqrt{e}=\frac{\pi}{2}$



(c) Use substitution: $\ln t=u; \frac{1}{t}\mathrm{d}t=\mathrm{d}u$

$$\int_e^{e^2}\frac{\mathrm{d}t}{t\ln t}=\int_1^2\frac{\mathrm{d}u}{u}=\ln|u|\Big]_1^2=\ln(2)-\ln(1)=\ln 2$$

(d) Use substitution: $9-x^2=t; -2x\,\mathrm{d}x=\mathrm{d}t$

$$\int_{-1}^2 3x\sqrt{9-x^2}\,\mathrm{d}x=\int_8^5\frac{3\sqrt{t}\mathrm{d}t}{-2}=\frac{3}{2}\int_5^8\sqrt{t}\mathrm{d}t=\frac{3}{2}\left(\frac{2}{3}t^{\frac{3}{2}}\right)\Bigg]_5^8=16\sqrt{2}-5\sqrt{5}$$

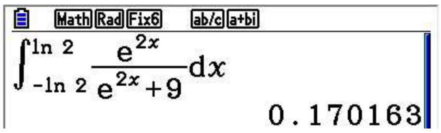
(e) Use substitution: $3+\cos x=t; -\sin x\,\mathrm{d}x=\mathrm{d}t$

$$\int_{-\frac{\pi}{3}}^{\frac{2\pi}{3}}\frac{\sin x}{\sqrt{3+\cos x}}\mathrm{d}x=\int_{3+\frac{1}{2}}^{3-\frac{1}{2}}\frac{-\mathrm{d}t}{\sqrt{t}}=\int_{\frac{5}{2}}^{\frac{7}{2}}\frac{\mathrm{d}t}{\sqrt{t}}=\left(2\sqrt{t}\right)\Big]_{\frac{5}{2}}^{\frac{7}{2}}=\sqrt{14}-\sqrt{10}$$

(f) $\ln x=t, \frac{1}{x}\mathrm{d}x=\mathrm{d}t \Rightarrow \int_e^{e^2}\frac{\ln x}{x}\mathrm{d}x=\int_1^2 t\,\mathrm{d}t=\frac{t^2}{2}\Big]_1^2=2-\frac{1}{2}=\frac{3}{2}$

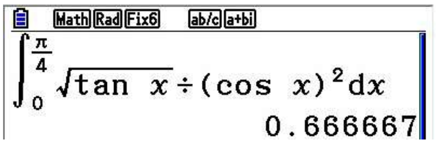
(g) Substitution: $e^{2x}+9=t;\quad 2e^{2x}dx=dt$

$$\int_{-\ln 2}^{\ln 2}\frac{e^{2x}}{e^{2x}+9}dx=\int_{\frac{37}{4}}^{13}\frac{\frac{1}{2}dt}{t}=\left(\frac{1}{2}\ln|t|\right)\Bigg]_{\frac{37}{4}}^{13}=\frac{1}{2}\left(\ln 13-\ln\left(\frac{37}{4}\right)\right)=\frac{1}{2}\ln\left(\frac{52}{37}\right)$$



(h) Substitution: $\tan x=t;\dfrac{dx}{\cos^2 x}=dt$

$$\int_0^{\frac{\pi}{4}}\frac{\sqrt{\tan x}}{\cos^2 x}dx=\int_0^1\sqrt{t}\,dt=\left(\frac{2}{3}t\sqrt{t}\right)\Bigg]_0^1=\frac{2}{3}$$



(i) Substitution: $x^2=t;2xdx=dt$

$$\int_0^{\sqrt{\pi}}7x\cos\left(x^2\right)dx=\int_0^{\pi}\frac{7}{2}\cos t\,dt=\left(\frac{7}{2}\sin t\right)\Bigg]_0^{\pi}=\frac{7}{2}\left(\underbrace{\sin\pi}_{0}-\underbrace{\sin 0}_{0}\right)=0$$

(j) Substitution: $\sqrt{x}=t;\dfrac{dx}{2\sqrt{x}}=dt$

$$\int_{\pi^2}^{4\pi^2}\frac{\sin\sqrt{x}}{\sqrt{x}}dx=\int_{\pi}^{2\pi}2\sin t\,dt=\left(-2\cos t\right)\Big]_{\pi}^{2\pi}=-2\left(\underbrace{\cos(2\pi)}_{1}-\underbrace{\cos(\pi)}_{-1}\right)=-4$$

(k) Substitution: $1-\sin 3t=x\Rightarrow-\cos 3t\times 3dt=dx$

$$\int_0^{\frac{\pi}{6}}\left(1-\sin 3t\right)\cos 3t\,dt=\int_1^0 x\times\left(-\frac{1}{3}\right)dx=\frac{1}{3}\int_0^1 x\,dx=\frac{1}{3}\times\frac{x^2}{2}\Bigg]_0^1=\frac{1}{6}$$

(l) Substitution: $\sin 2\theta=t;2\cos 2\theta d\theta=dt$

$$\int_0^{\frac{\pi}{4}}e^{\sin 2\theta}\cos 2\theta\,d\theta=\int_0^1 e^t\times\frac{1}{2}dt=\frac{1}{2}e^t\Bigg]_0^1=\frac{e-1}{2}$$

(m) Substitution: $\tan 2t=x;2\sec^2 2t\,dt=dx$

$$\int_0^{\frac{\pi}{8}}\left(3+e^{\tan 2t}\right)\sec^2 2t\,dt=\int_0^1\left(3+e^x\right)\times\frac{1}{2}dx=\frac{1}{2}\left(3x+e^x\right)\Bigg]_0^1=1+\frac{e}{2}$$

(n) Substitution: $e^{t^2} = x \Rightarrow 2te^{t^2}\,dt = dx$

$$\int_0^{\sqrt{\ln \pi}} 4t\,e^{t^2} \sin\left(e^{t^2}\right)dt = 2\int_1^{\pi} \sin x\,dx = 2\left(-\cos x\right)\Big]_1^{\pi} = 2 + 2\cos 1$$

3. **(a)** $$av\left(f\right) = \frac{1}{2-1}\int_1^2 x^4 dx = \frac{x^5}{5}\Bigg]_1^2 = \frac{32}{5} - \frac{1}{5} = \frac{31}{5}$$

(b) $$av\left(f\right) = \frac{1}{\frac{\pi}{2}-0}\int_0^{\frac{\pi}{2}} \cos x\,dx = \frac{2}{\pi}\left(\sin x\right)\Big]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\left(\sin\left(\frac{\pi}{2}\right) - \sin 0\right) = \frac{2}{\pi}$$

(c) $$av\left(f\right) = \frac{1}{\frac{\pi}{4}-\frac{\pi}{6}}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec^2 x\,dx$$

$$= \frac{12}{\pi}\left(\tan x\right)\Big]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{12}{\pi}\left(\underbrace{\tan\left(\frac{\pi}{4}\right)}_{1} - \underbrace{\tan\left(\frac{\pi}{6}\right)}_{\frac{\sqrt{3}}{3}}\right)$$

$$= \frac{12 - 4\sqrt{3}}{\pi}$$

(d) $$av\left(f\right) = \frac{1}{4-0}\int_0^4 e^{-2x} dx$$

$$= \frac{1}{4} \times \left(-\frac{1}{2}e^{-2x}\right)\Bigg]_0^4$$

$$= -\frac{1}{8}\left(e^{-8} - e^{0}\right)$$

$$= \frac{e^8 - 1}{8v^8}$$

4. **(a)** $$\frac{d}{dx}\int_2^x \frac{\sin t}{t}dt = \frac{\sin x}{x}$$

(b) $$\frac{d}{dt}\int_t^3 \frac{\sin x}{x}dx = \frac{d}{dt}\int_3^t \left(-\frac{\sin x}{x}\right)dx = -\frac{\sin t}{t}$$

(c) $$\frac{d}{dt}\int_{-\pi}^t \frac{\cos y}{1+y^2}dy = \frac{\cos t}{1+t^2}$$

5. **(a)** $\int_0^k \frac{dx}{3x+2} = \frac{1}{3}\ln(3x+2)\Big]_0^k = \frac{1}{3}\left(\ln(3k+2) - \ln 2\right) = \frac{1}{3}\ln\left(\frac{3k+2}{2}\right)$

(b) $\frac{1}{3}\ln\left(\frac{3k+2}{2}\right) = 1 \Rightarrow \ln\left(\frac{3k+2}{2}\right) = 3 \Rightarrow \frac{3}{2}k + 1 = e^3 \Rightarrow k = \frac{2(e^3-1)}{3}$

6. To evaluate $\int_0^1 x^p(1-x)^q\, dx$ we can use substitution. Substitute $u = 1-x \Rightarrow dx = -du$ and change the limits of integration to the new variable. Remember that when $x = 0$, then $u = 1$, and when $x = 1$, then $u = 0$.

$$\int_0^1 x^p(1-x)^q\, dx = -\int_1^0 (1-u)^p u^q du = \int_0^1 (1-u)^p u^q du$$

Now replace the variable name in the answer by x, and the result follows.

7. **(a)** We use the substitution method applied in question 7 above:

$1 - x = t \Rightarrow -dx = dt$

$$\int x(1-x)^k\, dx = \int (1-t)t^k(-dt) = \int \left(t^{k+1} - t^k\right) dt = \frac{t^{k+2}}{k+2} - \frac{t^{k+1}}{k+1} + c$$
$$= \frac{(1-x)^{k+2}}{k+2} - \frac{(1-x)^{k+1}}{k+1} + c, \quad k \in \mathbb{N}, c \in \mathbb{R}$$

(b) $\int_0^1 x(1-x)^k\, dx = \frac{(1-x)^{k+2}}{k+2} - \frac{(1-x)^{k+1}}{k+1}\Bigg]_0^1 = 0 - \left(\frac{1}{k+2} - \frac{1}{k+1}\right) = \frac{1}{(k+2)(k+1)}$

8. **(a)** $F(3) = \int_3^3 \sqrt{5t^2+2}\, dt = 0$

(b) $F'(x) = \frac{d}{dx}\int_3^x \sqrt{5t^2+2}\, dt = \sqrt{5x^2+2} \Rightarrow F'(3) = \sqrt{5\times 3^2 + 2} = \sqrt{47}$

(c) $F''(x) = \frac{d}{dx}\left(\sqrt{5x^2+2}\right) = \frac{10x}{2\sqrt{5x^2+2}} \Rightarrow F''(3) = \frac{10\times 3}{2\sqrt{5\times 3^2+2}} = \frac{15}{\sqrt{47}} = \frac{15\sqrt{47}}{47}$

9. If the function $f(x)$ is constant over the set of positive real numbers, then it is neither increasing nor decreasing. Thus, its derivative is equal to zero.

$$\frac{d}{dx}f(x) = \frac{d}{dx}\int_x^{3x}\frac{dt}{t} = \frac{d}{dx}\left(\int_x^k \frac{dt}{t} + \int_k^{3x}\frac{dt}{t}\right) = \frac{d}{dx}\left(\int_k^{3x}\frac{dt}{t} - \int_k^x \frac{dt}{t}\right)$$
$$= \frac{1}{3x}\times 3 - \frac{1}{x} = 0$$

10. In both cases, we will use the trapezoidal rule. We can either use a GDC or a spreadsheet for the calculations. Recall that you need to use the trapezoidal formula – we chose $n = 5$.

$$\int_a^b f(x)\mathrm{d}x \approx T_5 = \frac{\Delta x}{2}\left(y_0 + 2\left(y_1 + y_2 + y_3 + y_4\right) + y_5\right)$$
$$= \frac{b-a}{10}\left(y_0 + 2\left(y_1 + y_2 + y_3 + y_4\right) + y_5\right)$$

(a) In this case we have to approximate $\frac{1}{\sqrt{2\pi}}\int_1^3 \mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x$.

We first calculate

$$\int_1^3 \mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x \approx T_5 = \frac{2}{10}\left(f(1) + 2\left(f(1.4) + f(1.8) + f(2.2) + f(2.6)\right) + f(3)\right),$$ and

then multiply with $\frac{1}{\sqrt{2\pi}}$. Here is the output of a spreadsheet.

x	*f(x)*	*T*
1	0.6065	0.6065
1.4	0.3753	0.7506
1.8	0.1979	0.3958
2.2	0.0889	0.1778
2.6	0.0340	0.0681
3	0.0111	0.0111
		0.1604

(b) Similarly, we approximate $\frac{1}{\sqrt{2\pi}}\int_0^2 \mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x$.

$$\int_0^2 \mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x \approx T_5 = \frac{2}{10}\left(f(0) + 2\left(f(0.4) + f(0.8) + f(1.2) + f(1.6)\right) + f(2)\right)$$

and finally

x	*f(x)*	*T*
0	1.0000	1.0000
0.4	0.9231	1.8462
0.8	0.7261	1.4523
1.2	0.4868	0.9735
1.6	0.2780	0.5561
2	0.1353	0.1353
		0.4759

11. Since the function is symmetric about the *y-axis*, then it is enough to find the integral $\int_0^{400}\sqrt{1+\left(\frac{x}{1000}\right)^2}\,dx$ and multiply the result by 2. Given the large domain we do need to use a spreadsheet or program a GDC to do the calculations. Below is a part of a spreadsheet output.

We divided the interval[0, 400] into 40 subintervals, each of size 10.

Recall that the trapezoidal rule will evaluate the following

$$\int_0^{400}\sqrt{1+\left(\frac{x}{1000}\right)^2}\,dx \approx T_{40} = \frac{400-0}{2\times 40}\left(y_0+2\left(y_1+y_2+y_3+\cdots+y_{39}\right)+y_{40}\right)$$

The second column contains the values of the integrand at each point in the interval, and the last column corresponds to the corresponding trapezoidal values: $\left(y_0+2\left(y_1+y_2+y_3+\cdots+y_{39}\right)+y_{40}\right)$

The entry before last in the third column adds all the terms, and the last entry multiplies the sum by 5, which is the value of $\frac{400-0}{2\times 40}$ and by 2 as explained earlier.

So, the length is approximately 821 m.

x	*f(x)*	*T*
0	1	1
10	1.000049999	2.0001
20	1.00019998	2.0004
30	1.000449899	2.0009
↓	↓	↓
400	1.077032961	1.077033
Sum of the function values		82.08547
Sum * 5 * 2		820.8547

12. For convenience, we will use 6 intervals.

$$Area = \int_0^6 f(x)\,dx \approx T_6 = \frac{6}{2\times 6}\left(y_0+2\left(y_1+y_2+y_3+y_4+y_5\right)+y_6\right)$$

x	*f(x)*	*T*
0	3	3
1	6	12
2	5	10
3	4	8
4	5	10
5	6	12
6	3	3
		29

13. As you notice from diagram, we have 8 intervals. Consider the left corner of the pond to be at the origin.

$$\text{Area} = \int_0^{16} f(x)\mathrm{d}x \approx T_6 = \frac{\Delta x}{2}\left(y_0 + 2\left(y_1 + y_2 + \cdots + y_7\right) + y_8\right)$$
$$= \frac{2}{2}\left(0 + 2\left(6.2 + 7.2 + 6.8 + 5.6 + 5 + 4.8 + 4.8\right) + 0\right) = 80.8\ \mathrm{m}^2$$

14. The table implies that we can use 10 intervals, each of size 0.5 second. Thus,

$$\text{Distance} \approx \frac{1}{2\times 2}(0 + 2(4.67 + 7.34 + 8.86 + 9.73 + 10.22 + 10.51 + 10.67 + 10.76 + 10.81) + 10.81)$$
$$= 44.49\,\mathrm{m}$$

15. **Note**: on exams, you will be given information to enable you to use the appropriate model. No need to memorize these models. Refer to example 16.16 for this model.

(a) $$\begin{pmatrix}\text{Capital}\\ \text{value}\end{pmatrix} = \int_0^T r(t)\mathrm{e}^{-it}\mathrm{d}t$$

(b) $$\begin{pmatrix}\text{Well's}\\ \text{capital value}\end{pmatrix} = \int_0^{10} 240000\mathrm{e}^{-0.06t}\mathrm{d}t \approx \$1804753$$

16. To make a decision, you will need to compare the present value of an investment to the price you pay for it. The present value represents the value of the investment

$\begin{pmatrix}\text{Well's}\\ \text{capital value}\end{pmatrix} = P(T) = \int_0^5 100\mathrm{e}^{-0.1t}\mathrm{e}^{-0.1t}\mathrm{d}t = 500\left(1-\mathrm{e}^{-1}\right) \approx 316$, That is, it is worth 316 thousand dollars in Today's estimates. Yes, buy it since it is more worth than the price to be paid.

17. Remember that the mean value of a function is $av(f) = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x$ and considering $b = 12$ and $a = 1$, then the mean temperature is modelled by $\frac{1}{11}\int_1^{12} f(x)\mathrm{d}x$. We use Trap. method to estimate $\int_1^{12} f(x)\mathrm{d}x$ with $\Delta x = 1$

$$\int_1^{12} f(x)\mathrm{d}x \approx \frac{1}{2}\left(15 + 2\left(20 + 25 + 31 + 32.5 + 32 + 30 + 30 + 28 + 23 + 18\right) + 15\right)$$
$$= 284.5$$

Thus, mean temp = $\frac{284.5}{11} = 25.9\ °\mathrm{C}$. This is the average temperature. There is also the maximum and minimum that are usually reported. Also, some may consider the average to be $\frac{284.5}{12} = 23.7$ since the 284.5 is the total for 12 months and we average by dividing by 12.

For the following two questions, refresh your knowledge by looking at the discussion leading to example 16.16 and the example itself.

18. The initial gift is the *present value* of the income stream of £20 000 per year. Thus, the amount of the gift should be

$$\text{Present value } = \int_0^{10} 20000e^{-0.085t}\,dt \approx £134726$$

19. **(a)** Vending machine produces a total income = $\int_0^5 5000e^{0.04t}\,dt \approx £27675$

(b) The future value of the income stream at 12% compounded continuously is

$$FV = e^{0.12\times 5}\int_0^5 5000e^{0.04t}e^{-0.12t}\,dt \approx £37545$$

Total interest = £37545 − £27675 = £9870

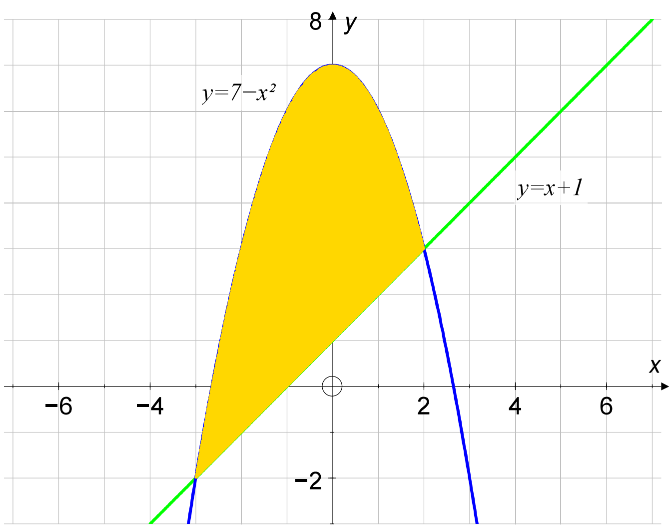
Exercise 16.3

1. We will alternate writing solutions with and without using a GDC.

(a) For this exercise, we will demonstrate two types of answers – without and with a GDC

Without a GDC:

Firstly, we sketch the line and the parabola, and then shade the enclosed area.



Now we need to find the points of intersection by solving the system of equations.

$$\left.\begin{matrix} y = x+1 \\ y = 7-x^2 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} y = x+1 \\ x+1 = 7-x^2 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} y = x+1 \\ x^2+x-6=0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} y = x+1 \\ (x-2)(x+3)=0 \end{matrix}\right\} \Rightarrow$$

$$\left.\begin{matrix} y_1 = 3, y_2 = -2 \\ x_1 = 2, x_2 = -3 \end{matrix}\right\} \Rightarrow (-3,-2) \text{ or } (2,3).$$

So, the integral that we need to calculate is:

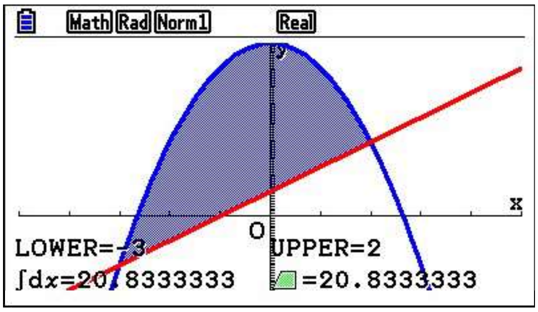
$$\int_{-3}^{2}\left|\left(7-x^2\right)-\left(x+1\right)\right|\mathrm{d}x=\int_{-3}^{2}\left(6-x-x^2\right)\mathrm{d}x=\left[6x-\frac{x^2}{2}-\frac{x^3}{3}\right]_{-3}^{2}$$

$$=\left(12-2-\frac{8}{3}\right)-\left(-18-\frac{9}{2}+9\right)=\frac{125}{6}$$

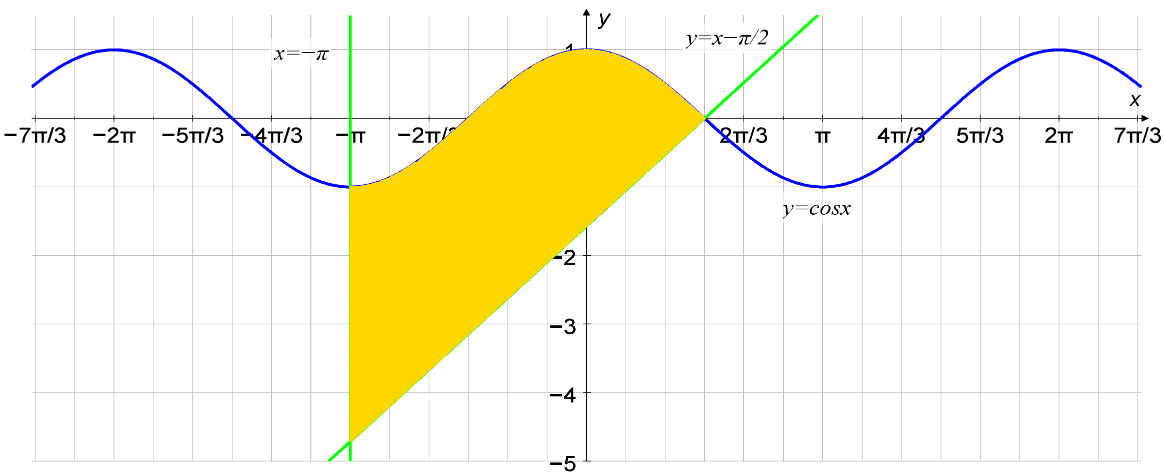
With GDC: You will not need to show every step of calculating intercepts and evaluating the integral. It is enough to write for example:

The area of the required region is bounded by the graphs of the two functions and hence it can be found by subtracting the areas between each of the functions and the *x*-axis between the vertical lines at the *x*-coordinates of their points of intersection.

$$\text{Area}=\int_{-3}^{2}\left|\left(7-x^2\right)-\left(x+1\right)\right|\mathrm{d}x\approx 20.83$$



(b) Firstly, we sketch the cosine curve, the oblique line and the vertical line, and then shade the enclosed area.



Now we have to find the point of intersection of the curve and the oblique line. By inspection, we can see that the point is $\left(\frac{\pi}{2}, 0\right)$; therefore, to find the area, we need to solve the following integral:

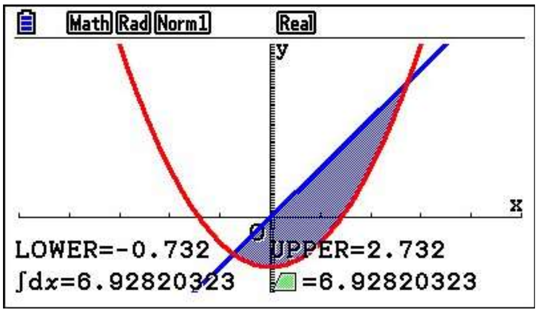
$$\int_{-\pi}^{\frac{\pi}{2}} \left|\cos x - \left(x - \frac{\pi}{2}\right)\right| \mathrm{d}x = \left[\sin x - \frac{x^2}{2} + \frac{\pi}{2}x\right]_{-\pi}^{\frac{\pi}{2}} = \left(1 - \frac{\pi^2}{8} + \frac{\pi^2}{4}\right) - \left(0 - \frac{\pi^2}{2} - \frac{\pi^2}{2}\right)$$

$$= 1 + \frac{\pi^2}{8} + \pi^2 = 1 + \frac{9\pi^2}{8}$$

(c) The two functions intersect at points with x-coordinates $1-\sqrt{3}$ and $1+\sqrt{3}$.

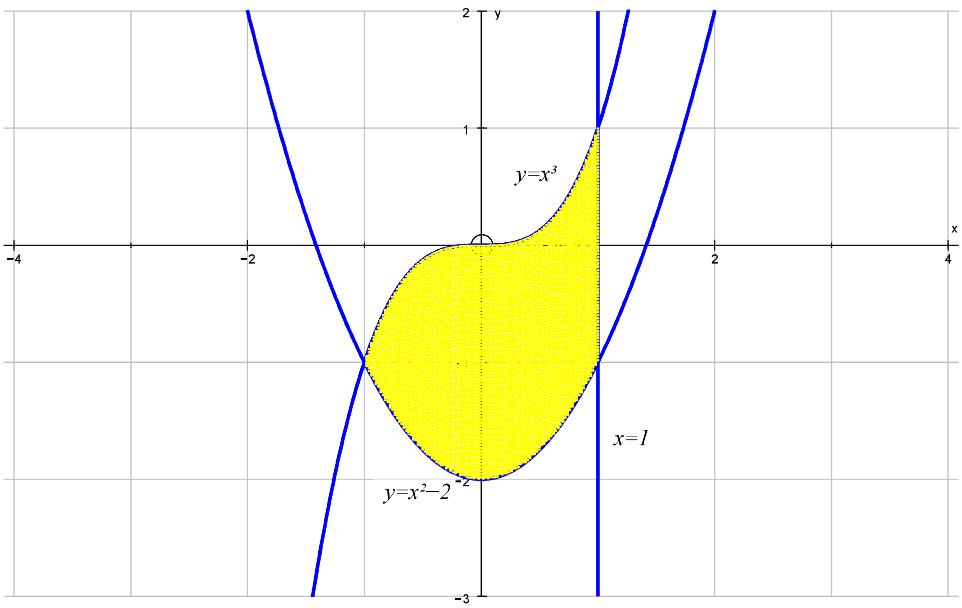
The area of the bounded region shown below is given by

$$\int_{1-\sqrt{3}}^{1+\sqrt{3}} \left|2x - \left(x^2 - 2\right)\right| \mathrm{d}x$$



The required area is 6.93 units^2.

(d) Sketching all the given curves and shading the enclosed area gives the following graph.



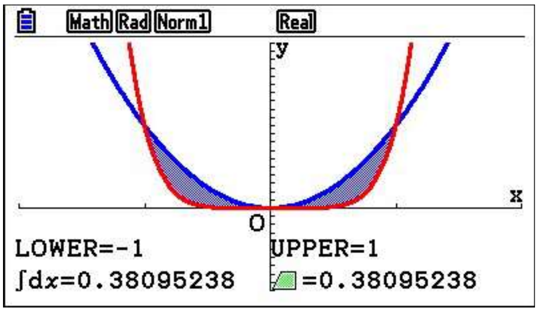
So, by inspection, we can see that the point of intersection of the curves is $(-1,-1)$. Therefore, the integral that we need to calculate to find the area is:

$$\int_{-1}^{1}\left|x^3-\left(x^2-2\right)\right|\mathrm{d}x=\int_{-1}^{1}\left(x^3-x^2+2\right)\mathrm{d}x=\left[\frac{x^4}{4}-\frac{x^3}{3}+2x\right]_{-1}^{1}=\frac{10}{3}$$

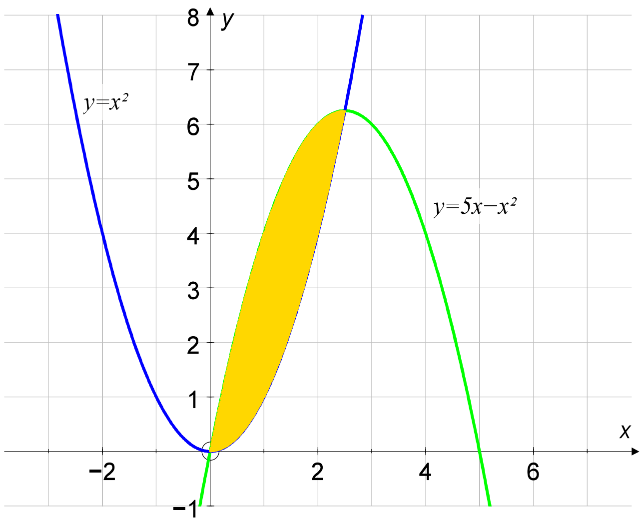
(e) The two functions intersect when $x^6=x^2$. Thus,

$x^6-x^2=x^2\left(x^2+1\right)\left(x^2-1\right)$ and the points of intersection are at the points with x-coordinates -1, 0 and 1. At $x = 0$, they are tangent to each other making the graph of $y=x^2$ above that of $y=x^6$. Thus the area is

$$\int_{-1}^{1}\left|x^2-x^6\right|\mathrm{d}x=\frac{8}{21}\approx 0.381$$



(f) Sketching the given curves and shading the enclosed area gives the following graph.



One point of intersection is obviously the origin, while the other looks like it has an x-coordinate of 2.5, but since we are not sure we will solve the simultaneous equations.

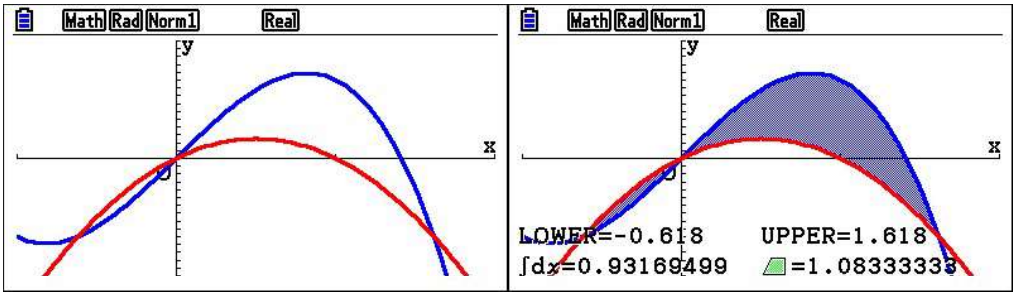
$$\left.\begin{array}{r} y = 5x - x^2 \\ y = x^2 \end{array}\right\} \Rightarrow x^2 = 5x - x^2 \Rightarrow 2x^2 - 5x = 0 \Rightarrow \begin{cases} x_1 = 0,\ x_2 = \dfrac{5}{2} \\ y_1 = 0,\ y_2 = \dfrac{25}{4} \end{cases}$$

So, the integral we have to calculate is:

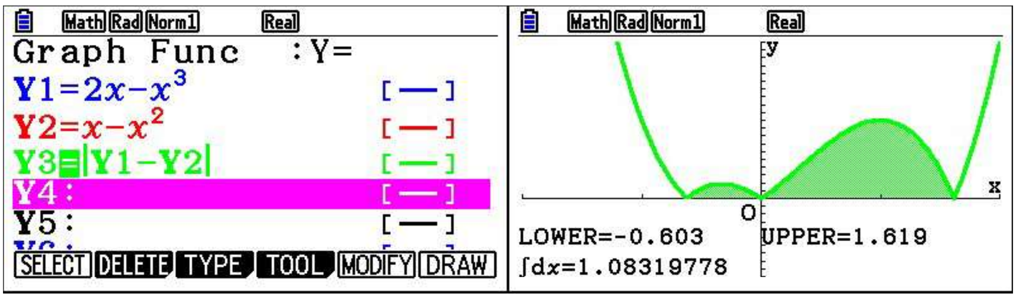
$$\int_0^{\frac{5}{2}} \left|\left(5x - x^2\right) - x^2\right| \mathrm{d}x = \int_0^{\frac{5}{2}} \left(5x - 2x^2\right) \mathrm{d}x = \left[5\times\frac{x^2}{2} - 2\times\frac{x^3}{3}\right]_0^{\frac{5}{2}} = \frac{125}{24}$$

(g) Note that there are two areas that are defined by the two curves and the curves alternate

between being the upper and lower graphs. In order to get both the areas enclosed by the curves shaded, we need to pay attention to the absolute value of the difference between their integrals. This can be done directly or some GDCs will also give the right answer. We demonstrate both cases below.

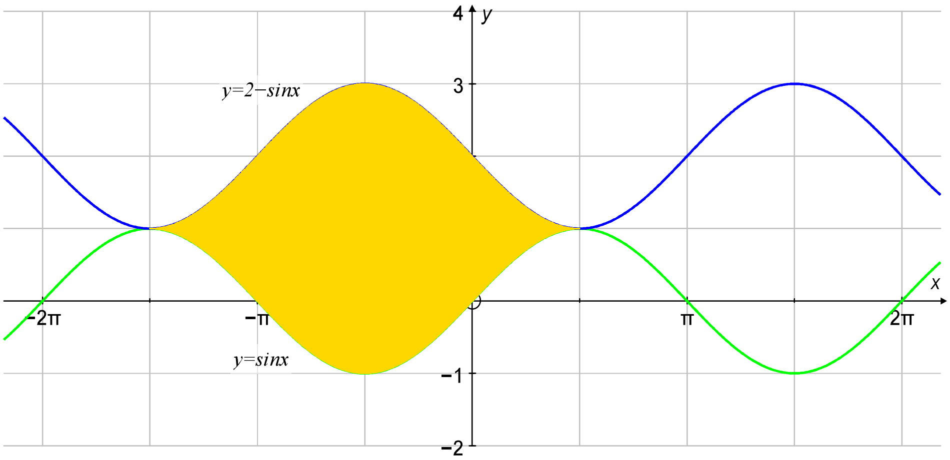


Writing the functions separately and then evaluating the absolute value of the difference:



So the required area is approximately 1.083 units^2.

(h) Sketching the given curves and shading the enclosed area for one period only gives the following graph.

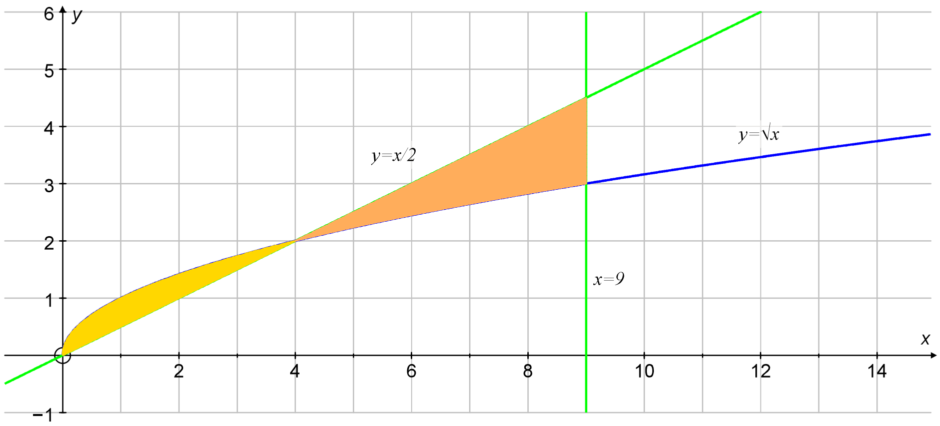


By inspection, we note that the points of intersection are $\left(-\frac{3\pi}{2},1\right)$ and $\left(\frac{\pi}{2},1\right)$, so the integral we have to calculate is:

$$\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}}\left|(2-\sin x)-\sin x\right|\mathrm{d}x=\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}}(2-2\sin x)\,\mathrm{d}x=\left[2x+2\cos x\right]_{-\frac{3\pi}{2}}^{\frac{\pi}{2}}$$

$$=\left(2\times\frac{\pi}{2}+2\cos\frac{\pi}{2}\right)-\left(2\times\left(-\frac{3\pi}{2}\right)+2\cos\left(-\frac{3\pi}{2}\right)\right)=4\pi$$

(i) Sketching the given curves and shading the enclosed area gives the following graph.

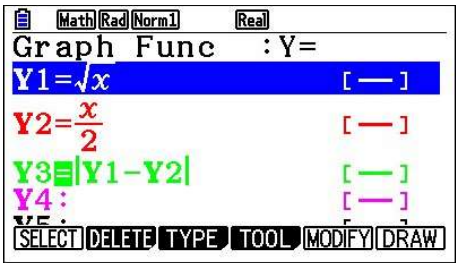


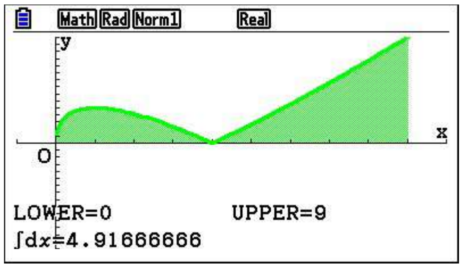
Again, note that there are two areas bounded by the curves and that the curves exchange positions of being upper and lower. By inspection, we can see that the points of intersection are the origin and (4, 2). Therefore, to find the total shaded area, we need to find the following integrals and add them up.

$$\text{Area} = \int_0^9 \left|\sqrt{x} - \frac{x}{2}\right| \mathrm{d}x = \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) \mathrm{d}x + \int_4^9 \left(\frac{x}{2} - \sqrt{x}\right) \mathrm{d}x$$

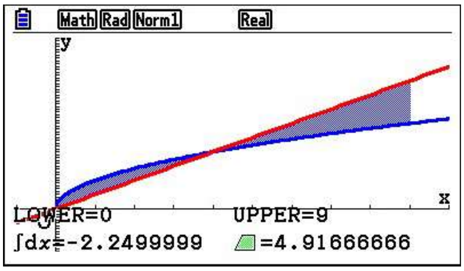
$$= \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^2}{4}\right]_0^4 + \left[\frac{x^2}{4} - \frac{2}{3}x^{\frac{3}{2}}\right]_4^9 = \left(\frac{16}{3} - 4\right) + \left(\frac{81}{4} - 18\right) = \frac{4}{3} + \frac{43}{12} = \frac{59}{12} \approx 4.92$$

GDC work demonstration:

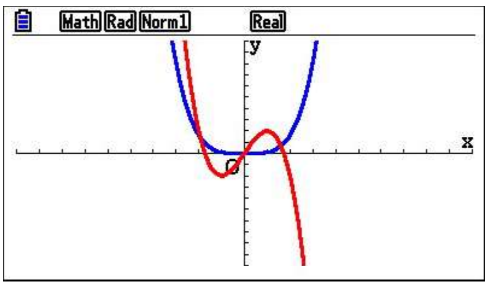




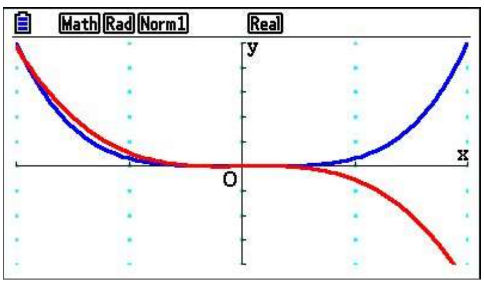
Or



(j) This question cannot be done algebraically since we cannot solve the equation of the fourth degree for finding the boundaries of the integral. Sketch the graphs first.

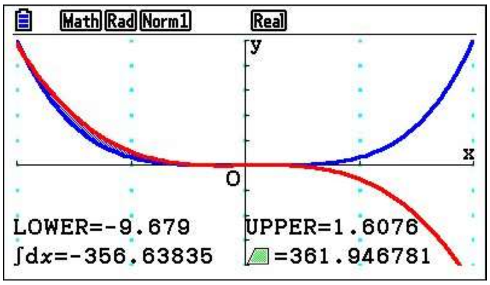


In cases similar to this, you have to pay attention that the graph you see may not be the whole one. We know that in general a quartic function decreases faster than a cubic function. Therefore, apart from the three points of intersection that we can see on the graph, we expect to find one more to the left. To find the fourth point of intersection, we need to adjust the window.



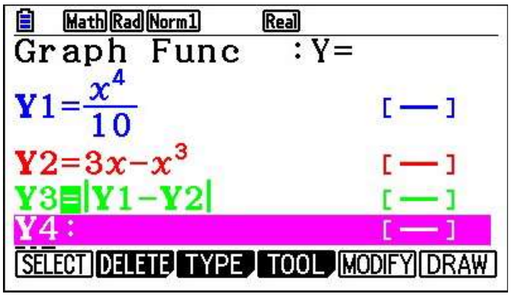
Apparently, there is another point of intersection on the left.

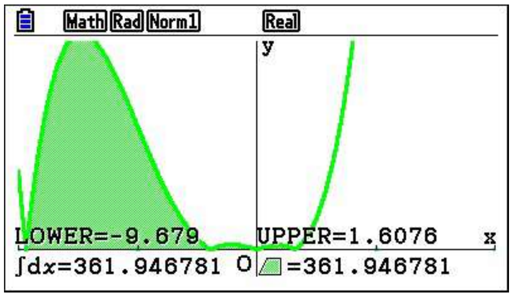
It is wise to have the GDC do the calculation of finding the area between curves



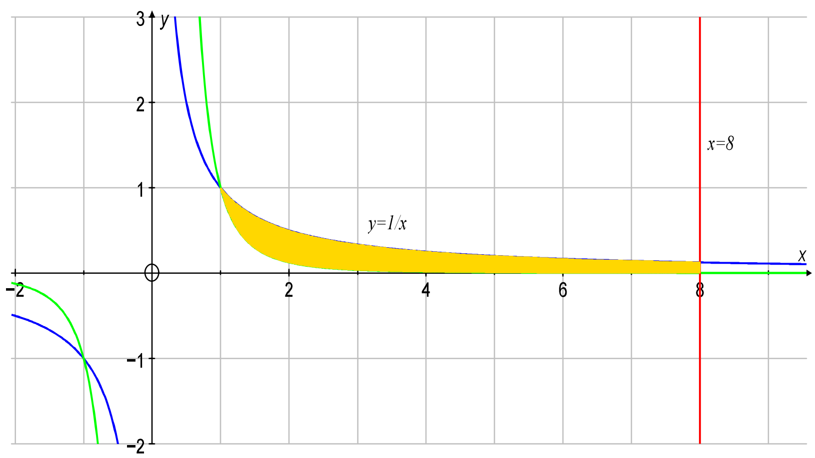
Area is 361.95.

Examiner Note: By using the absolute value function, we can skip all the points in between and simply calculate the integral from the first point on the left side until the last point on the right side. The final answer in the IB exam would be given correct to three significant figures, 362, if not otherwise stated in the question.





(k) Sketching the given curves and shading the enclosed area gives the following graph.

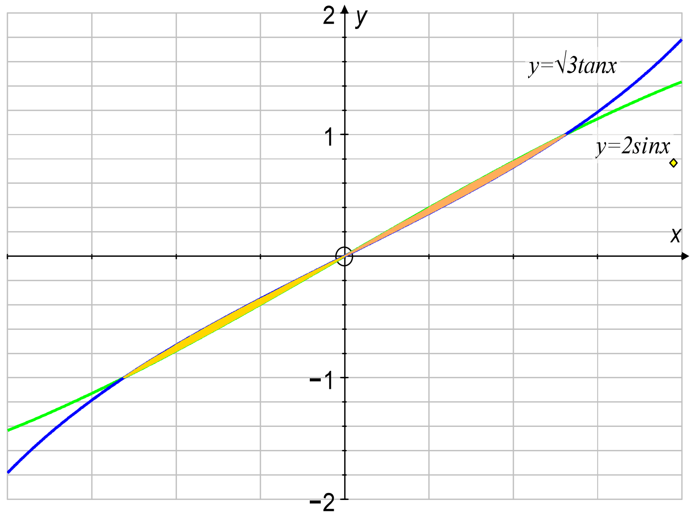


Points of intersection are given by

$\frac{1}{x}=\frac{1}{x^3}\Rightarrow x(x-1)(x+1)=0$ where the only two possible points are $(-1,\ -1)$ or $(1,\ 1)$.

$$\text{Area} = \int_1^8 \left|\frac{1}{x} - \frac{1}{x^3}\right| \mathrm{d}x = \left[\ln|x| + \frac{1}{2x^2}\right]_1^8 = \left(\ln 8 + \frac{1}{128}\right) - \left(\ln 1 + \frac{1}{2}\right) = 3\ln 2 - \frac{63}{128}$$

(l) Sketching the given curves for the restricted domain and shading the enclosed area gives the following graph.



To find the points of intersection, we need to solve the following equation:

$$2\sin x = \sqrt{3}\tan x \Rightarrow 2\sin x - \sqrt{3}\frac{\sin x}{\cos x} = 0 \Rightarrow \sin x\left(2 - \frac{\sqrt{3}}{\cos x}\right) = 0$$

$$\Rightarrow (\sin x = 0) \text{ or } \left(2 - \frac{\sqrt{3}}{\cos x} = 0\right) \Rightarrow (\sin x = 0) \text{ or } \left(\cos x = \frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow x_1 = 0, x_2 = -\frac{\pi}{6}, x_3 = \frac{\pi}{6}$$

Since we have two areas, one in the first quadrant and one in the third, bounded by the curves, and since both functions are odd (symmetrical with respect to the origin); therefore, the enclosed areas must each have the same area. So, the final area is obtained by finding the area of one region and multiplying it by 2.

$$\int_0^{\frac{\pi}{6}} \left(2\sin x - \sqrt{3}\tan x\right) \mathrm{d}x = \left[-2\cos x + \sqrt{3}\ln|\cos x|\right]_0^{\frac{\pi}{6}}$$

$$= \left(-2\cos\left(\frac{\pi}{6}\right) + \sqrt{3}\ln\left(\cos\left(\frac{\pi}{6}\right)\right)\right) - \left(-2\cos 0 + \sqrt{3}\ln(\cos 0)\right)$$

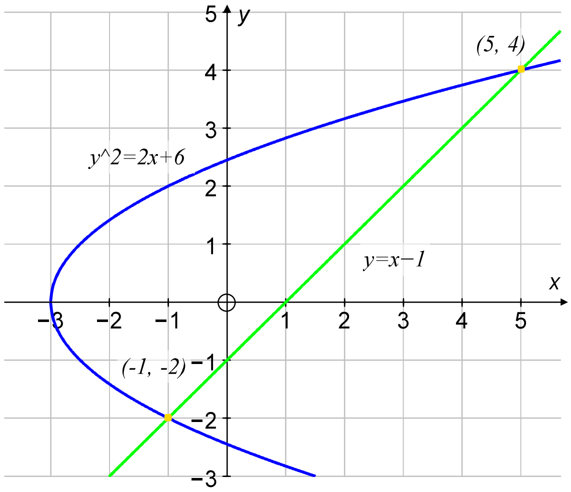
$$= -2 \times \frac{\sqrt{3}}{2} + \sqrt{3}\ln\left(\frac{\sqrt{3}}{2}\right) + 2 = 2 - \sqrt{3} + \sqrt{3}\left(\frac{1}{2}\ln 3 - \ln 2\right)$$

$$\text{Area} = 2\times\left[2-\sqrt{3}+\sqrt{3}\left(\frac{1}{2}\ln 3-\ln 2\right)\right] = 4-2\sqrt{3}+\sqrt{3}\left(\ln 3-2\ln 2\right)$$

note: $\int \tan x\,\mathrm{d}x = \int \frac{\sin x}{\cos x}\mathrm{d}x = \int \frac{-\mathrm{d}(\cos x)}{\cos x} = -\ln|\cos x| + c$

In questions **(m)** – **(q)**, we will integrate with respect to y since, in both equations of the curves, the variable x is linear and therefore it is easier to express it in terms of y.

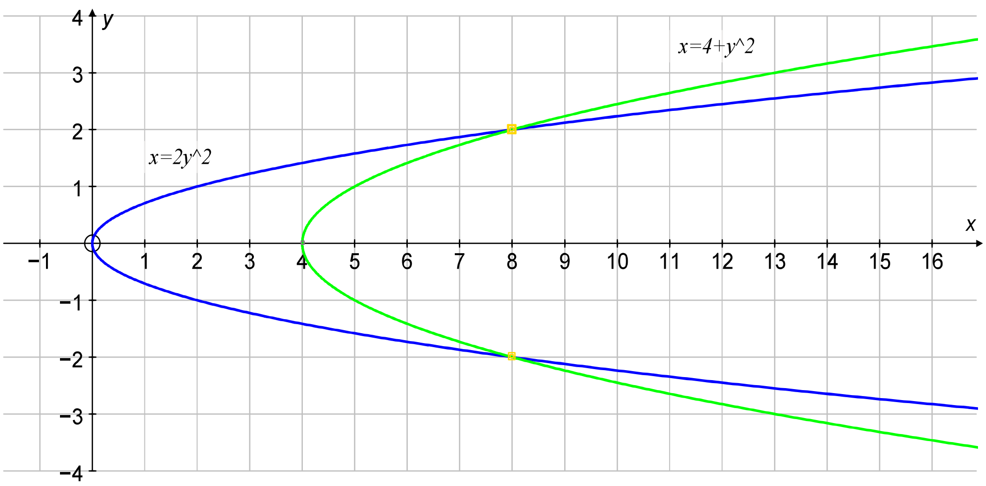
(m) We sketch the curves, find the points of intersection and swap the variable of integration.



$$\int_{-2}^{4}\left((y+1)-\left(\frac{1}{2}y^2-3\right)\right)\mathrm{d}y = \int_{-2}^{4}\left(y-\frac{1}{2}y^2+4\right)\mathrm{d}y = \left(\frac{y^2}{2}-\frac{y^3}{6}+4y\right)\Bigg]_{-2}^{4}$$

$$=\left(8-\frac{32}{3}+16\right)-\left(2+\frac{4}{3}-8\right)=18$$

(n) We sketch the curves, find the points of intersection and swap the variable of integration.

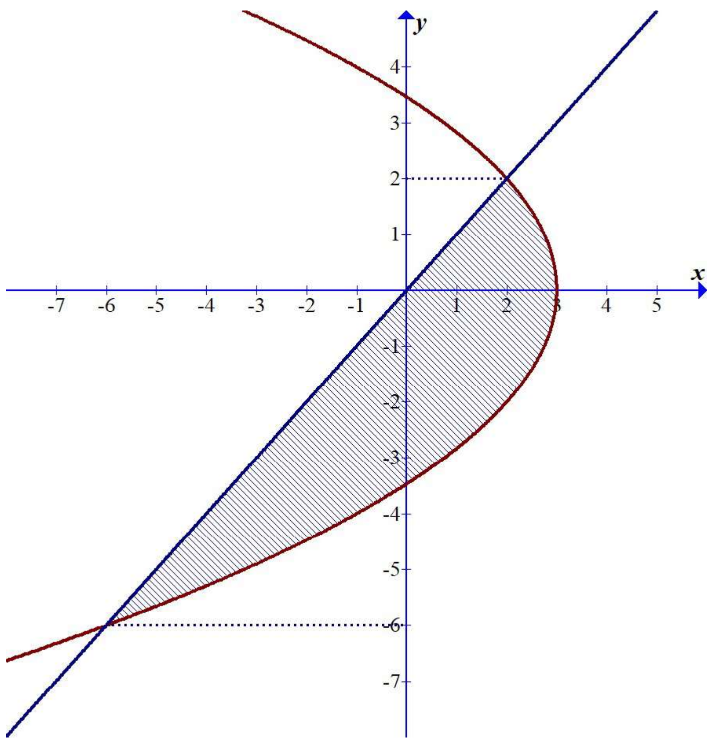


$$\int_{-2}^{2}\left(\left(4+y^2\right)-\left(2y^2\right)\right)\mathrm{d}y=\int_{-2}^{2}\left(4-y^2\right)\mathrm{d}y=\left[\left(4y-\frac{y^3}{3}\right)\right]_{-2}^{2}=\frac{32}{3}$$

Note that since the functions are symmetrical with respect to the x-axis, we could have used an integral from 0 to 2 and multiplied it by 2.

(o) To use a GDC, we need to express x explicitly in terms of y and then use the finite integral feature on the calculator.

$$\begin{cases}4x+y^2=12\\ y=x\end{cases}\Rightarrow\begin{cases}x=3-\dfrac{y^2}{4}\\ x=y\end{cases}$$

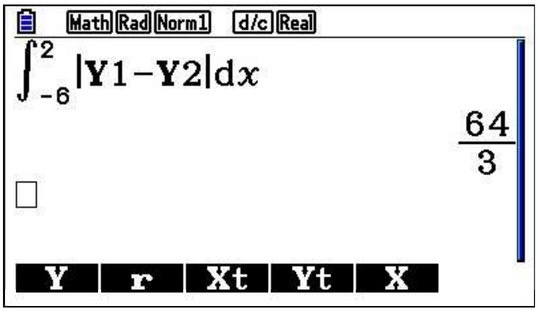


Next step is to partially solve the simultaneous equations to find the limits of integration. We will use x instead of y on the GDC.

Solving the system, we can either use GDC or directly. This system is simple

$$\left.\begin{matrix}x=3-\dfrac{y^2}{4}\\ x=y\end{matrix}\right\}\Rightarrow y^2+4y-12=0\Rightarrow y=-6\text{ or }y=2.\text{ Thus,}$$

$$\text{Area}=\int_{-6}^{2}\left|3-\frac{y^2}{4}-y\right|\mathrm{d}y=\frac{64}{3}$$



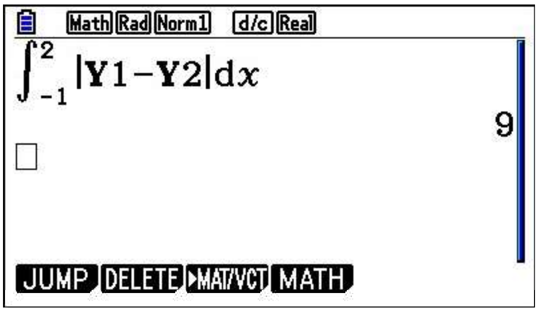
(p) To use a GDC, we need to express x explicitly in terms of y and then use the finite integral feature on the calculator.

$$\begin{cases} x - y = 7 \\ x = 2y^2 - y + 3 \end{cases} \Rightarrow \begin{cases} x = 7 + y \\ x = 2y^2 - y + 3 \end{cases}$$

Next step is to partially solve the simultaneous equations to find the limits of integration. We will use x instead of y on the GDC.

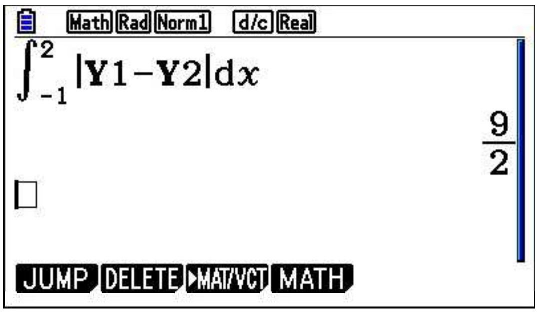
Solving the system, we can either use GDC or directly.

$$7 + y = 2y^2 - y + 3 \Rightarrow 2y^2 - 2y - 4 = 0 \Rightarrow y^2 - y - 2 = 0 \Rightarrow y = -1 \text{ or } y = 2$$



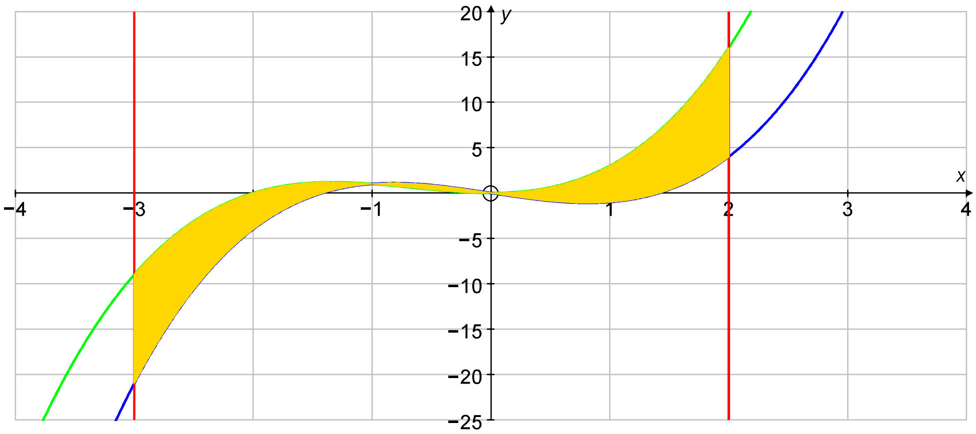
(q) Since the variable x is already expressed explicitly in terms of y, we simply need to partially solve the simultaneous equations to find the borders of integration.

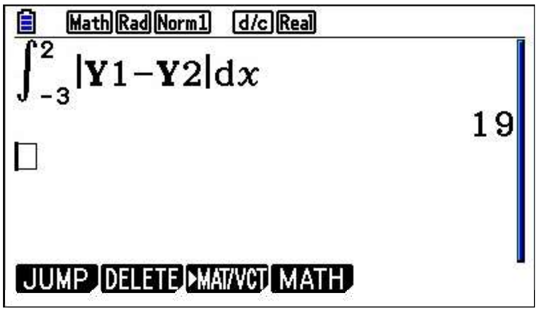
$$y^2 = 2y^2 - y - 2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow y = -1 \text{ or } y = 2$$



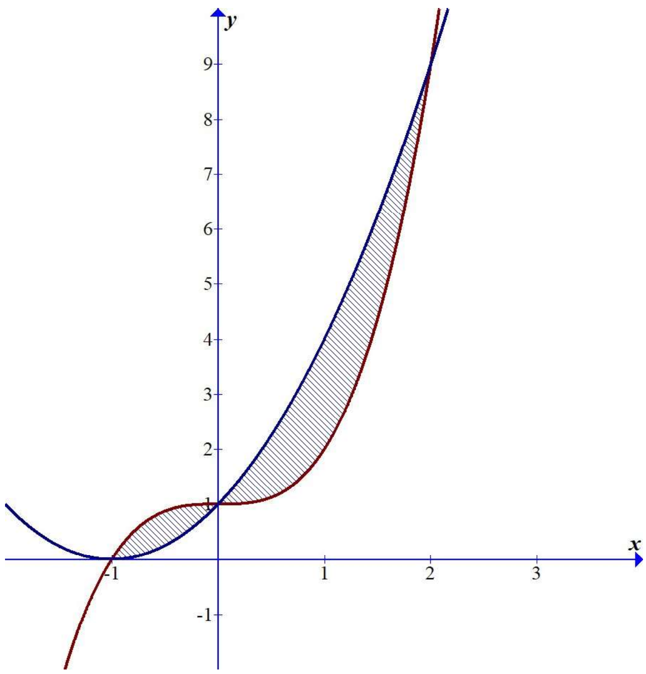
Note: We don't have to simplify the expressions and, if we are not sure which function is the upper one and which is the lower one, we don't need to spend too much time on graphing and identifying. We can simply use the absolute value of the difference of two functions; the result will always be positive and therefore it is the area between the curves.

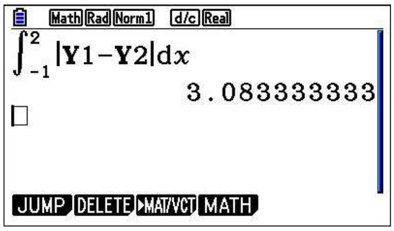
(r) In this case, since the boundaries of integration are given by the vertical lines, we don't even have to sketch the curves. Since there are multiple areas enclosed by the two curves (alternating upper and lower curves), we can simply apply the absolute value function.





(s) The sketch of the region is given below.



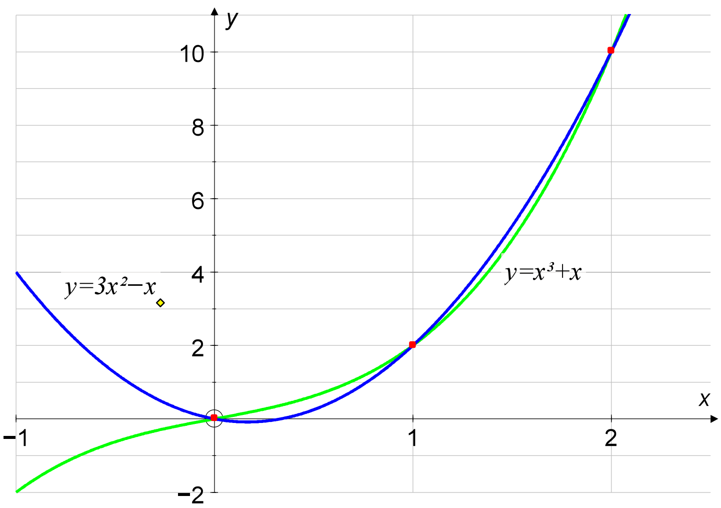


$$\text{Area} = \int_{-1}^{2}\left|(x^3+1)-(x+1)^2\right|\mathrm{d}x = 3.083$$

Even though the GDC does not offer an answer in fraction form, we can find the exact value by splitting the work into two intervals:

$$\text{Area} = \int_{-1}^{0}\left((x^3+1)-(x+1)^2\right)\mathrm{d}x + \int_{0}^{2}\left((x+1)^2-(x^3+1)\right)\mathrm{d}x = \frac{37}{12}$$

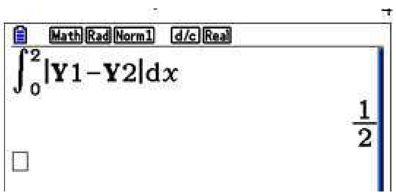
(t) Firstly, we need to sketch the functions and find the points of intersection.



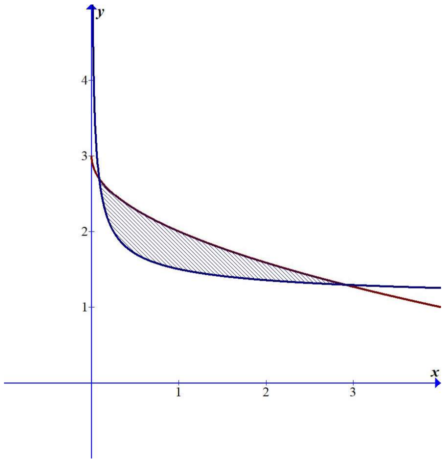
To find the points of intersection, we need to solve the system of simultaneous equations $\left.\begin{matrix} y = x^3 + x \\ y = 3x^2 - x \end{matrix}\right\} \Rightarrow x^3 + x = 3x^2 - x \Rightarrow x(x-1)(x-2) = 0$

Since there are two regions enclosed by the curves, we will need to split the integral into two integrals, exchanging the upper and the lower function.

$$\begin{aligned}\text{Area} &= \int_{0}^{1}\left((x^3+x)-(3x^2-x)\right)\mathrm{d}x + \int_{1}^{2}\left((3x^2-x)-(x^3+x)\right)\mathrm{d}x \\ &= \int_{0}^{1}\left(x^3-3x^2+2x\right)\mathrm{d}x + \int_{1}^{2}\left(-x^3+3x^2-2x\right)\mathrm{d}x \\ &= \frac{1}{4}-1+1+(-4)+8-4+\frac{1}{4}-1+1 = \frac{1}{2}\end{aligned}$$



(u) The sketch of the region is shown below

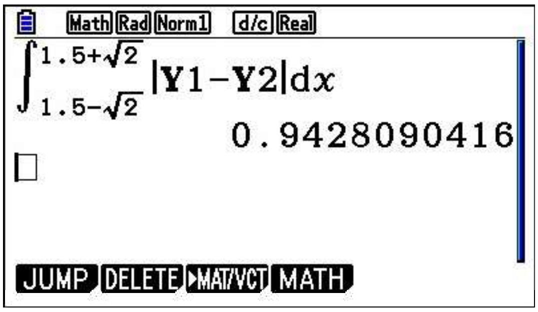


To find the intersections exactly, we solve the system of equations

$$3-\sqrt{x}=\frac{2\sqrt{x}+1}{2\sqrt{x}} \Rightarrow 2x-4\sqrt{x}+1=0 \Rightarrow x=\frac{3}{2}\pm\sqrt{2}$$

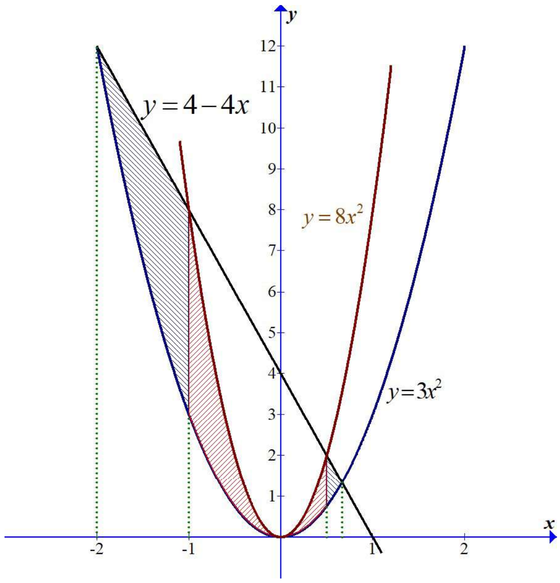
$$\text{Area}=\int_{\frac{3}{2}-\sqrt{2}}^{\frac{3}{2}+\sqrt{2}}\left|3-\sqrt{x}-\frac{2\sqrt{x}+1}{2\sqrt{x}}\right|\mathrm{d}x=\frac{2\sqrt{2}}{3}$$

Using a GDC



This is approximately equal to the exact answer above.

2. In order to calculate the area, we split it into 3 different intervals as shown below



The first one is between $y = 4 - 4x$ and $y = 3x^2$, the second between $y = 8x^2$ and $y = 3x^2$, and the third between $y = 4 - 4x$ and $y = 3x^2$.

We need to find the points of intersection of the curves so that we can establish the limits of integration by solving the corresponding systems of simultaneous equations.

$$\begin{cases} y = 3x^2 \\ y = 4 - 4x \end{cases} \Rightarrow 4 - 4x = 3x^2 \Rightarrow x_1 = -2, x_4 = \frac{2}{3}$$

$$\begin{cases} y = 8x^2 \\ y = 4 - 4x \end{cases} \Rightarrow 8x^2 + 4x - 4 = 0 \Rightarrow x_2 = -1, x_3 = \frac{1}{2}$$

Note that in both cases we did not calculate the y-values since, for the integration, we simply need the x-coordinates of the points of intersection.

The area in question is therefore

$$\text{Area} = \int_{-2}^{-1}\left(4 - 4x - 3x^2\right)\text{d}x + \int_{-1}^{\frac{1}{2}}\left(8x^2 - 3x^2\right)\text{d}x + \int_{\frac{1}{2}}^{\frac{2}{3}}\left(4 - 4x - 3x^2\right)\text{d}x$$

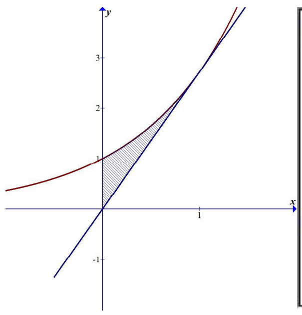
$$= \left(4x - 2x^2 - x^3\right)\Big]_{-2}^{-1} + \left(\frac{5x^3}{3}\right)\Bigg]_{-1}^{\frac{1}{2}} + \left(4x - 2x^2 - x^3\right)\Big]_{\frac{1}{2}}^{\frac{2}{3}} = \frac{269}{54}$$

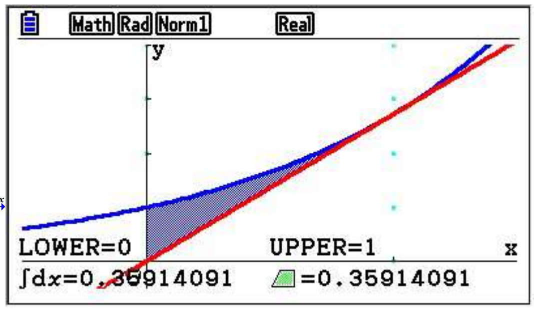
3. Firstly, we need to find the equation of the tangent at the point $(1, e)$.

$$y = \text{e}^x \Rightarrow y' = \text{e}^x \quad m = y'(1) = \text{e} \Rightarrow \text{Tangent}: y = \text{e}(x-1) + \text{e} \Rightarrow y = \text{e}x$$

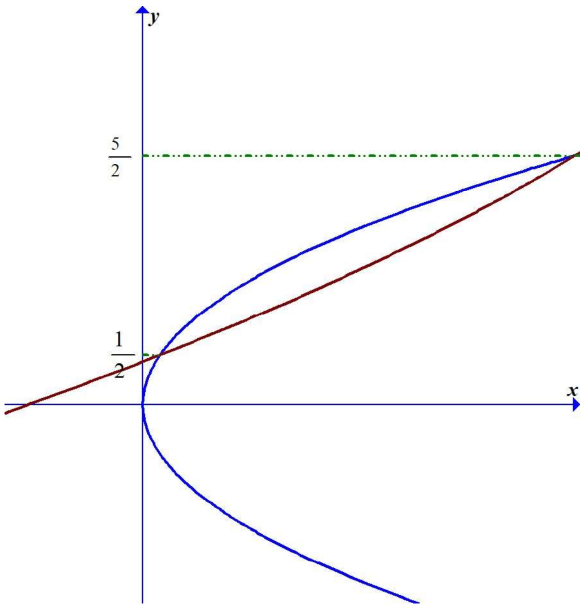
Since $y'' = \text{e}^x \Rightarrow y''(1) = \text{e} > 0$, we can conclude that the curve is above the tangent; therefore, to find the area of the region enclosed by the curves, we calculate the following integral.

$$\int_0^1 \left(e^x - ex\right) dx = \left[\left(e^x - e\frac{x^2}{2}\right)\right]_0^1 = \left(e - \frac{e}{2}\right) - (1 - 0) = \frac{e}{2} - 1$$





4. In this question, we can do all the calculations with respect to y since x is expressed as the subject in both equations.

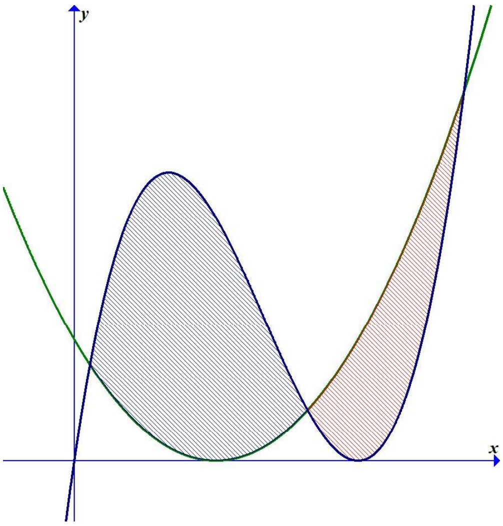


The y- coordinates of the points of intersection are found by solving

$$3y^2 = 12y - y^2 - 5 \Rightarrow 4y^2 - 12y + 5 = 0 \Rightarrow y_1 = \frac{1}{2}, y_2 = \frac{5}{2}$$

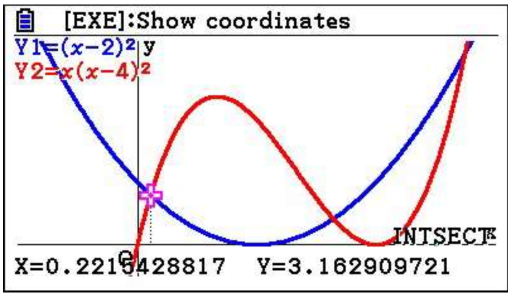
Therefore, $\text{Area} = \int_{\frac{1}{2}}^{\frac{5}{2}} \left|12y - y^2 - 5 - 3y^2\right| dy = \left[6y^2 - \frac{4}{3}y^3 - 5y\right]_{\frac{1}{2}}^{\frac{5}{2}} = \frac{16}{3}$

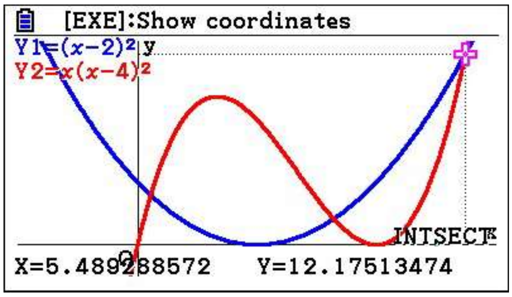
5. If we sketch the graphs, we get the following



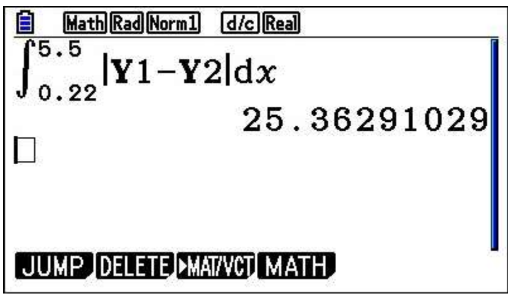
The x-coordinates of the points of intersection cannot be found exactly. We will use a GDC for this.

One method is first to find the points of intersection

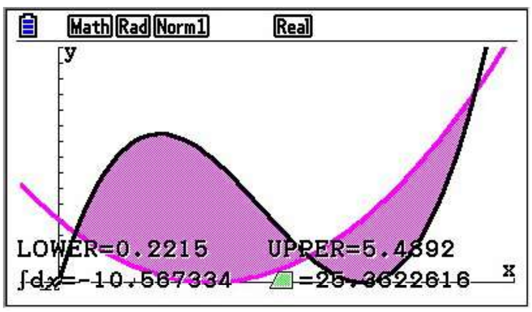




Then we find the area



However, some GDCs can do the whole process in one step after entering the functions.



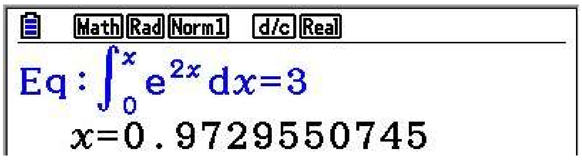
6. The area under this curve is given by

$$\int_0^m e^{2x}\,dx = \left(\frac{1}{2}e^{2x}\right)\Bigg]_0^m = \frac{1}{2}\left(e^{2m}-1\right)$$

Now, if the area is 3 square units, then

$$\frac{1}{2}\left(e^{2m}-1\right) = 3 \Rightarrow e^{2m}-1 = 6 \Rightarrow m = \frac{\ln 7}{2}$$

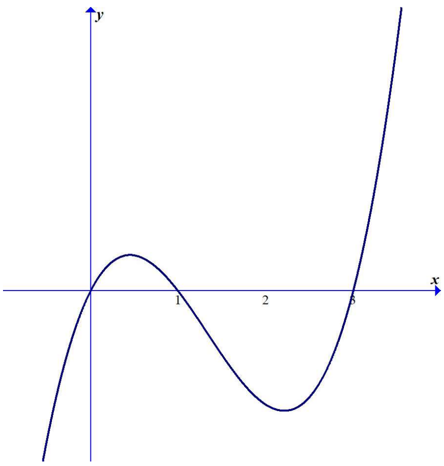
A GDC can also be set up to solve the equation



7. Firstly, we need to find the zeros of the function.

$$x^3-4x^2+3x=0 \Rightarrow x(x-1)(x-3)=0 \Rightarrow x_1=0, x_2=1, x_3=3$$

Sketch the graph

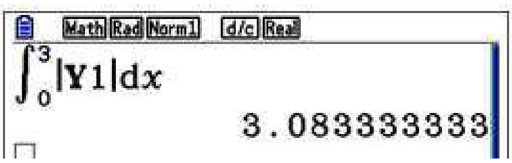


To calculate the area enclosed by the function and the x-axis, we need to take into account the fact that the region between the second and the third zero is below the x-axis and hence we need to take the opposite expression.

$$\text{Area} = \int_0^1 \left(x^3-4x^2+3x\right)dx + \int_1^3 \left(-x^3+4x^2-3x\right)dx$$

$$= \left(\frac{x^4}{4}-\frac{4x^3}{3}+\frac{3x^2}{2}\right)\Bigg]_0^1 + \left(-\frac{x^4}{4}+\frac{4x^3}{3}-\frac{3x^2}{2}\right)\Bigg]_1^3 = \frac{37}{12}$$

Using a GDC, we find the integral of the absolute value.



8. Parabola passes through(17, 37) $\Rightarrow y = ax^2 \Rightarrow 37 = a \times 17^2 \Rightarrow a = \dfrac{37}{17^2} \approx 0.128$

Thus, the parabolic cross section is $y = 0.128x^2 \Rightarrow Area = \int_0^{17} 0.1280x^2 \mathrm{d}x = 209.62$ m^2.

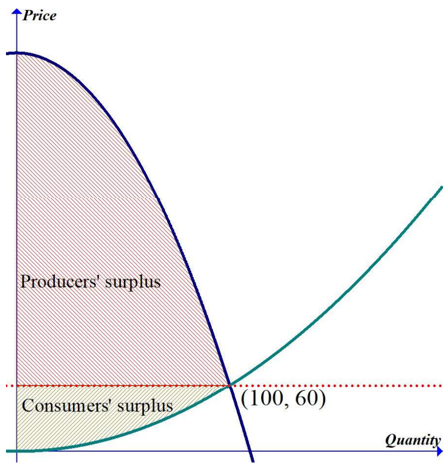
Rectangle area =5 × 37, Triangle area = ½ × 18 × 37

Total area of cross section = 727.62.

Thus volume needed is 727.62 × 175 =127333.5 m^3.

9. Before doing this problem, look at the explanation of market demand, consumers' and producers' surplus in the section.

(a) Market demand here corresponds to the quantity at which there is equilibrium, i.e. point of intersection between demand and supply functions:



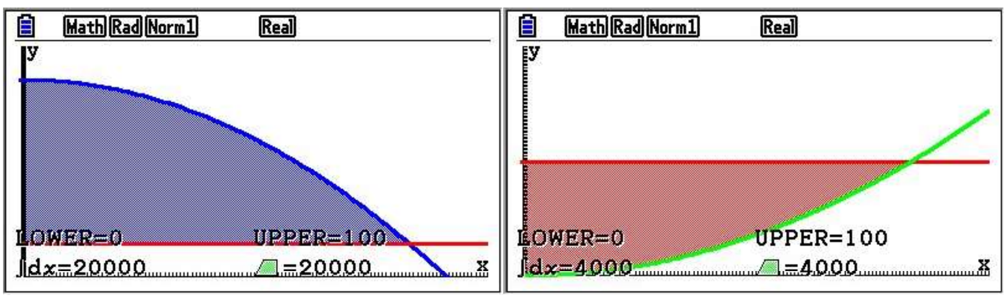
To find the point of intersection, you either equate the two expressions and solve for x:

$$360 - 0.03x^2 = 0.006x^2 \Rightarrow 0.036x^2 = 360 \Rightarrow x = 100, y = 60$$

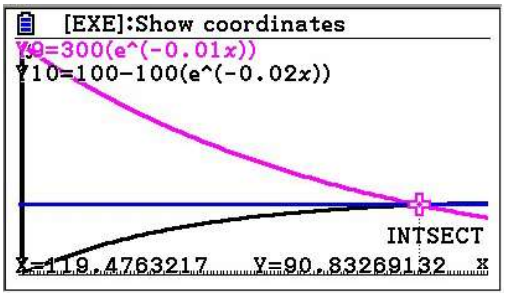
Market demand $x = 100$,

$$\begin{pmatrix}\text{Consumers'}\\ \text{surplus}\end{pmatrix} = \int_0^{100} \left(360 - 0.03x^2 - 60\right) \mathrm{d}x = 20000$$

$$\begin{pmatrix}\text{Producers'}\\ \text{surplus}\end{pmatrix} = \int_0^{100} \left(60 - 0.006x^2\right) \mathrm{d}x = 4000$$



(b) Market demand is at $(119.48, 90.83) \Rightarrow$ market demand $x = 119.48$



$$\begin{pmatrix}\text{Consumers'}\\ \text{surplus}\end{pmatrix} = \int_0^{119.48}\left(300e^{-0.01x} - 90.83\right)\mathrm{d}x \approx 10{,}065$$

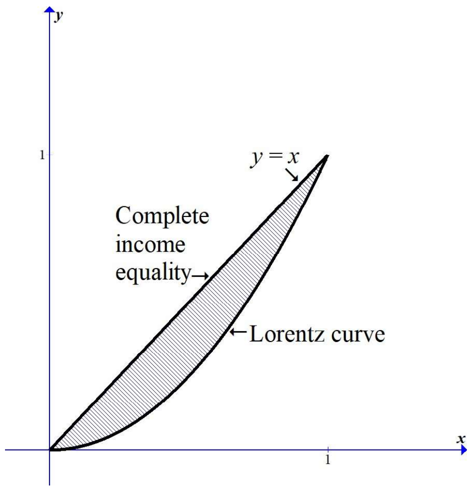
$$\begin{pmatrix}\text{Producers'}\\ \text{surplus}\end{pmatrix} = \int_0^{119.48}\left(90.83 - 100 + 100e^{-0.02x}\right)\mathrm{d}x \approx 3446$$

(c) Also, similarly, market demand is at (97.63, 150.69).

So, market demand is 97.63.

$$\begin{pmatrix}\text{Consumers'}\\ \text{surplus}\end{pmatrix} = \int_0^{97.63}\left(400e^{-0.01x} - 150.69\right)\mathrm{d}x \approx 10{,}220$$

$$\begin{pmatrix}\text{Producers'}\\ \text{surplus}\end{pmatrix} = \int_0^{97.63}\left(150.69 - 0.01x^{2.1}\right)\mathrm{d}x \approx 9966$$



10. This is the coefficient of inequality L discussed in the section and it is equal to

$$L=\frac{\int_0^1 (x-f(x))\,\mathrm{d}x}{\int_0^1 x\,\mathrm{d}x}=2\int_0^1 (x-f(x))\,\mathrm{d}x.$$

$$L=2\int_0^1 (x-f(x))\,\mathrm{d}x=2\int_0^1 x\mathrm{d}x-2\int_0^1 f(x)\mathrm{d}x=1-2\int_0^1 f(x)\mathrm{d}x$$

It is enough to estimate $\int_0^1 f(x)\mathrm{d}x$, multiply by 2 and then subtract from 1.

We will use Trap. method

$$\int_0^1 f(x)\mathrm{d}x\approx\frac{0.2}{2}\left(0+2\left(0.087+0.226+0.405+0.633\right)+1\right)=0.3702$$

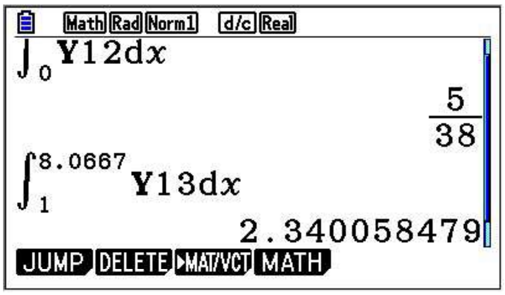
$$L=1-2\times 0.3702=0.2596$$

11. Recall that the mean, $\mu=E(X)=\sum_x xp(x)$, and thus in the continuous case

$$E(X)=\int_x xf(x)\mathrm{d}x.$$

(a) This is a piecewise function, so, the mean life of batteries is

$$\int_0^1 \frac{15x}{76}\left(x^4-2x^2+2\right)\mathrm{d}x+\int_1^{8\frac{1}{15}}-\frac{15x}{8056}(15x-121)\,\mathrm{d}x$$



The mean is $\left(2.34+\frac{5}{38}\right)\times 10\approx 24.7$ hours

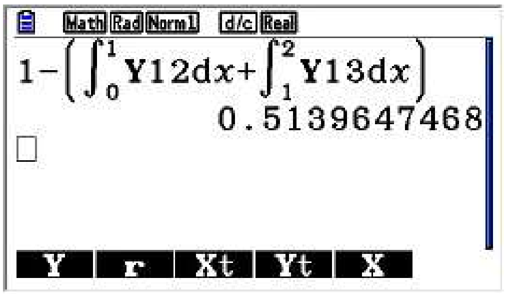
(b) 20 hours correspond to 2 in the given model

$$P\left(X\geq 2\right)=1-P\left(X<2\right)$$

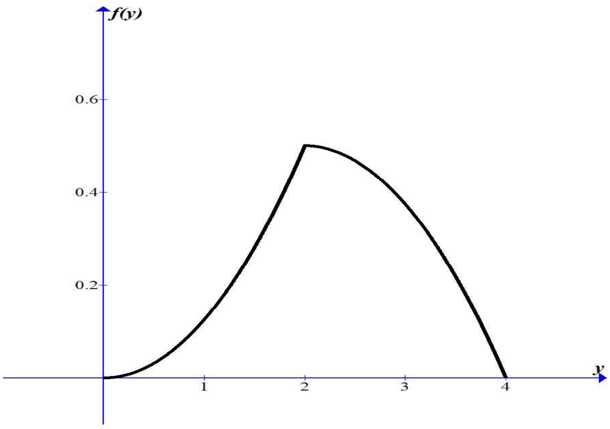
So, we need to find the integral up to 2 and subtract the result from 1.

Probability that a battery lasts at least 20 hours is

$$1-\left(\int_0^1 \frac{15}{76}\left(x^4-2x^2+2\right)\mathrm{d}x+\int_1^{8\frac{1}{15}}-\frac{15}{8056}(15x-121)\,\mathrm{d}x\right)=0.514$$



12. **(a)** Use your GDC



(b) The mean is the expected value.

$$\mathrm{E}(Y)=\int_{y} yf(y)\mathrm{d}y=\int_0^2 \frac{y}{8}y^2\mathrm{d}y+\int_2^4 \frac{y^2}{8}(4-y)\,\mathrm{d}y=\frac{7}{3},$$

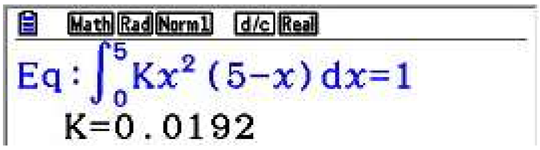
that is $\frac{7}{3}\times 100\approx 233$ barrels

(c) Since $\int_0^2 \frac{1}{8}y^2\mathrm{d}y=\frac{1}{3}>0.1$, then we only need to consider the function on the first interval.

The question can be interpreted as finding the number m such that

$\int_0^m \frac{1}{8}y^2\mathrm{d}y=0.1\Rightarrow m=1.339$, that is, 134 barrels.

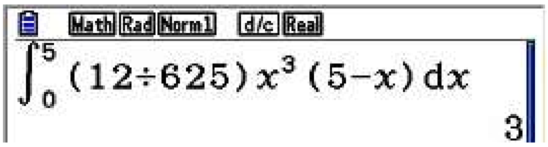
13. **(a)** To find k, we solve the equation: $\int_0^5 ky^2(5-y)\,\mathrm{d}y = 1$, the solution is $0.0192 = \frac{12}{625}$



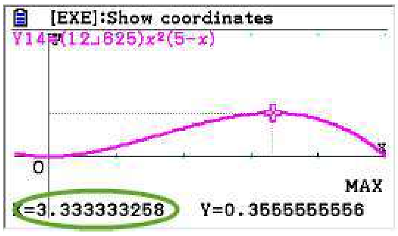
Alternatively, symbolic integration will also give the same result

$$\int_0^5 ky^2(5-y)\,\mathrm{d}y = 1 \Rightarrow k\left[\frac{5x^3}{3} - \frac{x^4}{4}\right]_0^5 = \frac{625}{12}k = 1 \Rightarrow k = \frac{12}{625}$$

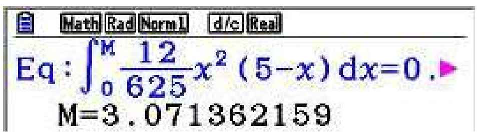
(b) Mean: $\frac{12}{625}\int_0^5 y^3(5-y)\,\mathrm{d}y = 3$,



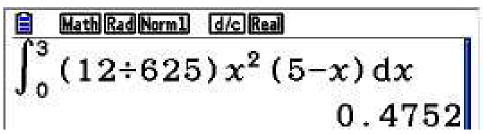
Mode: corresponds to the maximum of the function, mode = 3.3



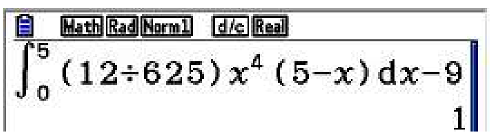
Median: corresponds to the value m such that $\frac{12}{625}\int_0^m y^2(5-y)\,\mathrm{d}y = 0.5$, $m = 3.1$



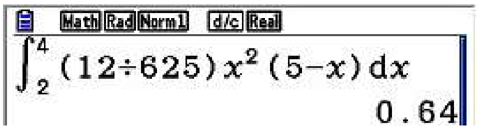
(c) $\frac{12}{625}\int_0^3 y^2(5-y)\,\mathrm{d}y = 0.475$,



(d) Variance $= \frac{12}{625}\int_0^5 y^4(5-y)\,\mathrm{d}y - 3^2 = 1$, so, standard deviation = 1.



(e) We are looking for $P(3-1<X<3+1)=P(2<X<4)=0.64$



14. **(a)** In this model, it is clear that the price is the response variable (dependent), while the quantity, x, is the explanatory variable (independent).

$$\mathrm{d}(x)=\int\frac{-6000}{(3x+50)^2}\mathrm{d}x=\frac{2000}{3x+50}+c$$

With the initial conditions: $8=\dfrac{2000}{3\times150+50}+c\Rightarrow c=4$

Thus, the price-demand equation is $\mathrm{d}(x)=\dfrac{2000}{3x+50}+4.$

(b) We solve the equation $6.50=\dfrac{2000}{3x+50}+4$ for the quantity x.

$x=250$, that is 250 000 bottles.

(c) $s(x)=\displaystyle\int\frac{300}{(3x+25)^2}\mathrm{d}x=\frac{-100}{3x+25}+c$

With the initial conditions:

$5=\dfrac{-100}{3\times75+25}+c\Rightarrow c=5.4\Rightarrow s(x)=\dfrac{-100}{3x+25}+5.4$

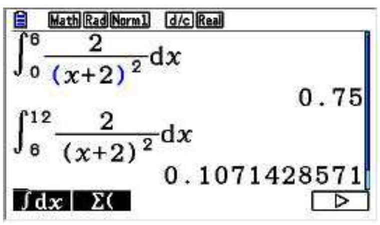
(d) Market equilibrium is where demand and supply are equal

$$\frac{2000}{3x+50}+4=\frac{-100}{3x+25}+5.4\Rightarrow x=484$$

$s(484)=\dfrac{-100}{3\times484+25}+5.4=5.33=d(484)$, that is equilibrium is at 484 000 bottles with a price of £5.33 each.

15. **(a)** $\displaystyle\int_0^6\frac{2}{(t+2)^2}\mathrm{d}t=-\left[\frac{2}{t+2}\right]_0^6=1-\frac{1}{4}=0.75$

(b) $\displaystyle\int_6^{12}\frac{2}{(t+2)^2}\mathrm{d}t=-\left[\frac{2}{t+2}\right]_6^{12}=0.11$



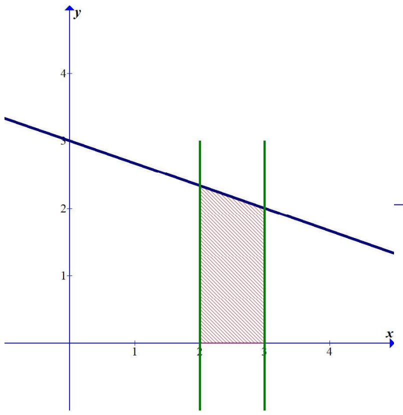
(c) $\int_0^x \frac{2}{(t+2)^2}\,\mathrm{d}t = -\left[\frac{2}{t+2}\right]_0^x = 1-\frac{2}{x+2} \Rightarrow \lim_{x\to\infty}\left(1-\frac{2}{x+2}\right)=1$

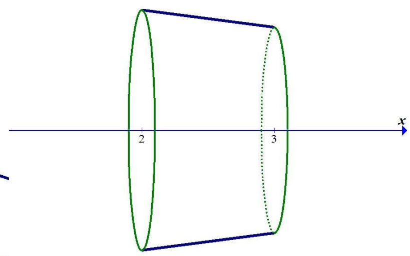
Exercise 16.4

We chose to exclude a few intermediate steps in calculations as the objective is to have you practice the new concepts. The algebraic manipulation steps are assumed to be mastered at this stage.

1. We will sketch the region and a typical disk with the created volume in the first 2 questions. We will leave the drawings for you to complete in the rest of the exercises. These exercises are meant to give you some practice in symbolic integration. However, most of your exam questions will assume that you use a GDC for such tasks.

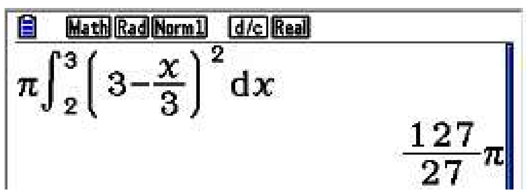
(a) We begin by sketching the lines and shading the region that is rotated about the x-axis.



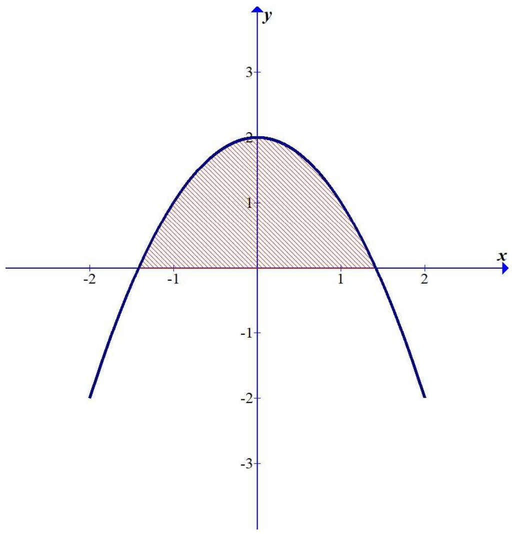


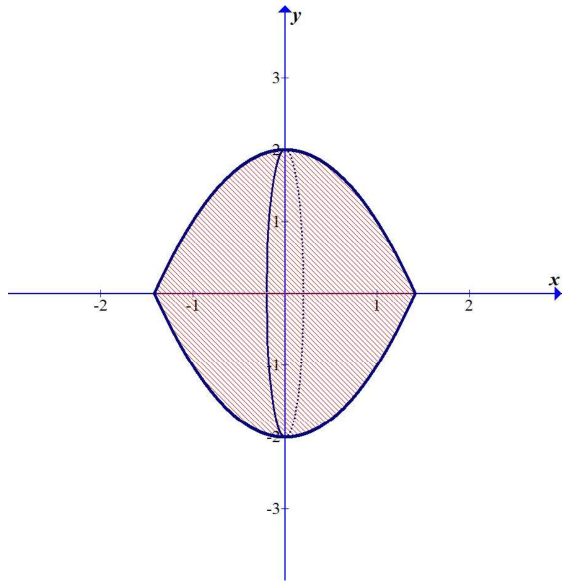
To find the volume, we need to evaluate the following integral:

$$V=\pi\int_2^3\left(3-\frac{x}{3}\right)^2\mathrm{d}x=\pi\int_2^3\left(9-2x+\frac{x^2}{9}\right)\mathrm{d}x=\pi\left[9x-x^2+\frac{x^3}{27}\right]_2^3=\frac{127}{27}\pi$$



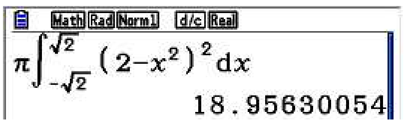
(b) We begin by sketching the parabola and shading the region between the x-axis ($y = 0$) and the parabola that is rotated about the x-axis. By inspection, we find that the parabola intersects the x-axis at the points $\left(-\sqrt{2},0\right)$ and $\left(\sqrt{2},0\right)$.



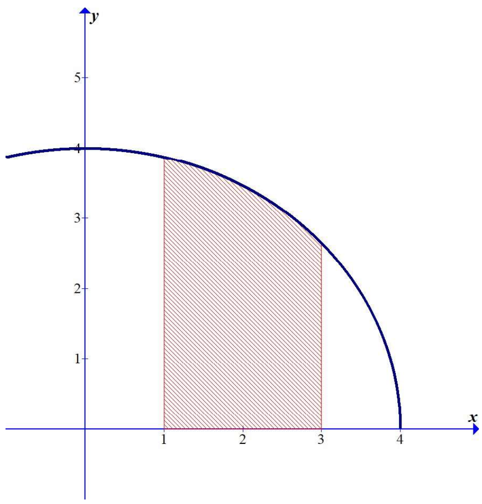


So, to find the volume of the solid, we need to evaluate the following integral:

$$V = \pi\int_{-\sqrt{2}}^{\sqrt{2}}\left(2-x^2\right)^2 \mathrm{d}x = \pi\int_{-\sqrt{2}}^{\sqrt{2}}\left(4-4x^2+x^4\right)\mathrm{d}x = \pi\left[4x-\frac{4}{3}x^3+\frac{x^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} = \frac{64\sqrt{2}}{15}\pi$$



(c) We begin by sketching the curves and shading the region that is rotated about the x-axis.



So, to find the volume of the solid, we need to evaluate the following integral:

$$V = \pi\int_1^3\left(\sqrt{16-x^2}\right)^2 \mathrm{d}x$$

On an exam paper, you only need to write down

$V = \pi\int_1^3\left(\sqrt{16-x^2}\right)^2 \mathrm{d}x = \frac{70}{3}\pi$, or just $V = \pi\int_1^3\left(\sqrt{16-x^2}\right)^2 \mathrm{d}x \approx 23.3\pi \approx 73.3$

(d) volume $= \pi\int_1^3 \frac{9}{x^2}\,dx = \pi\left[-\frac{9}{x}\right]_1^3 = 6\pi$

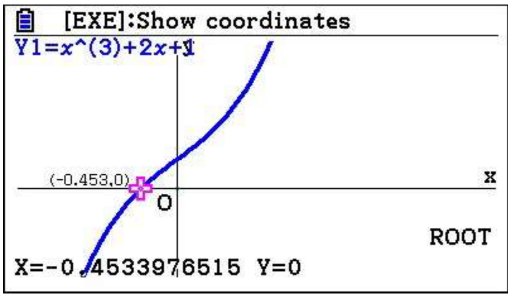
(e) $V = \pi\int_0^3 (3-x)^2\,dx = \pi\int_0^3 (9-6x+x^2)\,dx = \pi\left[9x - 6\times\frac{x^2}{2} + \frac{x^3}{3}\right]_0^3 = 9\pi$

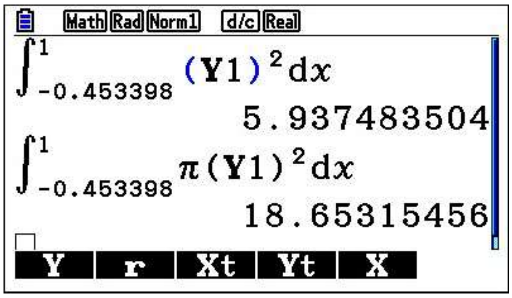
(f) volume $= \pi\int_0^{\pi} \left(\sqrt{\sin x}\right)^2 dx = \pi\int_0^{\pi} \sin x\,dx = \pi\left[-\cos x\right]_0^{\pi} = 2\pi$

(g) $V = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \left(\sqrt{\cos x}\right)^2 dx = \pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{3}} \cos x\,dx = \pi\left[\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{3}} = \pi\left(\frac{\sqrt{3}}{2}+1\right)$

(h) $V = \pi\int_{-2}^{2} (4-x^2)^2\,dx = \pi\int_{-2}^{2} (16-8x^2+x^4)\,dx = \pi\left[16x - 8\times\frac{x^3}{3} + \frac{x^5}{5}\right]_{-2}^{2} = \frac{512\pi}{15}$

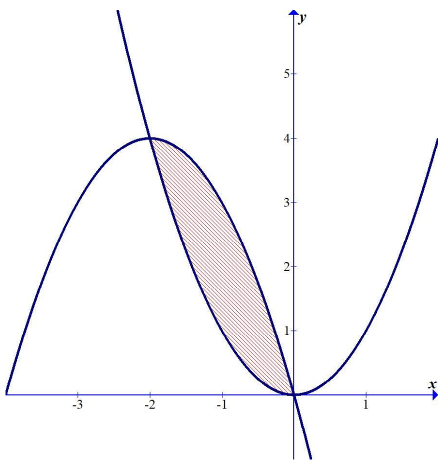
(i) This is apparently a GDC active question as we need to find the point of intersection between the cubic curve and the x-axis.





So, the volume is $\pi\int_{-0.4534}^{1} (x^3+2x+1)^2\,dx \approx 5.94\pi \approx 18.7$ correct to three significant figures.

(j) We sketch the parabolas and shade the region that is rotated about the x-axis. By inspection, we find that the points of intersection of the *curves* are $(-2,4)$ and $(0,0)$.

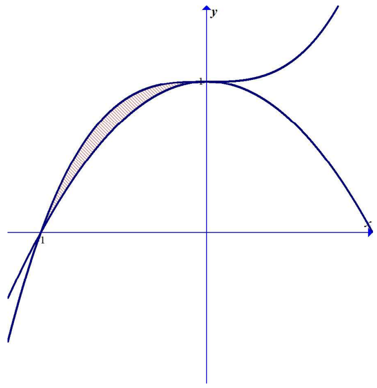


To find the volume, we need to evaluate the integrals of the upper function and the integral of the lower function, and then we subtract them to obtain the eventual answer. (can be done in one)

$$V = V_{upper} - V_{lower} = \pi\int_{-2}^{0}\left(-4x - x^2\right)^2 \mathrm{d}x - \pi\int_{-2}^{0}\left(x^2\right)^2 \mathrm{d}x$$

$$= \pi\int_{-2}^{0}\left(\left(-4x - x^2\right)^2 - \left(x^2\right)^2\right)\mathrm{d}x = \frac{256}{15}\pi - \frac{32}{5}\pi = \frac{32}{3}\pi$$

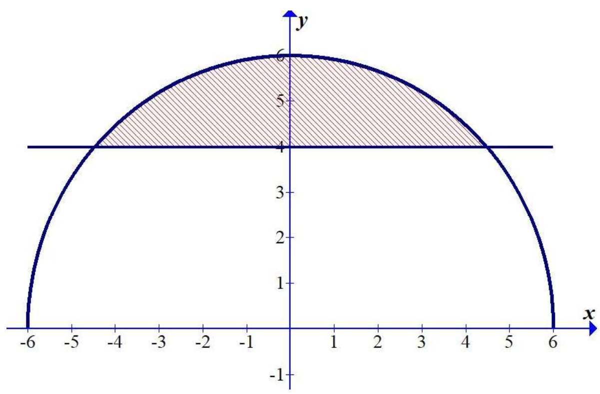
(k) We begin by sketching the curve, shading the region that is rotated about the x-axis.



$$V = \pi\int_{-1}^{0}\left(\left(x^3+1\right)^2 - \left(1-x^2\right)^2\right)\mathrm{d}x = \pi\int_{-1}^{0}\left(x^6 - x^4 + 2x^3 + 2x^2\right)\mathrm{d}x$$

$$= \pi\left(\frac{x^7}{7} - \frac{x^5}{5} + \frac{x^4}{2} + \frac{2x^3}{3}\right)\Bigg]_{-1}^{0} = \frac{23\pi}{210}$$

(l) We begin by sketching the curve, shading the region that is rotated about the x-axis.



Before we find the volume of revolution, we firstly need to find the limits of integration by solving the simultaneous equations.

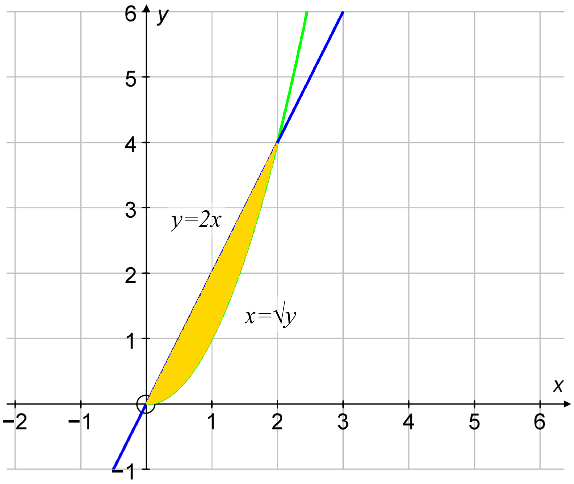
$$\begin{cases} y = \sqrt{36 - x^2} \\ y = 4 \end{cases} \Rightarrow 4 = \sqrt{36 - x^2} \Rightarrow x_1 = -2\sqrt{5},\ x_2 = 2\sqrt{5}$$

Since both functions are even (symmetrical with respect to the y-axis), we can simply calculate the integral from 0 to $2\sqrt{5}$ and then multiply it by 2.

$$V = 2\pi\int_0^{2\sqrt{5}}\left(\left(\sqrt{36-x^2}\right)^2 - 4^2\right)\mathrm{d}x = 2\pi\int_0^{2\sqrt{5}}\left(20-x^2\right)\mathrm{d}x = 2\pi\left[\left(20x-\frac{x^3}{3}\right)\right]_0^{2\sqrt{5}}$$

$$= 2\pi\left(40\sqrt{5}-\frac{40\sqrt{5}}{3}\right) = \frac{160\pi\sqrt{5}}{3}$$

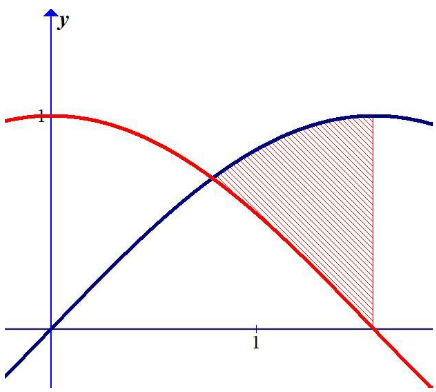
(m) We begin by sketching the curve, shading the region that is rotated about the x-axis.



To ease your calculations, note that $x=\sqrt{y} \Leftrightarrow y = x^2, x \geq 0$. By inspection, we can find the limits of integration.

$$V = \pi\int_0^2\left((2x)^2 - \left(x^2\right)^2\right)\mathrm{d}x = \pi\left[\left(\frac{4x^3}{3}-\frac{x^5}{5}\right)\right]_0^2 = \pi\left(\frac{32}{3}-\frac{32}{5}\right) = \frac{64\pi}{15}$$

(n) Sketch the curves and shade the region that is rotated about the x-axis.

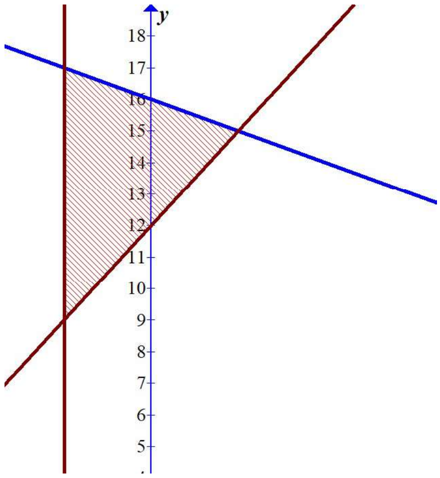


$$V = \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\sin^2 x - \cos^2 x\right)\mathrm{d}x = \pi\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(-\cos 2x\right)\mathrm{d}x = \pi\left[\left(-\frac{1}{2}\sin 2x\right)\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{2}$$

(o) $$V = \pi\int_1^3\left(\left(2x^2+4\right)^2 - x^2\right)\mathrm{d}x = \pi\int_1^3\left(4x^4+15x^2+16\right)\mathrm{d}x = \frac{1778\pi}{5}$$

(p) $V = \pi\int_1^3 \left(\sqrt{x^4+1}\right)^2 \mathrm{d}x = \pi\int_1^3 \left(x^4+1\right)\mathrm{d}x = \pi\left(\frac{x^5}{5}+x\right)\Bigg]_1^3 = \frac{252\pi}{5}$

(q) We begin by sketching the curve, shading the region that is rotated about the x-axis and find the point of intersection by inspection.

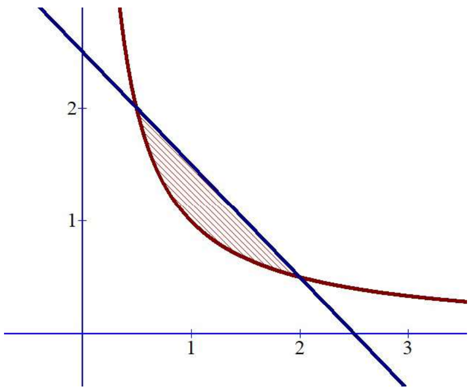


$$V = \pi\int_{-1}^{1}\left((16-x)^2-(3x+12)^2\right)\mathrm{d}x = \pi\int_{-1}^{1}\left(-8x^2-104x+112\right)\mathrm{d}x$$

$$= \pi\left(-\frac{8x^3}{3}-57x^2+112x\right)\Bigg]_{-1}^{1} = \frac{656\pi}{3}$$

(r) We begin by sketching the curve, shading the region that is rotated about the x-axis. We can find the points of intersection by solving the simultaneous equations.

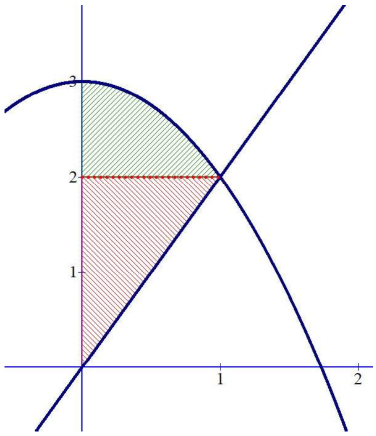
$$\begin{cases} y = \dfrac{1}{x} \\ y = \dfrac{5}{2}-x \end{cases} \Rightarrow \frac{1}{x} = \frac{5}{2}-x \Rightarrow 2x^2-5x+2=0 \Rightarrow x_1 = \frac{1}{2}, x_2 = 2$$



$$V = \pi\int_{\frac{1}{2}}^{2}\left(\left(\frac{5}{2}-x\right)^2-\left(\frac{1}{x}\right)^2\right)\mathrm{d}x = \pi\int_{\frac{1}{2}}^{2}\left(\frac{25}{4}-5x+x^2-\frac{1}{x^2}\right)\mathrm{d}x = \frac{9\pi}{8}$$

2. One method of solving a rotation about the y-axis involves swapping the variables of integration from x to y, which means that we need to express x in terms of y to find the volume of revolution (disk method).

A second method, known as the shell method, is described in the HL textbook. In most cases, the shell method is simpler. In the IB syllabus (this cycle), it is not required. Unless it is specifically requested that you use the "disk" method we recommend using the shell method.



(a) $V=\pi\int_0^1\left(\left(3-x^2\right)^2-\left(2x\right)^2\right)\mathrm{d}x=\pi\int_0^1\left(9-10x^2+x^4\right)\mathrm{d}x=\frac{88\pi}{15}$

(b) If we want to use the disk method, then we need to split the region into two regions and change the limits of integration from 0 to 2 for the first integral and from 2 to 3 for the second integral.

$$y=3-x^2\Rightarrow x=\sqrt{3-y},\quad y=2x\Rightarrow x=\frac{y}{2}$$

$$V=\pi\int_0^2\left(\frac{y^2}{4}\right)\mathrm{d}y+\pi\int_2^3\left(\sqrt{3-y}\right)^2\mathrm{d}y=\pi\left(\frac{y^3}{12}\right)\Bigg]_0^2+\pi\left(3y-\frac{y^2}{2}\right)\Bigg]_2^3$$

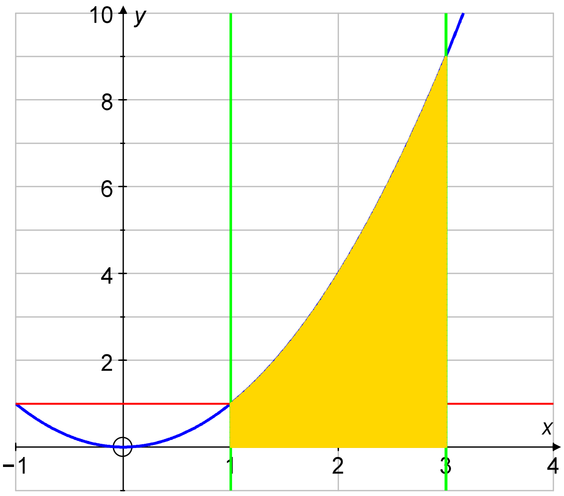
$$=\pi\left(\frac{2}{3}+9-\frac{9}{2}-6+2\right)=\frac{7\pi}{6}$$

The shell method:

$$V=2\pi\int_0^1\left(x\left(3-x^2-2x\right)\right)\mathrm{d}x=2\pi\int_0^1\left(3x-x^3-2x^2\right)\mathrm{d}x=2\pi\left(\frac{3x^2}{2}-\frac{x^4}{4}-\frac{2x^3}{3}\right)\Bigg]_0^1$$

$$=2\pi\left(\frac{3}{2}-\frac{1}{4}-\frac{2}{3}\right)=2\pi\times\frac{18-3-8}{12}=\frac{7\pi}{6}$$

3. **(a)** For demonstration purposes, we will do the calculations here using both methods.



Disk method: We need to express x in terms of y: $y = x^2 \Rightarrow x = \sqrt{y}$

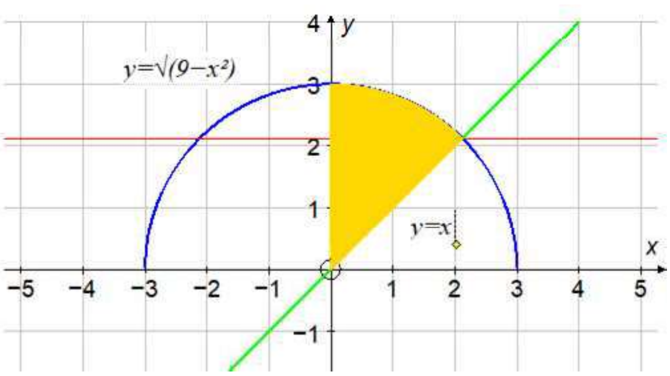
$$V = \pi\int_0^1 \left(3^2 - 1^2\right)\mathrm{d}y + \pi\int_1^9 \left(3^2 - \left(\sqrt{y}\right)^2\right)\mathrm{d}y = \pi\left(\left(8y\right)\Big]_0^1 + \left(9y - \frac{y^2}{2}\right)\Bigg]_1^9\right)$$

$$= \pi\left(8 + 81 - \frac{81}{2} - 9 + \frac{1}{2}\right) = 40\pi$$

The shell method:

$$V = 2\pi\int_1^3 \left(x\left(x^2\right)\right)\mathrm{d}x = 2\pi\int_1^3 x^3\mathrm{d}x = 2\pi\left(\frac{x^4}{4}\right)\Bigg]_1^3 = 2\pi\left(\frac{81}{4} - \frac{1}{4}\right) = 2\pi \times 20 = 40\pi$$

(b) **Disk method**



Note that these two curves intersect at the point with equal x- and y-coordinate:

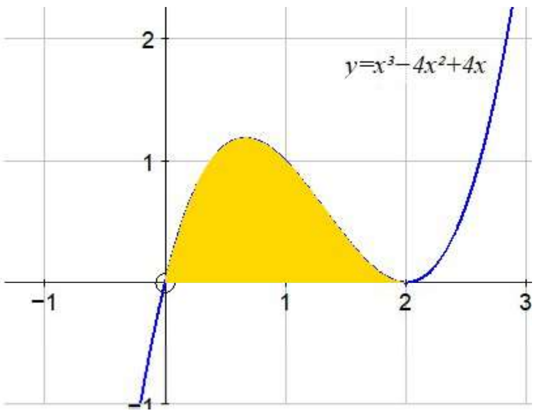
$\sqrt{9-x^2} = x \Rightarrow 9 - x^2 = x^2 \Rightarrow 9 = 2x^2 \Rightarrow x = \sqrt{\frac{9}{2}} = \frac{3\sqrt{2}}{2}$, and that x and y are symmetrical in the equation of the curve; therefore, $x = \sqrt{9-y^2}$.

We need to split the integration into two parts.

$$V = \pi\int_0^{\frac{3\sqrt{2}}{2}} y^2\,\mathrm{d}y + \pi\int_{\frac{3\sqrt{2}}{2}}^{3}\left(\sqrt{9-y^2}\right)^2\mathrm{d}y = \pi\left(\left[\left(\frac{y^3}{3}\right)\right]_0^{\frac{3\sqrt{2}}{2}} + \left[\left(9y - \frac{y^3}{3}\right)\right]_{\frac{3\sqrt{2}}{2}}^{3}\right)$$

$$= \pi\left(\frac{9\sqrt{2}}{4} + 27 - 9 - \frac{27\sqrt{2}}{2} + \frac{9\sqrt{2}}{4}\right) = 9\pi\left(2-\sqrt{2}\right)$$

(c) This question cannot be done by swapping the variables, so we will use the shell method.



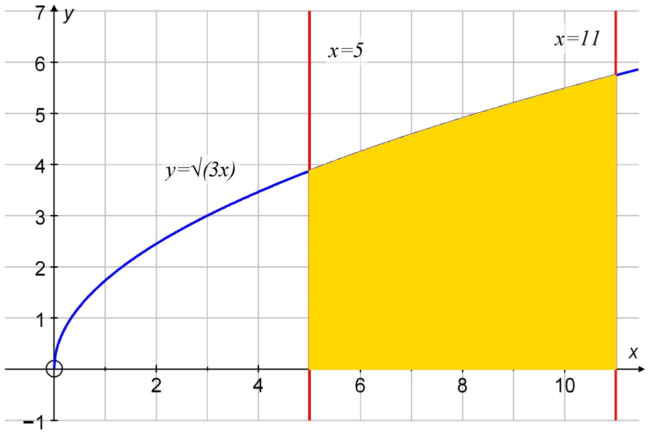
$$V = 2\pi\int_0^2\left(x\times\left(x^3-4x^2+4x\right)\right)\mathrm{d}x$$

$$= 2\pi\int_0^2\left(x^4-4x^3+4x^2\right)\mathrm{d}x$$

$$= 2\pi\left(\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right)\Bigg]_0^2$$

$$= 2\pi\left(\frac{32}{5} - 16 + \frac{32}{3}\right) = \frac{32\pi}{15}$$

(d) Shell method may be simpler.

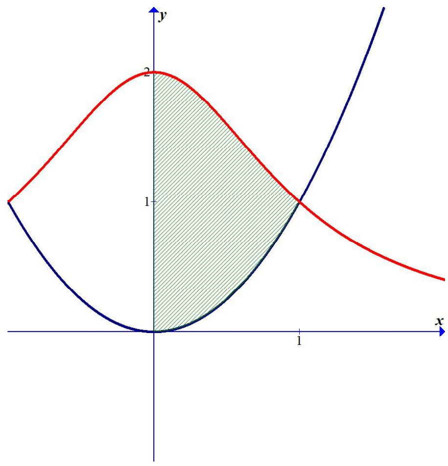


The shell method:

$$V = 2\pi\int_5^{11}\left(x\sqrt{3x}\right)\mathrm{d}x = 2\pi\sqrt{3}\int_5^{11} x^{\frac{3}{2}}\,\mathrm{d}x = 2\pi\sqrt{3}\left[\left(\frac{2}{5}x^{\frac{5}{2}}\right)\right]_5^{11} = \frac{484\pi\sqrt{33}}{5} - 20\pi\sqrt{15}$$

(e) Given the intensive work with expressing x in terms of y, we will use the shell method instead.

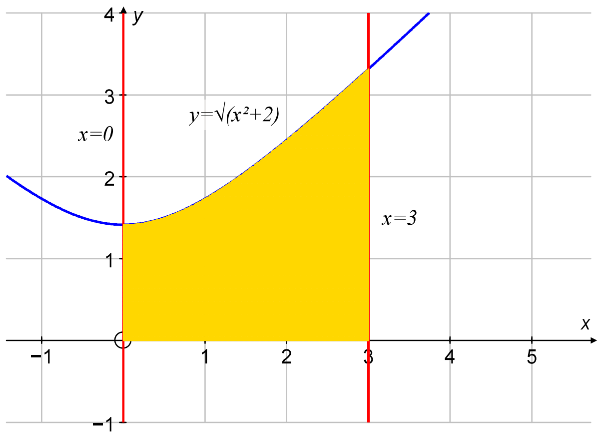
It is enough to rotate the region in the first quadrant to create the solid.



$$V = 2\pi\int_0^1\left(x\left(\frac{2}{1+x^2} - x^2\right)\right)\mathrm{d}x = 2\pi\int_0^1\left(\frac{2x}{1+x^2} - x^3\right)\mathrm{d}x = 2\pi\left[\ln\left(1+x^2\right) - \frac{x^4}{4}\right]_0^1$$

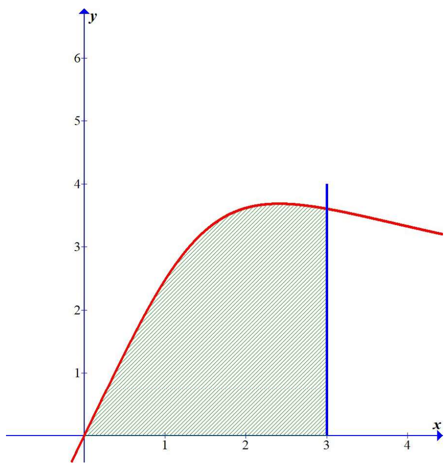
$$= 2\pi\left(\ln 2 - \frac{1}{4}\right) = 2\pi\ln 2 - \frac{\pi}{2}$$

(f) This one lends itself to the shell method too.



$$V = 2\pi\int_0^3\left(x\sqrt{x^2+2}\right)\mathrm{d}x = 2\pi\int_2^{11}\sqrt{t}\times\frac{1}{2}\mathrm{d}t = \pi\left[\left(\frac{2}{3}t^{\frac{3}{2}}\right)\right]_2^{11} = \frac{2\pi}{3}\left(11\sqrt{11}-2\sqrt{2}\right)$$

(g) Again, this question cannot be done by swapping the variables, so we will use the shell method.



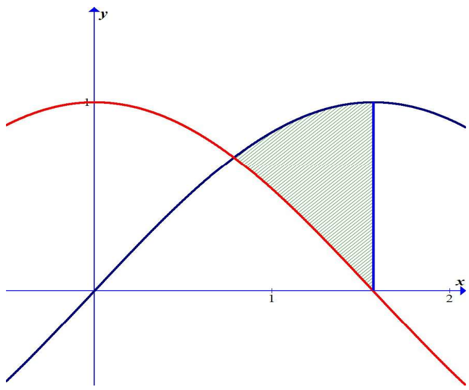
$$V = 2\pi\int_0^3\left(x\times\frac{7x}{\sqrt{x^3+7}}\right)\mathrm{d}x = 2\pi\int_0^3\left(\frac{7x^2}{\sqrt{x^3+7}}\right)\mathrm{d}x = 14\pi\int_7^{34}\frac{\frac{1}{3}\mathrm{d}t}{\sqrt{t}} = \frac{14\pi}{3}\left(2\sqrt{t}\right)\Big]_7^{34}$$

$$= \frac{28\pi}{3}\left(\sqrt{34}-\sqrt{7}\right)$$

We used the substitution $t = x^3 + 7$ and $\mathrm{d}t = 3x^2\mathrm{d}x$ for the integral.

(h) Using the disk method will involve expression containing powers of inverse trigonometric functions which will require lengthy calculations. We will use the shell method instead.

These two curves meet at the point $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$.



$$V = 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (x\sin x - x\cos x)\,dx$$

This requires finding the following integrals using integration by parts

$\int x\sin x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + c$, and

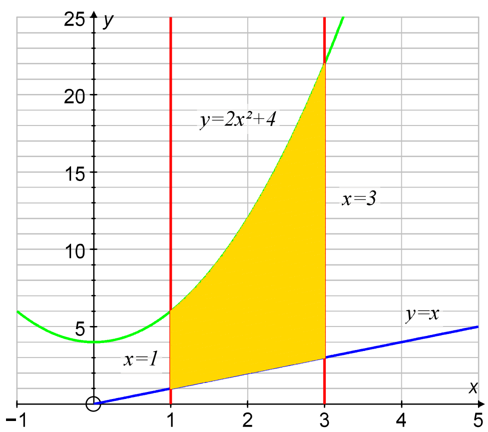
$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + c$

Now,

$$V = 2\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (x\sin x - x\cos x)\,dx = 2\pi \left(-x\cos x + \sin x - x\sin x - \cos x\right)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= 2\pi\left(1 - \frac{\pi}{2} + \frac{\pi\sqrt{2}}{4}\right) = \pi\left(2 - \pi + \frac{\pi\sqrt{2}}{2}\right)$$

(i) Using the disk method first. Points of intersection can be found by inspection and left for you to verify.



$$V = \pi\left(\int_1^3 \left(y^2 - 1\right)\mathrm{d}y + \int_3^6 \left(3^2 - 1^2\right)\mathrm{d}y + \int_6^{22}\left(9 - \left(\sqrt{\frac{y}{2}} - 2\right)^2\right)\right)\mathrm{d}y$$

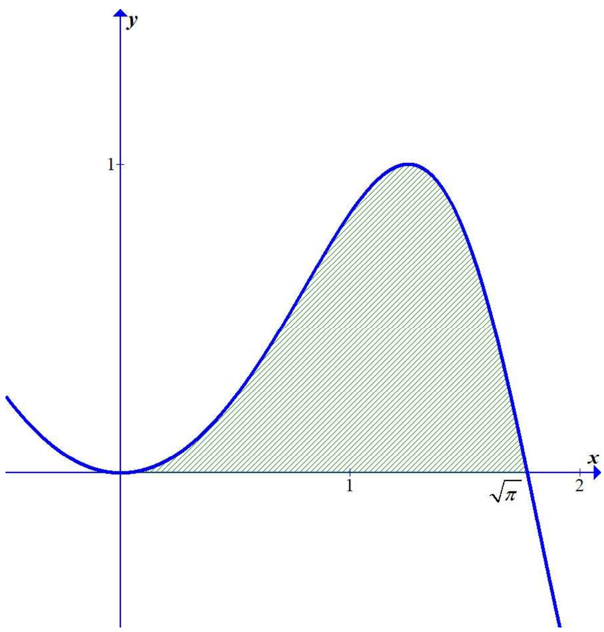
$$= \pi\left(\left[\left(\frac{y^3}{3} - y\right)\right]_1^3 + \left(8y\right)\Big]_3^6 + \left[\left(11y - \frac{y^2}{4}\right)\right]_6^{22}\right) = \frac{284\pi}{3}$$

The shell method:

$$V = 2\pi\int_1^3 \left(x\left(2x^2 + 4 - x\right)\right)\mathrm{d}x = 2\pi\int_1^3 \left(2x^3 + 4x - x^2\right)\mathrm{d}x = 2\pi\left[\left(\frac{x^4}{2} + 2x^2 - \frac{x^3}{3}\right)\right]_1^3$$

$$= 2\pi\left(\frac{81}{2} + 18 - 9 - \frac{1}{2} - 2 + \frac{1}{3}\right) = \frac{284\pi}{3}$$

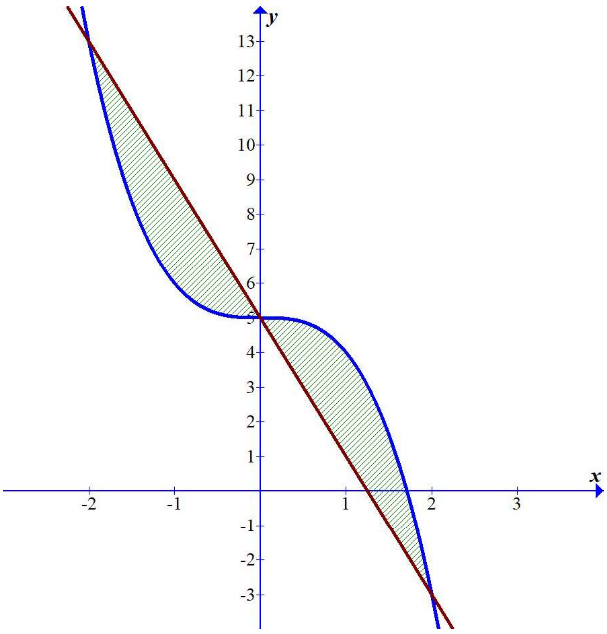
(j) This question cannot be done by swapping the variables, so we will use the shell method.



$$V = 2\pi\int_0^{\sqrt{\pi}} \left(x\sin\left(x^2\right)\right)\mathrm{d}x = 2\pi\int_0^{\pi} \sin t \times \frac{1}{2}\mathrm{d}t = \pi\left(-\cos t\right)\Big]_0^{\pi} = 2\pi$$

(k) The shell method is more straight forward.

Due to symmetry of the graphs, it will be enough to evaluate the integral over the interval [0, 2] and multiply by 2.



$$V = 4\pi\int_0^2 \left(x\left(5 - x^3 - 5 + 4x\right)\right)\mathrm{d}x = 4\pi\int_0^2 \left(4x^2 - x^4\right)\mathrm{d}x = 4\pi\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_0^2 = \frac{256\pi}{15}$$

4. We will use the shell method here as it will not require that we swap variables and deal with two functions.

$V = 2\pi\int_2^{10} xf(x)\mathrm{d}x$; we use Trap. Method to estimate the integral

$$\Delta x = \frac{10-2}{8} = 1;$$

$$\int_2^{10} xf(x)\mathrm{d}x \approx \frac{1}{2}\left(2\cdot f(2) + 2\left(3\cdot f(3) + 4\cdot f(4) + \cdots + 9\cdot f(9)\right) + 10\cdot f(10)\right)$$

$$= \frac{1}{2}\left(2\cdot 0 + 2\left(3\cdot 1.5 + 4\cdot 2 + \cdots + 9\cdot 3\right) + 10\cdot 0\right) \approx 127$$

Volume $\approx 2\pi \times 127 \approx 798$ cubic units.

5. **(a)** As the diagram shows, the cross section of the dam is made up of three parts. The parabolic part generated by a parabola whose vertex is at $(-95, 0)$ and passes through the point in the upper corner with coordinates $(-68, 40)$. The volume generated by this part, and the other parts is half of the full rotation.

The equation of the parabola is of the form $y = a(x+k)^2$ because its vertex is not at the origin, but 95 units to the left. So, the equation is $y = a(x+95)^2$ and since it passes through $(-68, 40)$, then $a = 0.0549$. Thus, the equation is $y = 0.0549(x+95)^2$ and the volume generated by this part is

$$V_1 = \frac{1}{2}\left|2\pi\int_{-95}^{-68} 0.0549x(x+95)^2\right| \mathrm{d}x$$

The middle part is generated by $y = 40$ and hence

$$V_2 = \frac{1}{2}\left|2\pi\int_{-68}^{-59} 40x\right| \mathrm{d}x$$

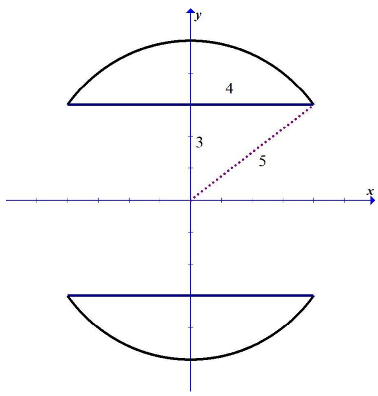
The third part is generated by the segment between $(-35, 0)$ and $(-59, 40)$ and whose equation is $y = -\frac{5}{3}(x+35)$ and hence

$$V_3 = \frac{1}{2}\left|2\pi\int_{-59}^{-35} -\frac{5x}{3}(x+35)\right| \mathrm{d}x$$

Volume of concrete required is $V_1 + V_2 + V_3 \approx 233310\,\mathrm{m}^3$.

(b) Number of trucks $= \dfrac{233310}{8.5} \approx 27449$ trucks

6. Due to symmetry of the sphere we can consider it to be rotated around the x-axis. The cross section of the sphere is a circle whose equation is $x^2 + y^2 = 25 \Rightarrow y = \sqrt{25 - x^2}$ is the upper part. If this part is rotated around the x-axis, it will generate the whole sphere. However, when we bore the whole inside the sphere, the part of the sphere that remains is restricted between –4 and 4. The hole is generated by rotating $y = 3$ around the x- axis.



The volume remaining for the ring is

$$V = \pi\int_{-4}^{4}\left(\left(\sqrt{25-x^2}\right)^2 - 3^2\right)\mathrm{d}x = \frac{256\pi}{3} \approx 268.08$$

7. We take the disk approach for the paraboloid.

$$V = \pi\int_0^h x^2\mathrm{d}y = \pi\int_0^h 2py\,\mathrm{d}y = \pi py^2\Big|_0^h = \pi ph^2$$

r is the radius of the cylinder and is on the parabola $\Rightarrow r^2 = 2ph$ and volume of cylinder is

$$\pi r^2 h = \pi\cdot 2ph\cdot h = 2\pi ph^2$$

Therefore, the volume of the paraboloid is one-half that of the circumscribed cylinder.

8. In general, the volume would be $V = \int_0^{18} A(x)\mathrm{d}x$ where $A(x)$ is the area of a cross section. Since the function is given by a table and not a formula, we will use the trapezoidal rule with 6 intervals to estimate the volume:

$$\Delta x = \frac{18}{6} = 3,$$

$$\Rightarrow V = \frac{3}{2}\left(141 + 2\left(82+35+25+14+5\right)+0\right) \approx 694.5 \text{ m}^3$$

Cost is therefore $694.5\times 9.80 = €6806.1$

9. We consider rotation around the y-axis, and use disk method expressing x in terms of y.

The point of intersection of the cylinder with the paraboloid has x-coordinate of 3, thus,

$$y = 25 - 3^2 = 16$$

Remaining volume is $V = \pi\int_0^{10}\left(x^2 - 3^2\right)\mathrm{d}y = \pi\int_0^{10}\left(25 - y - 9\right)\mathrm{d}y = 128\pi \approx 402 \text{ cm}^3$.

If we had used cylindrical shells instead, then

$$V = 2\pi\int_3^5 x\left(25 - x^2\right)\mathrm{d}x = 128\pi$$

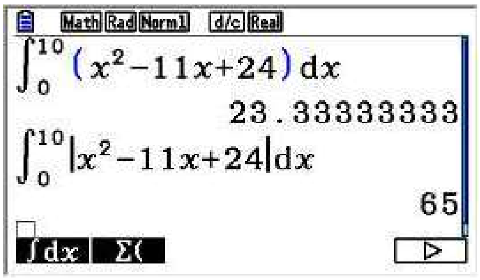
Exercise 16.5

1. **(a)** To find the displacement (net distance), we evaluate the following integral:

$$\text{Displacement} = \int_0^{10}\left(t^2 - 11t + 24\right)\mathrm{d}t = \left[\frac{t^3}{3} - 11\times\frac{t^2}{2} + 24t\right]_0^{10} = \frac{1000}{3} - 550 + 240 = \frac{70}{3}\ \text{m}$$

To find the total distance travelled, we find $\int_0^{10}\left|t^2 - 11t + 24\right|\mathrm{d}t$.

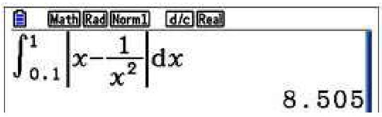
We can evaluate both using GDC



(b) To find the net distance, we evaluate the following integral:

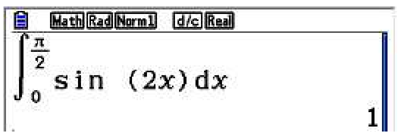
$$\int_{0.1}^{1}\left(t - \frac{1}{t^2}\right)\mathrm{d}t = \left[\frac{t^2}{2} + \frac{1}{t}\right]_{0.1}^{1} = 1.5 - 10.005 = -8.505\text{, i.e., 8.5 m left.}$$

To find the total distance travelled, we find $\int_0^{10}\left|t - \frac{1}{t^2}\right|\mathrm{d}t = 8.5\ \text{m}$



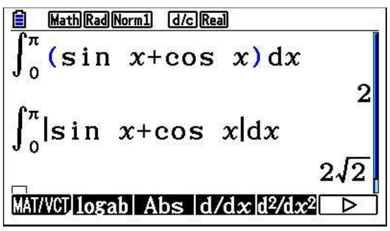
(c) We note that the whole graph is above the x-axis within the given interval, and therefore the displacement and the total distance travelled are the same, i.e.

$$\int_0^{\frac{\pi}{2}}\sin(2t)\,\mathrm{d}t = -\frac{1}{2}\cos(2t)\Big|_0^{\frac{\pi}{2}} = \frac{1}{2} + \frac{1}{2} = 1\ \text{m}.$$

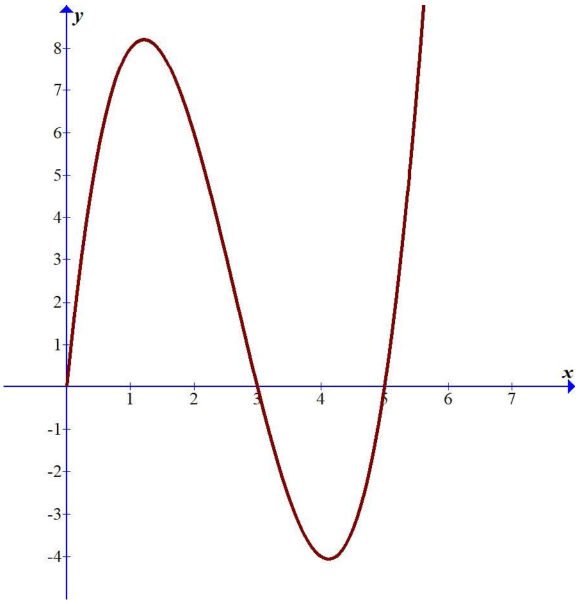


(d) Net distance $= \int_0^{\pi}(\sin t + \cos t)\,\mathrm{d}t = -\cos t + \sin t\Big|_0^{\pi} = -(-1) + 1 = 2\ \text{m}$

Net and total distance – with a GDC

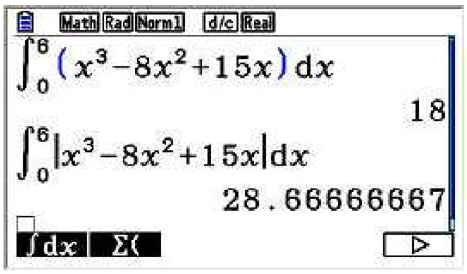


(e) By looking at the curve $v(t) = t^3 - 8t^2 + 15t = t(t-3)(t-5)$, note that within the interval]0, 6[there are two zeroes, 3 and 5; therefore, the velocity changes its direction twice. To find the (net distance) displacement, we evaluate the integral within that interval; but to find the total distance travelled, we take the integral of the absolute value since the curve is below the x-axis in the interval]3, 5[.



$$\text{Displacement} = \int_0^6 \left(t^3 - 8t^2 + 15t\right) dt = \left[\frac{t^4}{4} - 8 \times \frac{t^3}{3} + 15 \times \frac{t^2}{2}\right]_0^6 = 18 \text{ m}$$

$$\text{Total distance} = \int_0^6 \left|t^3 - 8t^2 + 15t\right| dt = \frac{86}{3} \approx 28.7 \text{ m}$$



(f) Note that the function is always positive on the given interval and therefore net distance and total distance travelled are going to be the same.

$$\int_0^1 \left(\sin\left(\frac{\pi t}{2}\right) + \cos\left(\frac{\pi t}{2}\right)\right) dt = \left[\frac{2}{\pi}\left(-\cos\left(\frac{\pi t}{2}\right)\right) + \frac{2}{\pi}\sin\left(\frac{\pi t}{2}\right)\right]_0^1$$

$$= \frac{2}{\pi}(0 + 1 + 1 - 0) = \frac{4}{\pi} \text{ m}$$

2. **(a)** $v(t)=\int a(t)\,\mathrm{d}t \Rightarrow v(t)=\int 3\,\mathrm{d}t = 3t+c, c\in\mathbb{R}$.

To find the value of the constant, we use the initial velocity:
$v(0)=0 \Rightarrow 3\times 0 + c = 0 \Rightarrow c = 0$. So, the velocity function is: $v(t)=3t$.

Since there is no change in the direction, net and total distance are the same

$$\text{Distance} = \int_0^2 3t\,\mathrm{d}t = \left[3\frac{t^2}{2}\right]_0^2 = \left(3\times\frac{2^2}{2}\right) - 0 = 6\text{ m}$$

(b) $v(t)=\int a(t)\,\mathrm{d}t \Rightarrow v(t)=\int (2t-4)\,\mathrm{d}t = 2\frac{t^2}{2}-4t+c = t^2-4t+c, c\in\mathbb{R}$.

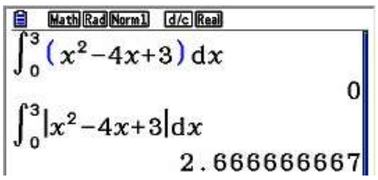
Also, $v(0)=3 \Rightarrow 0^2-4\times 0+c=3 \Rightarrow c=3$.

So, the velocity function is: $v(t)=t^2-4t+3$.

$$\text{net distance} = \int_0^3 (t^2-4t+3)\,\mathrm{d}t = \left[\frac{t^3}{3}-4\times\frac{t^2}{2}+3t\right]_0^3 = (9-18+9)-0=0$$

Note that the zeros of the velocity parabola are 1 and 3; therefore, there is a change in the direction at 1.

$$\text{Total distance travelled} = \int_0^3 \left|t^2-4t+3\right|\mathrm{d}t = \frac{8}{3} \approx 2.67\text{ m}$$



(c) $v(t)=\int a(t)\,\mathrm{d}t \Rightarrow v(t)=\int \sin t\,\mathrm{d}t = -\cos t + c$.

Also, $v(0)=0 \Rightarrow -\cos(0)+c=0 \Rightarrow -1+c=0 \Rightarrow c=1$.

So, the velocity function is: $v(t)=1-\cos t$.

The cosine value is always between –1 and 1 so we see that the velocity function is never negative. There is no change in direction and therefore the displacement and the total distance travelled are the same.

$$\int_0^{\frac{3\pi}{2}} (1-\cos t)\,\mathrm{d}t = [t-\sin t]_0^{\frac{3\pi}{2}} = \left(\frac{3\pi}{2}-\sin\left(\frac{3\pi}{2}\right)\right)-0 = \frac{3\pi}{2}+1 \approx 5.71\text{ m}$$

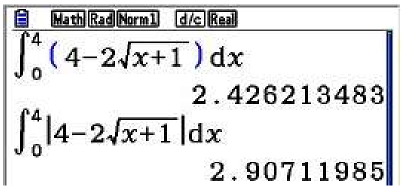
(d) $v(t)=\int a(t)\,\mathrm{d}t \Rightarrow v(t)=\int \frac{-1}{\sqrt{t+1}}\,\mathrm{d}t=-2\sqrt{t+1}+c, c\in\mathbb{R}$.

Also, $v(0)=2\Rightarrow -2\sqrt{0+1}+c=2\Rightarrow -2+c=2\Rightarrow c=4$.

So, the velocity function is: $v(t)=-2\sqrt{t+1}+4$.

$$\text{Displacement}=\int_0^4\left(4-2\sqrt{t+1}\right)\mathrm{d}t=\left[4t-\frac{4}{3}(t+1)^{\frac{3}{2}}\right]_0^4=\frac{52-20\sqrt{5}}{3}\approx 2.43\text{ m}$$

$$\begin{aligned}\text{Total distance travelled}&=\int_0^4\left|4-2\sqrt{t+1}\right|\mathrm{d}t\\&=\int_0^3\left(4-2\sqrt{t+1}\right)\mathrm{d}t+\int_3^4\left(2\sqrt{t+1}-4\right)\mathrm{d}t\\&\approx 2.91\text{ m}\end{aligned}$$



(e) $v(t)=\int a(t)\,\mathrm{d}t \Rightarrow v(t)=\int\left(6t-\frac{1}{(t+1)^3}\right)\mathrm{d}t=3t^2+\frac{1}{2(t+1)^2}+c, c\in\mathbb{R}$.

Also, $v(0)=2\Rightarrow 0+\frac{1}{2}+c=2\Rightarrow c=\frac{3}{2}$. So, the velocity function is:

$v(t)=3t^2+\frac{1}{2(t+1)^2}+\frac{3}{2}$.

The velocity function consists of a sum of positive terms and as such the velocity function is never negative. There is no change in direction and therefore net and the total distance travelled are the same.

$$\text{Distance}=\int_0^2\left(3t^2+\frac{1}{2(t+1)^2}+\frac{3}{2}\right)\mathrm{d}t=\left[t^3-\frac{1}{2(t+1)}+\frac{3t}{2}\right]_0^2=\frac{34}{3}\approx 11.3\text{ m}$$

3. **(a)** $v=9.8t+5\Rightarrow s(t)=\int(9.8t+5)\,\mathrm{d}t=9.8\times\frac{t^2}{2}+5t+c=4.9t^2+5t+c, c\in\mathbb{R}$

$s(0)=10\Rightarrow 10=c$. The position of the object at time t is given by:
$s(t)=4.9t^2+5t+10$

(b) $v=32t-2\Rightarrow s(t)=\int(32t-2)\,\mathrm{d}t=16t^2-2t+c, c\in\mathbb{R}$

$s(0.5)=4\Rightarrow 4=16\times 0.25-2\times 0.5+c\Rightarrow c=1$

The position of the object at time t is given by: $s(t)=16t^2-2t+1$

(c) $v=\sin(\pi t)\Rightarrow s(t)=\int\sin(\pi t)\,\mathrm{d}t=-\frac{1}{\pi}\cos(\pi t)+c, c\in\mathbb{R}$

$s(0)=0\Rightarrow 0=-\frac{1}{\pi}\underbrace{\cos 0}_{1}+c\Rightarrow c=\frac{1}{\pi}$

The position of the object at time t is given by: $s(t)=-\frac{1}{\pi}\cos(\pi t)+\frac{1}{\pi}$

(d) $v=\frac{1}{t+2}, t>-2,\quad s(t)=\int\frac{\mathrm{d}t}{t+2}=\ln(t+2)+c, t>-2$

$s(-1)=\frac{1}{2}\Rightarrow\frac{1}{2}=\ln(1)+c\Rightarrow c=\frac{1}{2}$

The position of the object at time t is given by: $s(t)=\ln(t+2)+\frac{1}{2}$

4. **(a)** $a=e^t\Rightarrow v(t)=\int e^t\,\mathrm{d}t=e^t+c, c\in\mathbb{R}\quad v(0)=20\Rightarrow 20=e^0+c\Rightarrow c=19$

$v(t)=e^t+19\Rightarrow s(t)=\int(e^t+19)\,\mathrm{d}t=e^t+19t+k, k\in\mathbb{R}\quad s(0)=5\Rightarrow k=4$

So, the position of the object at time t is given by: $s(t)=e^t+19t+4$.

(b) $a=9.8\Rightarrow v(t)=\int 9.8\,\mathrm{d}t=9.8t+c,\quad v(0)=-3\Rightarrow -3=0+c\Rightarrow c=-3$

$v(t)=9.8t-3\Rightarrow s(t)=\int(9.8t-3)\,\mathrm{d}t=9.8\times\frac{t^2}{2}-3t+k, s(0)=0\Rightarrow k=0$

So, the position of the object at time t is given by: $s(t)=4.9t^2-3t$

(c) $a=-4\sin 2t\Rightarrow v(t)=\int -4\sin 2t\,\mathrm{d}t=-4\left(-\frac{1}{2}\right)\cos 2t+c, v(0)=2\Rightarrow c=0$

$v(t)=2\cos 2t\Rightarrow s(t)=\int 2\cos 2t\,\mathrm{d}t=2\times\frac{1}{2}\sin 2t+k, s(0)=-3\Rightarrow c=-3$

So, the position of the object at time t is given by: $s(t)=\sin(2t)-3$.

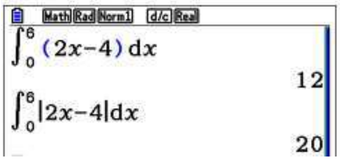
(d) $a=\frac{9}{\pi^2}\cos\left(\frac{3t}{\pi}\right)\Rightarrow v(t)=\int\frac{9}{\pi^2}\cos\left(\frac{3t}{\pi}\right)\mathrm{d}t=\frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right)+c, v(0)=0\Rightarrow c=0$

$v(t)=\frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right)\Rightarrow s(t)=\int\frac{3}{\pi}\sin\left(\frac{3t}{\pi}\right)\mathrm{d}t=-\cos\left(\frac{3t}{\pi}\right)+k, s(0)=-1\Rightarrow k=0$

So, the position of the object at time t is given by: $s(t)=-\cos\left(\frac{3t}{\pi}\right)$.

In questions 5–8, we denote displacement of a point by s and total distance travelled by d.

5. **(a)** $v(t) = 2t - 4 \Rightarrow s = \int_0^6 (2t-4)\,\mathrm{d}t,\ d = \int_0^6 |2t-4|\,\mathrm{d}t$

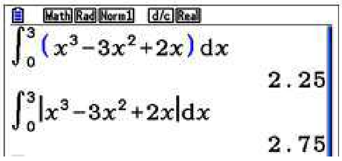


The displacement is 12 m and the total distance travelled is 20 m.

(b) $v(t) = |t-3| \Rightarrow s = \mathrm{d} = \int_0^5 |t-3|\,\mathrm{d}t$

In this case, both the displacement and the total distance travelled are 6.5 m since the function is always positive.

(c) $v(t) = t^3 - 3t^2 + 2t \Rightarrow s = \int_0^3 (t^3 - 3t^2 + 2t)\,\mathrm{d}t,\ d = \int_0^3 |t^3 - 3t^2 + 2t|\,\mathrm{d}t$



So, the displacement is 2.25 m and the total distance travelled is 2.75 m.

(d) $v(t) = \sqrt{t} - 2 \Rightarrow s = \int_0^3 (\sqrt{t} - 2)\,\mathrm{d}t = \left.\frac{2t^{3/2}}{3} - 2t\right|_0^3 = 2\sqrt{3} - 6 \approx -2.54,$

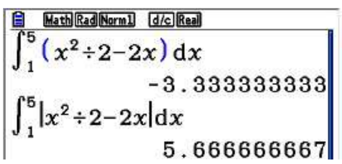
$$d = \int_0^3 |\sqrt{t} - 2|\,\mathrm{d}t = \int_0^3 (2 - \sqrt{t})\,\mathrm{d}t = \left. 2t - \frac{2t^{3/2}}{3}\right|_0^3 = 6 - 2\sqrt{3} \approx 2.54,$$

Since the function is always negative on the given interval, the displacement is –2.54 m, whereas the total distance travelled is 2.54 m.

6. **(a)** $a(t) = t - 2 \Rightarrow v(t) = \int (t-2)\,\mathrm{d}t = \frac{t^2}{2} - 2t + c, v(0) = 0 \Rightarrow 0 = c \Rightarrow v(t) = \frac{t^2}{2} - 2t$

$$s = \int_1^5 \left(\frac{t^2}{2} - 2t\right)\mathrm{d}t = \left.\left(\frac{t^3}{6} - t^2\right)\right]_1^5 = \left(\frac{125}{6} - 25\right) - \left(\frac{1}{6} - 1\right) = -\frac{10}{3} \approx -3.33 \text{ m}$$

$$d = \int_1^5 \left|2t - \frac{t^2}{2}\right|\mathrm{d}t = \frac{17}{3} \approx 5.67 \text{ m}$$



(b) $a(t)=\frac{1}{\sqrt{5t+1}} \Rightarrow v(t)=\int\frac{\mathrm{d}t}{\sqrt{5t+1}} \Rightarrow v(t)=\frac{2}{5}\sqrt{5t+1}+c,\quad v(0)=2 \Rightarrow c=\frac{8}{5}$

Since the function $v(t)=\frac{2}{5}\sqrt{5t+1}+\frac{8}{5}$ is always positive, the displacement and the distance travelled are the same.

$$d=s=\int_0^3\left(\frac{2}{5}\sqrt{5t+1}+\frac{8}{5}\right)\mathrm{d}t=\left(\frac{4}{75}(5t+1)^{\frac{3}{2}}+\frac{8}{5}t\right)\Bigg]_0^3=\frac{84}{25}+\frac{24}{5}=\frac{204}{25}=8.16 \text{ m}$$

(c) $a(t)=-2 \Rightarrow v(t)=-2t+c \quad v(0)=3 \Rightarrow c=3 \Rightarrow v(t)=-2t+3$

$$s=\int_1^4(-2t+3)\,\mathrm{d}t=\left(-t^2+3t\right)\Big]_1^4=-16+12+1-3=-6 \text{ m}$$

$$d=\int_1^4|-2t+3|\,\mathrm{d}t=\frac{13}{2}=6.5 \text{ m}$$

7. (a) $s=\int_1^3(9.8t-3)\,\mathrm{d}t=4.9t^2-3t\Big|_1^3=33.2$

(b) $s=\int_1^3(9.8t-3)\,\mathrm{d}t=33.2$

(c) $s=\int_1^3(9.8t-3)\,\mathrm{d}t=33.2$

Note: The displacement does not depend on the initial conditions since the displacement is the integral of the velocity function.

8. (a) $v(t)=s'(t)=50-20t$

(b) The maximum displacement takes place when the object stops and starts returning towards point O $\Rightarrow v(t)=0$.

$$v(t)=0 \Rightarrow 50-20t=0 \Rightarrow t=\frac{5}{2},\quad s\left(\frac{5}{2}\right)=\frac{2125}{2}=1062.5 \text{ m}$$

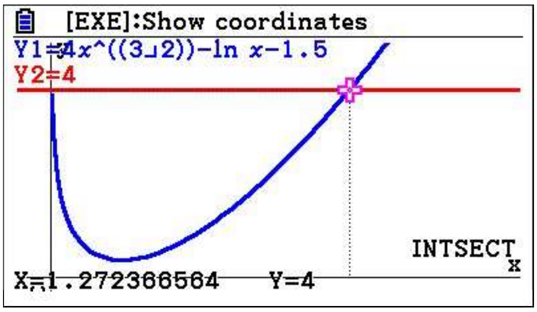
9. $$v(t)=\begin{cases}5t, & 0\le t<1\\ 6\sqrt{t}-\frac{1}{t}, & t\ge 1\end{cases}$$

For the distance to be 4 cm, we actually need to look at the integral of the velocity function. The first part of the integral between 0 and 1 is: $\int_0^1 5t\,\mathrm{d}t=\left(5\frac{t^2}{2}\right)\Big]_0^1=\frac{5}{2}$;

therefore, we need to find the time after 1 second in which the particle covers another 1.5 cm. We first need to find the correct displacement function so that the motion is continuous. Thus, our initial condition for the second part of the displacement function is that $s(1)=2.5$.

$s(t)=\int\left(6\sqrt{t}-\frac{1}{t}\right)\mathrm{d}t=4t^{3/2}-\ln(t)+c; s(1)=\frac{5}{2}\Rightarrow c=-\frac{3}{2}$, thus, the displacement function after 1 second is $s(t)=4t^{3/2}-\ln(t)-\frac{3}{2}$.

The solution can be estimated using GDC



So, the time at which the particle is 4 cm from the starting point, is 1.27 seconds.

10. The velocity of a projectile fired upwards is going to be influenced by gravity and hence the deceleration is going to slow the projectile down. The deceleration we are going to use is 9.81 m s^{-2}. (answers may slightly differ if we use 9.80 for deceleration)

(a) It will reach the maximum height when $v = 0$.

$v(t)=49-9.81t\Rightarrow v(t)=0\Rightarrow 49-9.81t=0\Rightarrow t=\frac{49}{9.81}\approx 4.995$; so, we can say 5 seconds.

(b) To find the maximum height, we need to find the height formula.

$$v(t)=49-9.81t\Rightarrow h(t)=\int(49-9.81t)\mathrm{d}t=49t-9.81\frac{t^2}{2}+c$$

$$h(0)=150\Rightarrow c=150\Rightarrow h(t)=49t-9.81\frac{t^2}{2}+150$$

The maximum height is reached when the velocity is zero, i.e., at t = 5s. Thus, the maximum height is $h(5)$ = 272.4 metres.

(c) Since the parabola is symmetrical with respect to the vertical axis of symmetry that passes through the vertex, we can say that the time taken to reach the maximum height will be doubled. So, the answer is approximately 10 seconds. Alternatively, you can solve the equation $h(t) = 150$.

$$49t-9.81\frac{t^2}{2}+150=150\Rightarrow 49t-9.81\frac{t^2}{2}=0\Rightarrow t_1=0$$

or $t=\frac{98}{9.81}\approx 9.9898\approx 10$ s

(d) This is simply $v(10)=49-9.81\times 10\approx -49.1\ \mathrm{ms}^{-1}$

The velocity will be approximately −49 m s^{-1}.

(e) The projectile hits the ground when its height is zero.

$$49t - 9.81\frac{t^2}{2} + 150 = 0 \Rightarrow t \approx 12.447 \text{ s}$$

So, the projectile will take 12.4 seconds to hit the ground.

(f) This is simply $|v(12.447)| = |49 - 9.81 \times 12.447| \approx 73.11 \text{ ms}^{-1}$

So, the speed at impact is 73.1 ms^{-1}.

11. By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)\,\mathrm{d}t$.

We use Trap. Method with $n = 6$ and $\Delta t = \frac{6-0}{6} = 1$.

$$\int_0^6 a(t)\,\mathrm{d}t \approx \frac{1}{2}\left(0 + 2(0.5 + 2.1 + 5 + 6.5 + 4.8) + 0\right) \approx 19\, ms^{-1}$$

12. With $\theta_0 = 40°$ and $f(t) = \frac{1}{\sqrt{1-(k\sin t)^2}}$, we use $n = 10$, $\Delta t = \frac{\frac{\pi}{2} - 0}{10} = \frac{\pi}{20}$ and obtain

$$T_n = \frac{\Delta t}{2}\left(1 + 2\left[f(1) + \cdots + f(9)\right] + \frac{1}{\sqrt{1-(0.342\sin 40)^2}}\right) \approx 1.57$$

We multiply by $4\sqrt{\frac{1}{9.8}}$ to find the period T of the pendulum to be approx. 2 seconds.

13. If deceleration is 4 ms^{-2}, then velocity is $v(t) = \int -4\,\mathrm{d}t = -4t + c$, but $v_0 = 44$, then $c = 44$.

So, $v(t) = -4t + 44$.

The car comes to rest when $v = 0 \Rightarrow v(t) = -4t + 44 = 0 \Rightarrow t = 11\text{ s}$

Now, distance travelled is $d(t) = \int_0^{11} v(t)\,\mathrm{d}t = \int_0^{11} (-4t + 44)\,\mathrm{d}t = 242\text{ m}$.

14. First, we should have all measures in the same units. Thus, the truck is initially at a distance of 1500 m and going with a speed of $\frac{50 \times 1000}{3600} = 13.89\,\text{ms}^{-1}$

Distance of truck from booth = $vt + s_0 = 13.89t + 1500$.

Car is accelerating from 0, and distance from booth is also zero, i.e., $v_0 = 0$ and $s_0 = 0$.

Now, $v(t) = at = 4t \Rightarrow s(t) = 2t^2$.

Car catches up with truck when they are at the same distance from the booth

$$2t^2 = 1500 + 13.89t \Rightarrow t \approx 31\text{ s}$$

Distance from booth:

$\text{car}: 2t^2 = 2\times 31^2 \approx 1922$

$\text{truck}: 1500 + 13.89\times 31 \approx 1930$

(discrepancy due to approximations made: $t = 31.1$ for example)

15. Description of this athletes route is given below

$$a(t) = \begin{cases} 4 & t \le 2 \\ 0 & t > 2 \end{cases} \Rightarrow v(t) = \begin{cases} 4t & t \le 2 \\ 8 & t > 2 \end{cases}$$

The distance covered during the first 2 seconds is $s_1(t) = 2t^2 = 2\times 2^2 = 8$

After the first 2 seconds, the athlete will therefore cover $s_2(t) = 8t$ metres

Therefore, to cover the 100 metres, $s_2(t)$ must be the remaining 92 metres.

The time after the first 2 seconds is then $\frac{92}{8} = 11.5$ s. Total time for the 100 m is 13.5 s

16. **(a)** Calculations are similar to the example with a change in the angle to 6°.

The vertical component of the velocity is

$v(0) = 54\sin(-6°) = -5.645 \text{ m s}^{-1}$

This means

$y'(t) = \int -9.8\mathrm{d}t = -9.8t + c \Rightarrow -5.645 = y'(0) = c \Rightarrow y'(t) = -9.8t - 5.645$

Since the initial height is 2.44 m,

$y(t) = \int y'(t)\mathrm{d}t = \int(-4.9t - 5.645)\,\mathrm{d}t = -4.9t^2 - 5.645t + c \Rightarrow y(t) = -4.9t^2 - 5.645t + 2.44$

The horizontal component is determined by $x''(0) = 0$ with initial velocity

$x'(0) = 54\cos(-6°) \approx 53.704 \text{ ms}^{-1}$

So $x(t) = \int 53.7\mathrm{d}t = 53.7t$

To clear the net, the height of the ball, y must be at least 0.914 m when $x = 11.90$ m

That means $x(t) = 53.7t = 11.90 \Rightarrow t = 0.22160$. At this time, the height is

$y(0.22160) = -4.9(0.22160)^2 - 5.645(0.22160) + 2.44 \approx 0.948 > 0.914.$

Thus, the ball is high enough to clear the net.

The second condition is that we need $x \le 18.29$ when the ball hits the ground ($y = 0$). $y(t) = -4.9t^2 - 5.645t + 2.44 = 0 \Rightarrow t \approx 0.3349$

However, $x(0.3349) = 53.7(0.3349) \approx 17.98 < 18.29$. The ball is in

(b) with a speed of 52 m s^{-1}, the calculations are:

$v(0) = 52\sin(-5°) = -4.532 \text{ m s}^{-1}$

This means $y'(t) = -9.8t + c \Rightarrow -4.532 = y'(0) = c \Rightarrow y'(t) = -9.8t - 4.532$

Since the initial height is 2.44 m,

$y(t) = -4.9t^2 - 4.532t + c \Rightarrow y(t) = -4.9t^2 - 4.532t + 2.44$

The horizontal component is determined by $x''(0) = 0$ with initial velocity

$x'(0) = 52\cos(-5°) \approx 51.802 \text{ m s}^{-1}$

So $x(t) = \int 51.8\,\mathrm{d}t = 51.8t$

To clear the net, the height of the ball, y must be at least 0.914 m when x = 11.90 m

That means $x(t) = 51.8t = 11.90 \Rightarrow t = 0.22973$. At this time, the height is

$y(0.22973) = -4.9(0.22973)^2 - 4.532(0.22973) + 2.44 \approx 1.14 > 0.914$. Thus, the ball is high enough to clear the net.

The second condition is that we need $x \le 18.29$ when the ball hits the ground ($y = 0$).

$y(t) = -4.9t^2 - 4.532t + 2.44 = 0 \Rightarrow t \approx 0.3812$

However, $x(0.3812) = 51.8(0.3812) \approx 19.75 > 18.29$. The ball is out.

Chapter 16 practice questions

1. **(a)** Gradient of the tangent through the origin and passing through the point (p,q) is $\frac{q}{p}$.

However, (p,q) is a point on $y = \mathrm{e}^x \Rightarrow q = \mathrm{e}^p$. Also, the line being tangent to $y=\mathrm{e}^x$ at (p,q) will have a gradient of e^p

Thus $\mathrm{e}^p = \frac{q}{p} = \frac{\mathrm{e}^p}{p} \Rightarrow p = 1$ and $q = \mathrm{e}$

Examiner's Note: Even though you might not know how to find the parameter in part **(a)** it is advisable to proceed with part **(b)** and attempt to write the definite integral.

(b) Tangent has the equation $y = ex$, and hence the area is

$$\int_0^1 (\mathrm{e}^x - \mathrm{e}x)\,\mathrm{d}x = \left(\mathrm{e}^x - \frac{1}{2}\mathrm{e}x^2\right)\Big|_0^1 = \frac{1}{2}\mathrm{e} - 1 \text{ square units}$$

2. **(a)** P is the y-intercept of $y = 2\mathrm{e}^x \Rightarrow P(0,2)$.

Q is the point of intersection between $y = 2\mathrm{e}^x$ and $y = \mathrm{e}^{2-x}$ Thus,

$$2\mathrm{e}^x = \mathrm{e}^{2-x} \Rightarrow 2 = \mathrm{e}^{2-2x} \Rightarrow x = 1 - \frac{1}{2}\ln 2 = 1 - \ln\sqrt{2}$$

$$y = 2\mathrm{e}^{1-\ln\sqrt{2}} = \mathrm{e}\sqrt{2} \Rightarrow Q\left(1 - \ln\sqrt{2}, \mathrm{e}\sqrt{2}\right)$$

R is the intersection between $y = 2$ and $y = e^{2-x}$. Thus,

$$\mathrm{e}^{2-x} = 2 \Rightarrow 2 - x = \ln 2 \Rightarrow x = 2 - \ln 2 \Rightarrow R(2 - \ln 2, 2)$$

(b) The solid is made up of two parts: one between $y = 2e^x$ and $y = 2$, and one between $y = \mathrm{e}^{2-x}$ and $y = 2$.

$$V = \pi\int_0^{1-\frac{\ln 2}{2}} \left((2\mathrm{e}^x)^2 - 2^2\right)\mathrm{d}x + \pi\int_{1-\frac{\ln 2}{2}}^{2-\ln 2} \left((\mathrm{e}^{2-x})^2 - 2^2\right)\mathrm{d}x$$

3. $\int_{0.5}^{a} \frac{\mathrm{d}x}{x^2} = -\frac{1}{x}\Big|_{0.5}^{a} = 2 - \frac{1}{a} = \frac{5}{3} \Rightarrow a = 3$

4. **(a)** $y=\ln x \Rightarrow y'=\frac{1}{x}$. At the point (e, 1) the slope of the tangent is: $m=y'(\mathrm{e})=\frac{1}{\mathrm{e}}$

The tangent can be found by using the formula for the tangent:

$y=f'(x_1)(x-x_1)+y_1$, where (x_1, y_1) is a particular point on the graph of the function.

$$y=\frac{1}{\mathrm{e}}(x-\mathrm{e})+1 \Rightarrow y=\frac{1}{\mathrm{e}}x-1+1 \Rightarrow y=\frac{1}{\mathrm{e}}x.$$

Since (0, 0) satisfies this equation, then the origin is on this line.

(b) direct application of derivative rules.

$$(x\ln x-x)'=\ln x+x\times\frac{1}{x}-1=\ln x+1-1=\ln x$$

(c) Note that the shaded region can be split into two regions.

The first region is a triangle bounded by the tangent line, x-axis and the vertical line $x=1$. Since that is a right-angled triangle, the area is calculated as: $A_{\text{Triangle}}=\frac{1\times\frac{1}{\mathrm{e}}}{2}=\frac{1}{2\mathrm{e}}$. In order to find the area of the second region, we evaluate the following integral:

$$\int_1^{\mathrm{e}}\left(\frac{1}{\mathrm{e}}x-\ln x\right)\mathrm{d}x=\left[\frac{1}{\mathrm{e}}\times\frac{x^2}{2}-(x\ln x-x)\right]_1^{e}=\frac{1}{2}\mathrm{e}-\frac{1}{2\mathrm{e}}-1$$

Now, the total area is the sum of those two areas; therefore,

$$A=\frac{1}{2\mathrm{e}}+\frac{1}{2}\mathrm{e}-\frac{1}{2\mathrm{e}}-1=\frac{1}{2}\mathrm{e}-1.$$

5. **(a)** **(i)** $s(t)=800+100t-4t^2 \Rightarrow s(5)=800+100\times5-4\times5^2=1200\text{ m}$

Distance travelled = 1200 – 800 = 400 m.

(ii) $v(t)=s'(t) \Rightarrow v(t)=100-8t \Rightarrow v(5)=100-8\times5=60\text{ m |s}^{-1}$

(iii) $v(t)=36 \Rightarrow 100-8t=36 \Rightarrow 64=8t \Rightarrow t=8\text{ s}$

(iv) $s(8)=800+100\times8-4\times8^2=1344\text{ m}$

(b) Firstly, we need to find the time at which the plane stops after touchdown:

$$v(t)=0 \Rightarrow 100-8t=0 \Rightarrow t=\frac{100}{8}=12.5\text{ s}$$

Now we need to find the distance the plane will travel after touchdown:

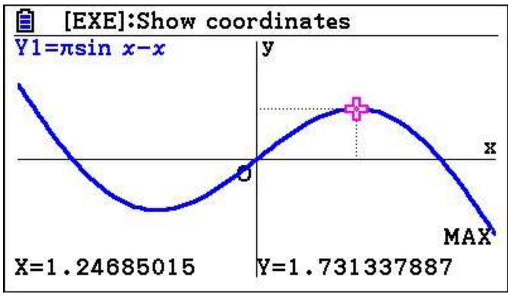
$$s\left(\frac{25}{2}\right)-s(0)=800+100\times\frac{25}{2}-4\times\left(\frac{25}{2}\right)^2-800=1250-625=625\text{ m}$$

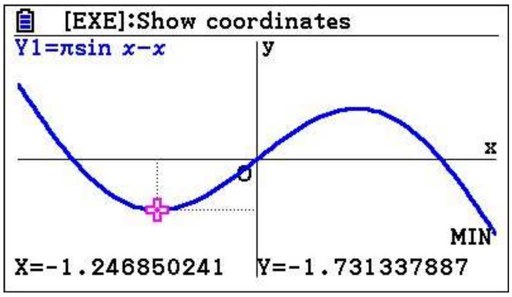
We notice from part **(a)(iv)**, the remaining runway length is $2000-1344=656$ m; therefore, there is enough runway to stop the plane if it makes a touchdown before point *P*.

6. **Note: Parts a) and b) cannot be solved without using a calculator.**

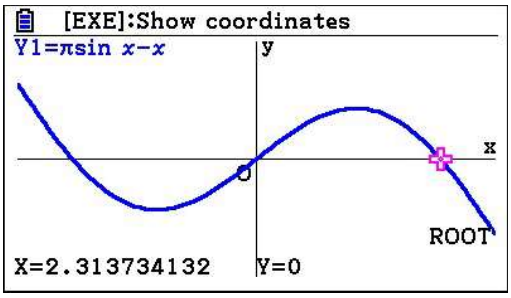
(a) To draw the function, we input the function into GDC and use its features to make your

estimates. Samples giving the maximum and minimum are given below





(b) The solution is approximately *x* 2.314 as shown



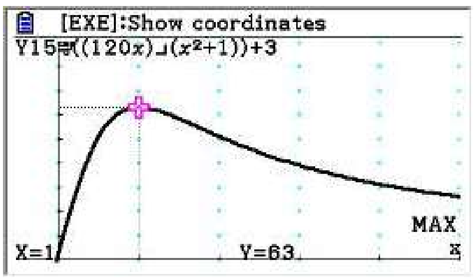
(c) $\int(\pi\sin x-x)\,\mathrm{d}x=-\pi\cos x-\dfrac{x^2}{2}+c, c\in\mathbb{R}$

$$\text{Area}=\int_0^1(\pi\sin x-x)\,\mathrm{d}x=\left[-\pi\cos x-\frac{x^2}{2}\right]_0^1=\pi(1-\cos 1)-\frac{1}{2}\approx 0.944$$

7. **(a)** To find the maximum rate, we find the derivative and equate it to zero.

$r'(t)=\dfrac{120(1-t^2)}{(t^2+1)^2}=0\Rightarrow t=1$, that is after one year, the production rate is the greatest.

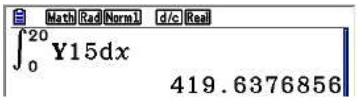
We can also graphically see that at $t=1$, production is greatest.



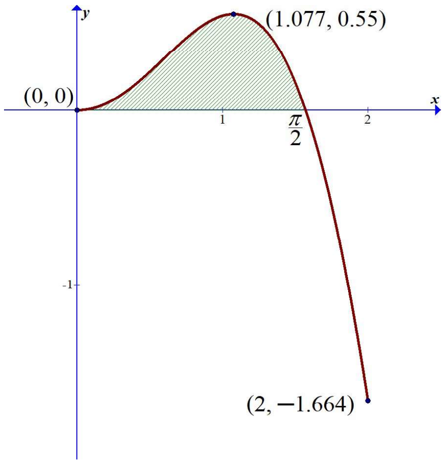
(b) $Q(t) = \int r(t)\mathrm{d}t = 60\ln\left(t^2+1\right)+3t+C$ and with initial conditions. $C = 0$

(c) $60\ln\left(t^2+1\right)+3t = 250 \Rightarrow t \approx 6.7$ years.

(d) $Q(t) = \int_0^{20} r(t)\mathrm{d}t = \left[60\ln\left(t^2+1\right)+3t\right]_0^{20} = 60\ln 401 + 60 \approx 420$ thousand barrels



8. For parts **(a)** **(i)**, (ii) and **(c)** we can use a GDC.



(a) **(i, ii)** See the diagram above.

(b) $x^2\cos x = 0, x > 0 \Rightarrow \cos x = 0 \Rightarrow x = \frac{\pi}{2}$

(c) **(i)** See diagram above.

(ii) $Area = \int_0^{\frac{\pi}{2}} x^2 \cos x \,\mathrm{d}x$

(d) $\int_0^{\frac{\pi}{2}} x^2\cos x\,\mathrm{d}x = \left[x^2\sin x + 2x\cos x - 2\sin x\right]_0^{\frac{\pi}{2}}$

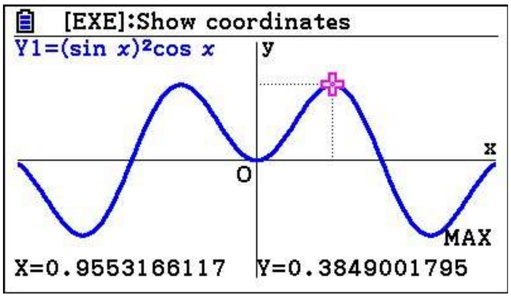
$$= \left(\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) + 2\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - 2\sin\left(\frac{\pi}{2}\right)\right) - 0$$

$$= \frac{\pi^2}{4} - 2 \approx 0.467$$

9. **(a)** Since $\sin^2 x$ is periodic of period π and $\cos x$ is periodic of period 2π, then the period of this function is 2π.

Moreover $f(x+2\pi) = \sin^2(x+2\pi)\cos(x+2\pi) = \sin^2(x)\cos(x) = f(x)$, so the fundamental period of f is 2π.

(b) By looking at the graph, we estimate that the range would be $[-0.4, 0.4]$.



(c) **(i)** $f'(x) = (2\sin x\cos x)\cos x + \sin^2 x(-\sin x) = 2\sin x\cos^2 x - \sin^3 x$.

(ii) $f'(x) = 0 \Rightarrow 2\sin x\cos^2 x - \sin^3 x = 0 \Rightarrow \sin x\left(2\cos^2 x - \sin^2 x\right) = 0$

Since the value of sine cannot be equal to 0 at *A*, we can conclude that:

$2\cos^2 x - \sin^2 x = 0 \Rightarrow 2\cos^2 x - \left(1 - \cos^2 x\right) = 0$

$$\Rightarrow 3\cos^2 x - 1 = 0 \Rightarrow \cos x = \sqrt{\frac{1}{3}}$$

(iii) Point *A* is at the maximum of the function, and thus,

$$f(x) = \sin^2(x)\cos(x) = \left(1 - \frac{1}{3}\right) \times \sqrt{\frac{1}{3}} = \frac{2}{3\sqrt{3}} \text{ or } \frac{2\sqrt{3}}{9}$$

(d) $f(x) = 0 \Rightarrow \sin x = 0$ or $\cos x = 0 \Rightarrow x = 0, x = \pi$ or $x = \frac{\pi}{2}$, so, the *x*-coordinate of point *B* is $\frac{\pi}{2}$.

(e) **(i)** $\int \sin^2(x)\cos(x)\,\mathrm{d}x = \int (\sin x)^2 \mathrm{d}(\sin x) = \frac{\sin^3 x}{3} + c, c \in \mathbb{R}$

(ii) $\int_0^{\frac{\pi}{2}} f(x)\,\mathrm{d}x = \left[\frac{\sin^3 x}{3}\right]_0^{\frac{\pi}{2}} = \frac{\sin^3\left(\frac{\pi}{2}\right)}{3} - \frac{\sin^3 0}{3} = \frac{1}{3}$

(f) $f''(x) = 0 \Rightarrow 9\cos^3 x - 7\cos x = 0 \Rightarrow \cos x\left(9\cos^2 x - 7\right) = 0$. Since the *x*-coordinate of *C* is less than $\frac{\pi}{2}$ the second factor must be equal to 0.

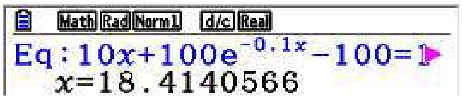
$$9\cos^2 x - 7 = 0 \Rightarrow \cos^2 x = \frac{7}{9} \Rightarrow \cos x = \frac{\sqrt{7}}{3} \Rightarrow x = \arccos\left(\frac{\sqrt{7}}{3}\right) \approx 0.491$$

10. **(a)** $S(t)=\int S'(t)\,dt=\int\left(10-10e^{-0.1t}\right)dt=10t+100e^{-0.1t}+c$

With initial condition that $S(0)=0$, $S(t)=10t+100e^{-0.1t}-100,\ 0\le t\le 24$

(b) $S(12)=10\times12+100e^{-0.1\times12}-100\approx$ €50 million

(c) $100=10t+100e^{-0.1t}-100\Rightarrow$ between 18 and 19 months



11. **(a)** Using scatter plot, second degree polynomial appears appropriate.

The equation by running regression is $M'(t)=4009+423.2t+51.2t^2$.

An appropriate model since r^2 = 95%. (A cubic model is also possible)

(b) $M(t)=\int_2^{10}\left(4009+423.2t+51.2t^2\right)dt\approx 69316$

12. **(a)** A cubic model seems to fit best with r^2 = 99.3%.

$y=-0.01214+0.6177x-2.235x^2+2.615x^3$

This is an estimate and may have some discrepancy, since it does not pass through (0, 0), nor (1, 1)

(b) Gini index $=2\int_0^1\left(x-\left(-0.01214+0.6177x-2.235x^2+2.615x^3\right)\right)dx\approx 0.2945$

13. Assuming that the supplies are freely falling, then with an initial height of 78 m, the height at any point is given by $h(t)=78-\frac{1}{2}gt^2=78-4.9t^2$

The supplies will reach the ground when the height is zero

$h(t)=78-4.9t^2=0\Rightarrow t\approx 4$ seconds.

The horizontal speed is given to be 30.5 m s^{-1}, and thus the horizontal distance the supplies will go before reaching the ground is $d=vt=30.5\times4\approx122$ m

14. We know that acceleration is the derivative of velocity with respect to time, t.

That is, $a=\frac{dv}{dt}=\frac{d^2s}{dt^2}$. However, in this exercise s is implicitly defined with respect to t, and velocity is expressed in terms of s. We can use the chain rule for this:

$$a=\frac{dv}{dt}=\frac{dv}{ds}\times\frac{ds}{dt}=\frac{dv}{ds}\times v\Rightarrow a=\frac{3(2s-1)-2(3s+2)}{(2s-1)^2}\times\frac{3s+2}{2s-1}=\frac{-7(3s+2)}{(2s-1)^3}$$

$$s=2\Rightarrow a=\frac{-7(6+2)}{(4-1)^3}=-\frac{56}{27}$$

15. **(a)** The production should be stopped when the marginal revenue and marginal cost are equal. This is so, because at that level every extra unit produced will bring in revenue as much as it costs, which is economically not feasible.

$C'(t) = R'(t) \Rightarrow 2t = 5te^{-0.1t^2} \Rightarrow t \approx 3$ years

(b) Area = $\int_0^3 \left(R'(t) - C'(t)\right)\mathrm{d}t = \int_0^3 \left(5te^{-0.1t^2} - 2t\right)\mathrm{d}t \approx 5.836$

The total profit over the useful life of the game is approximately €5836.

16. **(a)** Drawing scatter plots indicate that the present model is a power function and the new one is a quadratic model. Running regression for both, we get the following present: $y = x^{2.3}$, New: $y = 0.4x + 0.6x^2$ both with $r^2 = 1$.

(b) Gini index for the present system: 0.394

Gini index for the new system: 0.2

Interpretation: Yes, income will be more equally distributed after the changes.

17. We are given that $a(t) = -\frac{1}{20}t + 2, v(0) = 0$.

$$v(t) = \int\left(-\frac{1}{20}t + 2\right)\mathrm{d}t = -\frac{1}{40}t^2 + 2t + c, c \in \mathbb{R} \quad v(0) = 0 \Rightarrow c = 0 \Rightarrow v(t) = -\frac{1}{40}t^2 + 2t$$

$$d = \int_0^{60}\left|-\frac{1}{40}t^2 + 2t\right|\mathrm{d}t = \left[\left(-\frac{t^3}{120} + t^2\right)\right]_0^{60} = -1800 + 3600 = 1800\,\mathrm{m}.$$

18. Firstly, we need to find the zeros of the parabola.

$y = a^2 - x^2 \Rightarrow y = (a - x)(a + x) \Rightarrow x_1 = -a, x_2 = a$

The area of the rectangle is $A_R = 2ah$, where h is the height of the rectangle.

The area under the parabola is calculated by the following integral.

$$A_P = \int_{-a}^{a}\left(a^2 - x^2\right)\mathrm{d}x = \left[\left(a^2x - \frac{x^3}{3}\right)\right]_{-a}^{a} = \left(a^3 - \frac{a^3}{3}\right) - \left(-a^3 + \frac{a^3}{3}\right) = \frac{4}{3}a^3$$

Since the two areas must be the same, we can find the height of the rectangle:

$$2ah = \frac{4}{3}a^3 \Rightarrow h = \frac{2}{3}a^2$$

So, the dimensions of the rectangle are: $2a$ by $\frac{2}{3}a^2$.

19. **(a)** $f_k(x) = x\ln x - kx, x>0 \Rightarrow f_k{}'(x) = \ln x + 1 - k, x > 0$

(b) If the function is increasing, the first derivative is positive; therefore:

$\ln x + 1 - k > 0 \Rightarrow \ln x > k - 1 \Rightarrow x > \mathrm{e}^{k-1}, x \in \left]\mathrm{e}^{k-1}, +\infty\right[$

The question asks us to find the interval over which $f(x)$ is increasing; therefore, the value of k is 1 and the interval is: $x > \mathrm{e}^{-1} = \frac{1}{\mathrm{e}}, x \in \left]\frac{1}{\mathrm{e}}, +\infty\right[$.

(c) **(i)** $f_k'(x) = \ln x + 1 - k = 0 \Rightarrow \ln x = k - 1 \Rightarrow x = \mathrm{e}^{k-1}$

(ii) $f_k(x) = x\ln x - kx = 0 \Rightarrow x(\ln x - k) = 0 \Rightarrow x = 0$ or $\ln x - k = 0$

So, the other x-intercept is at: $\ln x - k = 0 \Rightarrow \ln x = k \Rightarrow x = \mathrm{e}^k$

(d) The area between the curve and the x-axis is given by $\int_0^{e^k} |x\ln x - kx|\,\mathrm{d}x$.

The integral can be evaluated partly using integration by parts and the rest is just the power rule:

$$\int x\ln x\,\mathrm{d}x = = \frac{x^2}{2}\ln x - \int\left(\frac{x^2}{2}\times\frac{1}{x}\right)\mathrm{d}x = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,\mathrm{d}x = \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$$

$$\int_0^{\mathrm{e}^k} |x\ln x - kx|\,\mathrm{d}x = \left|\left[\frac{x^2}{2}\ln x - \frac{x^2}{4} - k\frac{x^2}{2}\right]_0^{e^k}\right| = \left|\frac{\mathrm{e}^{2k}}{4}\left(2\ln e^k - 1 - 2k\right)\right| = \frac{\mathrm{e}^{2k}}{4}$$

So, the area enclosed by the curve and the x-axis is: $\frac{\mathrm{e}^{2k}}{4}$.

(e) $A(\mathrm{e}^k, 0), m = f_k'(\mathrm{e}^k) = \ln \mathrm{e}^k + 1 - k = 1 \Rightarrow T: y = 1\times(x - \mathrm{e}^k) + 0 \Rightarrow y = x - \mathrm{e}^k$

(f) The y-intercept of the tangent is $-\mathrm{e}^k$, so the area of the triangle enclosed by the tangent and the coordinate axes is: $A = \frac{1}{2}\left|\mathrm{e}^k \times (-\mathrm{e}^k)\right| = \frac{\mathrm{e}^{2k}}{2} = 2\times\frac{\mathrm{e}^{2k}}{4}$, which is twice the area enclosed by the curve.

(g) $k = 1 \Rightarrow x_1 = \mathrm{e}, k = 2 \Rightarrow x_1 = \mathrm{e}^2, k = 3 \Rightarrow x_1 = \mathrm{e}^3, k = 4 \Rightarrow x_1 = \mathrm{e}^4, \cdots$

To verify the statement, we are going to take two consecutive x-intercepts, for k and $k + 1$: $\frac{x_{k+1}}{x_k} = \frac{\mathrm{e}^{k+1}}{\mathrm{e}^k} = \mathrm{e}$. The ratio is constant and therefore the intercepts form a geometric sequence with common ratio e.

20. We are given the rate of consumption per year, thus, the total consumption within t years from the start, 2016 is given by the integral

$$C(t) = \int 5.3e^{0.01t}\,dt = 530e^{0.01t} + c\,;\ C(0) = 0 \Rightarrow c = -530$$

If consumption continues at the same rate and no new reserves are discovered, then 130 million tons will be exhausted by year t that satisfies the equation:

$530e^{0.01t} - 530 = 130 \Rightarrow t \approx 22$ years, i.e., in year 2038.

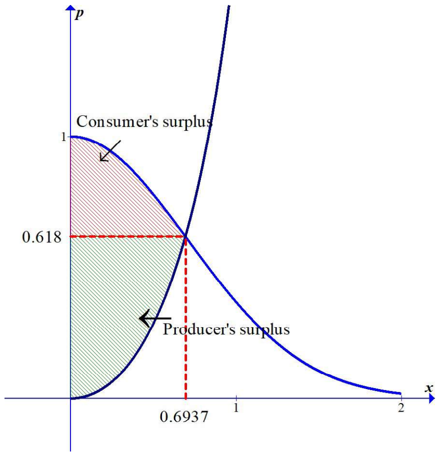
21. The mine's capital value is its present value as discussed in the chapter.

The present value formula gives us the following:

$$\begin{pmatrix}\text{Mine's}\\ \text{capital value}\end{pmatrix} = \int_0^{20} 560000\sqrt{t}\, e^{-0.05t}\,dt \approx \$189,805,52$$

(answers may slightly differ due to approximations.)

22. **(a)** The consumers' surplus is the area between the line corresponding to market equilibrium and the demand function. The producers' surplus is the area between the supply function and market equilibrium.

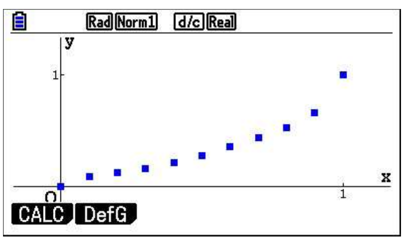


(b) GDC: (0.6937, 0.618), i.e, 694 units at 6.18 Euros.

(c) consumer's: $\int_0^{0.6937} \left(e^{-x^2} - 0.618\right) dx = 0.168$

Producer's: $\int_0^{0.6937} \left(0.618 - \left(e^{x^2} - 1\right)\right) dx = 0.299$

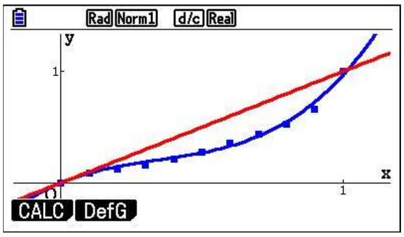
23. **(a)** Here is a scatter plot where the horizontal line represents the present of lowest-paid income recipients and the vertical axis the percent of income those groups receive.



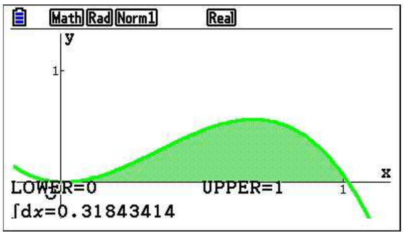
(b) A cubic model is a good fit since $r^2 = 0.991$. the model is

$y = 1.858x^3 - 1.931x^2 + 1.040x - 0.005$

(c) Below is a Lorentz curve for the data. We have to accept slight discrepancies since this is not an exact function.



(d) $L = 2\int_0^1 (x - f(x))\,\mathrm{d}x \approx 0.3184$ as shown in GDC output.



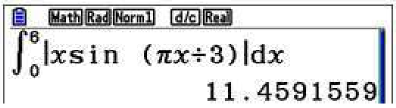
24. **(a)** $v(t) = 0 \Rightarrow t\sin\left(\frac{\pi}{3}t\right) = 0 \Rightarrow t = 0$ or $\frac{\pi}{3}t = k\pi \Rightarrow t = 3k, k \in \mathbb{Z}$

Using the restricted domain, we can calculate the values of t:
$t = 0$ or $t = 3$ or $t = 6$.

(b) **(i)** We use the absolute value of the integral for the total distance.

Total distance travelled $= \int_0^6 \left| t\sin\left(\frac{\pi}{3}t\right) \right| \mathrm{d}t$

(ii) We use GDC for this part.

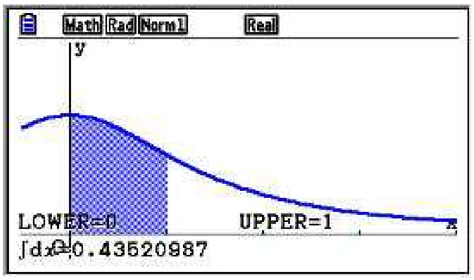


So, the total distance travelled is 11.5 m (correct to three significant figures).

Note: If not using a GDC, we should split the integral into two parts, from 0 to 3 and from 3 to 6, where the last one has a negative value and we take its opposite value. The anti-derivative can be found by using integration by parts.

25. **(a)** Since the velocity function is positive for all values, there is no need to consider absolute value:

$$\text{Distance travelled} = \int_0^1 v(t)\,\mathrm{d}t = \int_0^1 \frac{1}{2+t^2}\,\mathrm{d}t = \frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\right) \approx 0.435 \text{ m}$$



(b) $a = \frac{\mathrm{d}v}{\mathrm{d}t} \Rightarrow a(t) = \frac{-2t}{\left(2+t^2\right)^2}$

26. **(a)** Clothing: $FV = \mathrm{e}^{0.04\times5}\int_0^5 120000\mathrm{e}^{-0.04t}\,\mathrm{d}t \approx 664208$

Computer: $FV = \mathrm{e}^{0.04\times5}\int_0^5 100000\mathrm{e}^{0.05t}\mathrm{e}^{-0.04t}\,\mathrm{d}t \approx 626227$

The clothing store is the better investment over 5 years.

(b) Clothing: $FV = \mathrm{e}^{0.04\times10}\int_0^{10} 120000\mathrm{e}^{-0.04t}\,\mathrm{d}t \approx 1475474$

Computer: $FV = \mathrm{e}^{0.04\times10}\int_0^{10} 100000\mathrm{e}^{0.05t}\mathrm{e}^{-0.04t}\,\mathrm{d}t \approx 1568966$

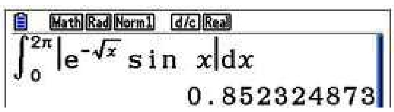
The computer store is the better investment over 10 years.

27. Total distance is given by the integral of the absolute value. $v(t) = 6t^2 - 6t, t \geq 0 \Rightarrow$

$$\text{distance} = \int_0^2 \left|6t^2 - 6t\right|\mathrm{d}t = \int_0^1 \left(6t - 6t^2\right)\mathrm{d}t + \int_1^2 \left(6t^2 - 6t\right)\mathrm{d}t$$

$$= \left(3t^2 - 2t^3\right)\Big]_0^1 + \left(2t^3 - 3t^2\right)\Big]_1^2 = 6 \text{ m}$$

28. Same as before-absolute value.



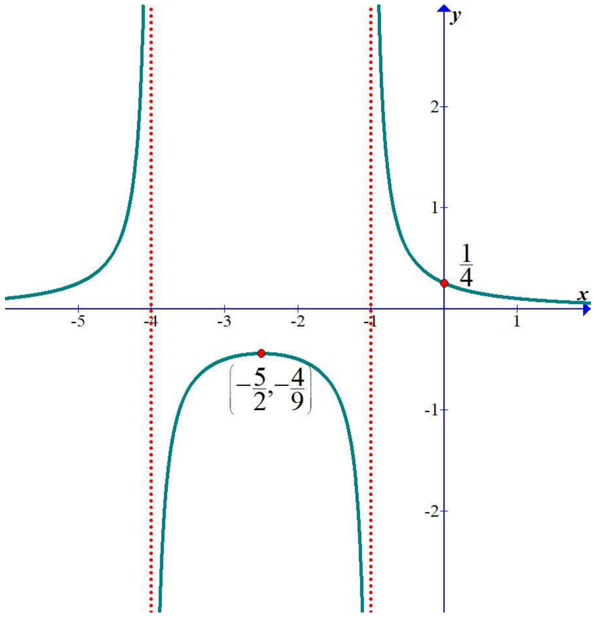
The total distance travelled is 0.852 m (correct to three significant figures).

29. **(a)** The function can be simplified and written as follows using partial fractions

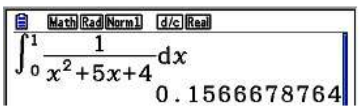
$$f(x)=\frac{1}{(x+4)(x+1)}=\frac{1}{3(x+1)}-\frac{1}{3(x+4)}$$

Since the numerator's degree is less than the denominator's, it has the x-axis as a horizontal asymptote. The vertical asymptotes are clearly $x=-1$ and $x=-4$.

When $x=0, y=\frac{1}{4}$. Also $f'(x)=-\frac{2x+5}{(x+4)^2(x+1)^2}\Rightarrow\left(-\frac{5}{2},-\frac{4}{9}\right)$ is an extreme value.

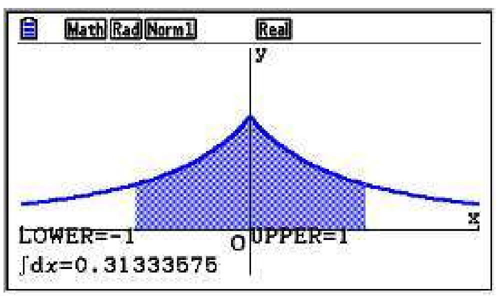


(b) $$\int_0^1\frac{dx}{(x+4)(x+1)}=\frac{1}{3}\int_0^1\frac{dx}{(x+1)}-\frac{1}{3}\int_0^1\frac{dx}{(x+4)}=\frac{1}{3}\ln\left(\frac{x+1}{x+4}\right)\Bigg|_0^1=\ln\sqrt[3]{\frac{8}{5}}$$



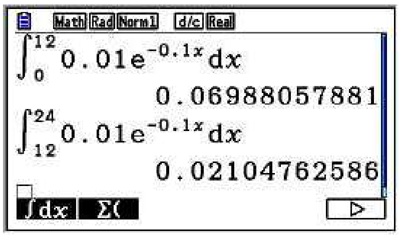
(c) As is clear from the graph, the required area is twice the area of the function bounded by the y-axis, the curve and x-axis.

Therefore, the required area is $2\ln\sqrt[3]{\frac{8}{5}}\approx 0.313$



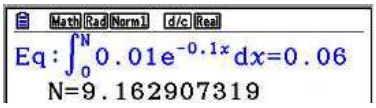
30. **(a)** $P(t \le 12) = \int_0^{12} 0.01\mathrm{e}^{-0.1t}\,\mathrm{d}t \approx 0.070$

(b) $P(12 \le t \le 24) = \int_{12}^{24} 0.01\mathrm{e}^{-0.1t}\,\mathrm{d}t \approx 0.021$



(c) Solve for n such that $P(t \le n) \le 0.06$:

$\int_0^{n} 0.01\mathrm{e}^{-0.1t}\,\mathrm{d}t \le 0.06 \Rightarrow n = 9$



31. We will use the trapezoidal method with $n = 12$.

$\Delta x = 50, n = 12$

$T_n = \frac{50}{2}\left(0 + 2(165 + 192 + \cdots + 215) + 0\right) \approx 91150\ \mathrm{m}^2$

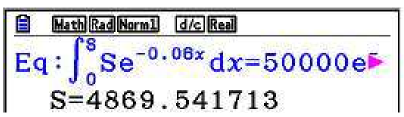
32. **(a)** A possible model using GDC: $y = -0.00083t^3 + 0.0531t^2 + 0.1434t + 58.2689$, $r^2 = 0.99$, which means that 99% of changes in consumption can be explained by this model.

(b) $\int_{21}^{31}\left(-0.00083t^3 + 0.0531t^2 + 0.1434t + 58.2689\right)\mathrm{d}t \approx 832.0782$ million barrels

33. **(a)** This is a continuous income flow. WE need to solve for the annual rate S:

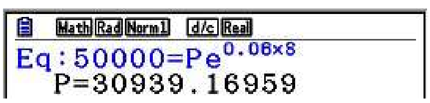
$FV = \mathrm{e}^{0.06\times 8}\int_0^{8} S\mathrm{e}^{-0.06t}\,\mathrm{d}t = 50000 \Rightarrow S \approx £4870$ per year

Use your GDC Solver.



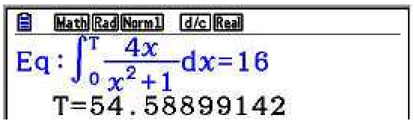
(b) This is a present value calculation. $P(T) = \int_0^{8} 4870\mathrm{e}^{-0.06t}\,\mathrm{d}t \approx 30942$ or equivalently, since we know the future value:

$50000 = P\mathrm{e}^{0.06\times 8} \Rightarrow P \approx 30939.17$ the discrepancy is due to our rounding up the answer in **(a)**.

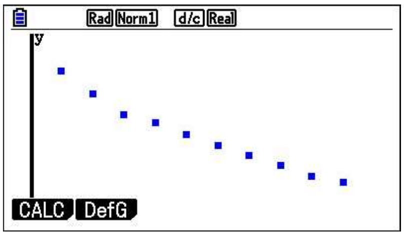


34. Solve the following equation for T:

$\int_0^T \frac{4t}{t^2+1}\,\mathrm{d}t = 16 \Rightarrow T$ is between 54 and 55 years.



35. **(a)** See scatterplot below



This appears to be an exponential decay. If we run Exponential regression we get: Depreciation $\approx -976.3\mathrm{e}^{-0.0998t}$ with $r^2 = 0.996$ indicating a very good fit. (an attempt to fit a cubic model will also have a high r^2.

However, when it comes to estimating the value of the machine, we observe that its value first decreases but it then increases, which does not fit a realistic situation. A quadratic model is also a good fit, but depreciation starts to increase in absolute value.)

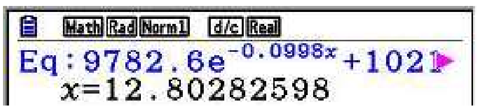
(b) The model should give the total depreciation after t years and that amount should be subtracted from the original value of the machinery.

$\text{Value}(t) \approx \int -976.3\mathrm{e}^{-0.0998t}\mathrm{d}t = 9782.6\mathrm{e}^{-0.0998t} + C$

With initial value of 20000, the model will be: $\text{Value}(t) \approx 9782.6\mathrm{e}^{-0.0998t} + 10217$

(c) Maintenance = $\int 200e^{0.2t}\mathrm{d}t = 1000\mathrm{e}^{0.2t} + c$, with initial condition of £1000, c = 0.

Solve $9782.6\mathrm{e}^{-0.0998t} + 10217 = 1000\mathrm{e}^{0.2t}$ for t. t = 12.8 years



Exercise 17.1

1. **(a)** Descriptive statistics. The focus is on the sample.

(b) Inferential statistics. We are making an inference from the sample of 30 students to the whole students' population.

(c) Inferential statistics. From a sample of 12 students, we infer the average hours spent gaming of the whole school.

(d) Descriptive statistics. The bar chart is just based on the voter support found.

2. Repeated sample of dentists in the regions the sugarless chewing gum is sold could ensure reliability.

3. The validity of the requirement is difficult to establish. Even if there is a correlation, there would be little causation as the relevance of the mathematical understanding in the nursing program is not evident.

4. **(a)** Retesting would increase the total sample size and may produce results closer to what might have been expected.

(b) Parallel testing may address any issues about the way the survey questions were phrased.

5. To address reliability, we could retest at other malls and location, other times of the day and other times of the week, months, year. This would ensure better randomisation.

6. To ensure the reliability of the finding retest will need to be done at other low frequency times of the day.

7. To check the validity of the claim, we could ask other students if they find it appealing.

Exercise 17.2

1.

Unbiased estimate: 4.3. To get the unbiased estimator of the variance, σ^2, we multiply it by $\frac{n}{n-1}$, which is always greater than 1. To get the unbiased estimator of the standard deviation, σ, we will multiply it by $\sqrt{\frac{n}{n-1}}$, which is also greater than 1. Hence, the unbiased estimator of the standard deviation is always greater than the standard deviation.

2. Unbiased estimate of the population mean = sample mean. Hence, $\bar{x} = 15$

Unbiased estimate of the population variance, $s_{n-1}^2 = \frac{n}{n-1} \cdot s_n^2$. Hence, $s_{n-1}^2 = 7.5$

3. Using GDC with values and frequency list.

Unbiased estimate of population mean = sample mean. Hence, $\bar{x} = 45.7$

Unbiased estimate of the population variance, $s_{n-1}^2 = \frac{n}{n-1} \cdot s_n^2$.

Hence, $s_{n-1}^2 = 36.2$ (3 s,f,)

4. Using GD. C

Unbiased estimate of population mean = sample mean. Hence, $\bar{x} = 501$ (3 s.f.)

Unbiased estimate of the population variance, $s_{n-1}^2 = \frac{n}{n-1} \cdot s_n^2$.

Hence, $s_{n-1}^2 = 36.6$ (3 s.f.)

5. The unbiased estimator of the variance: $\frac{n}{n-1} \cdot \sigma^2 = \frac{15}{14} \cdot 6.1^2 = 39.9$ (3 s.f.)

6. Solve the equation for *n*.

$\frac{n}{n-1} \cdot 5.22 = 5.31$.

$5.22 \cdot n = 5.31 \cdot (n-1)$

$0.09 \cdot n = 5.31$ and $n = 59$.

7. Solve the equation for n

$\frac{n}{n-1} < 1.01$ or $n > \frac{1.01}{0.01}$. Hence a sample size of at least 102.

Exercise 17.3

1. Answers will vary. Here is a sample produced by a software package

n	X1	X2	…	X30
1	1	6	…	1
2	5	4	…	1
3	2	2	…	1
4	0	3	…	6
5	6	4	…	7
6	1	6	…	4
7	2	4	…	2
8	2	0	…	2
9	0	8	…	0
10	0	6	…	6
11	1	2	…	4
12	4	8	…	4
13	4	7	…	3
14	7	1	…	9
15	2	6	…	5
16	5	2	…	9
17	3	5	…	8
18	2	2	…	0
19	7	2	…	9
20	3	3	…	2
$\bar{x}$	**2.85**	**4.05**	**…**	**4.15**

Looking at the last row, we estimate the population mean by find the mean of all the values and it is equal to 4.42 and the variance is 0.26.

2. Using GDC, unbiased estimate of the population mean:

$\mu = 57.4$ kg.

From GDC, $\sigma = 8.01$ kg.

Using, $\sigma_{\bar{X}}^2 = \dfrac{\sigma^2}{n}$.

Hence, unbiased estimate of the population variance,

$\sigma^2 = \sigma_{\bar{X}}^2 \cdot n = 8.01^2 \cdot 21 = 1350\,(3 \text{ s.f.})$

3. Using $\sigma_{\bar{X}}^2 = \dfrac{\sigma^2}{n}$ and $\sigma_{\bar{X}}^2 = 0.03\,\text{s}$. We calculate $\sigma^2 = \sigma_{\bar{X}}^2 \cdot n = 0.03 \cdot 30 = 0.9$

4. Using $\sigma_{\bar{X}}^2 = \dfrac{\sigma^2}{n}$ and $\sigma_{\bar{X}} = 6.2\,\text{min}$. We calculate $\sigma^2 = \sigma_{\bar{X}}^2 \cdot n = 6.2^2 \cdot 10 = 384\,(3 \text{ s.f.})$

5. Data are re-written as {90, 85, 80, 85, 80, 81, 82, 70, 72, 76, 73, 76, 73, 74, 68, 65, 68, 65, 63, 67, 66, 63}

The estimate for the mean carbohydrate intake in grams is $\mu = 73.7$ g.

Using GDC, we find $\sigma_{\bar{X}}^2 = 7.69$ g. (3 s.f.).

Hence, we calculate $\sigma^2 = \sigma_{\bar{X}}^2 \cdot n = 7.69^2 \cdot 22 = 1300\,(3 \text{ s.f.})$.

An estimate for the population variance from $\bar{X}$ is 1300.

Exercise 17.4

1. The standard normal distribution has mean of 0 and standard deviation of 1.

(a) (−1.48, 1,48)

(b) (−1.96, 1.96)

(c) (−2.58, 2.58)

2. **(a)** **(i)** $\hat{p} = \frac{8}{80} = 0.1$

(ii) Using the GDC to find the inverse normal of a 90% confidence interval, gives us the value of 1.64 (3 s.f.). So,

$$0.1 - 1.64 \cdot \sqrt{\frac{0.1 \cdot 0.9}{80}} \le p \le 0.1 + 1.64 \cdot \sqrt{\frac{0.1 \cdot 0.9}{80}}$$ or

(0.0488, 0,155) for the interval.

(iii) margin of error $= CI \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}$.

Hence, margin of error: $1.64 \cdot \sqrt{\frac{0.1 \cdot 0.9}{80}} = 0.0553$ (3 s.f.)

(b) This will reduce the margin of error as the sample is bigger; indeed, n is in the denominator of the $CI \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}$. The new confidence interval for the proportion is calculated: $0.1 - 1.65 \cdot \sqrt{\frac{0.1 \cdot 0.9}{400}} \le p \le 0.1 + 1.65 \cdot \sqrt{\frac{0.1 \cdot 0.9}{400}}$ or (0.0753, 0.125) for the interval.

3. **(a)** To calculate the 95% CI of the proportion of the population, p, we calculate: $0.65 + 1.96 \cdot \sqrt{\frac{0.65 \cdot 0.35}{400}} \le p \le 0.65 + 1.96 \cdot \sqrt{\frac{0.65 \cdot 0.35}{400}}$, which gives as interval (0.603, 0.697)

(b) **(i)** The standard error for the proportion is calculated by $\sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}$.

So, $SE_p = \sqrt{\frac{0.65 \cdot 0.35}{400}} = 0.0238$ (3 s.f.)

(ii) The margin of error is calculated by $CI \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}$, in this case with a 95% CI: $1.96 \cdot \sqrt{\frac{0.65 \cdot 0.35}{400}} = 0.0467$ (3 s.f.)

4. **(a)** Margin of error at 95% CI is $1.96\cdot\sqrt{\frac{\hat{p}\cdot(1-\hat{p})}{300}}$. To maximise the margin of error we need to maximize the product $\hat{p}\cdot(1-\hat{p})$. In the interval [0,1], $\hat{p}\cdot(1-\hat{p})$ is maximum when $\hat{p}=0.5$ (maximum of the parabola opening down with vertex at 0.5). Hence, maximum margin of error is

$$1.96\cdot\sqrt{\frac{0.05\cdot 0.5}{300}}=0.0566\,(3\text{ s.f.})$$

(b) Using GDC and the inverse normal function (88% CI has lower bound at 0.06 and upper bound at 0.94). Hence,

$$\text{margin of error} = CI\cdot\sqrt{\frac{\hat{p}\cdot(1-\hat{p})}{n}}=1.55\cdot\sqrt{\frac{0.5\cdot 0.5}{300}}=0.0449$$

5. CI on true population is estimated with $\bar{x}-CI\cdot\frac{\sigma}{\sqrt{n}}\le\mu\le\bar{x}+CI\cdot\frac{\sigma}{\sqrt{n}}$.

In this case, $30000-1.96\cdot\frac{2287}{\sqrt{30}}\le\mu\le 30000+1.96\cdot\frac{2287}{\sqrt{30}}$ or (29182, 30818).

6. CI on true population is estimated with $\bar{x}-CI\cdot\frac{\sigma}{\sqrt{n}}\le\mu\le\bar{x}+CI\cdot\frac{\sigma}{\sqrt{n}}$.

In this case, $45-1.96\cdot\frac{3.7}{\sqrt{30}}\le\mu\le 45+1.96\cdot\frac{3.7}{\sqrt{30}}$ or (43.7, 46.3).

The previous mean was 4.7 min., and now we are 95% confident that the new mean is within 43.7 min. and 46.3 min. Hence, we can say that the changes were effective in decreasing the mean manufacturing time.

7. CI on true population is estimated with $\bar{x}-CI\cdot\frac{\sigma}{\sqrt{n}}\le\mu\le\bar{x}+CI\cdot\frac{\sigma}{\sqrt{n}}$.

In this case, $115-1.96\cdot\frac{5.1}{\sqrt{10}}\le\mu\le 115+1.96\cdot\frac{5.1}{\sqrt{10}}$ or (112, 118).

In Golf, less is more! Hence, there is improvement from earlier as we are 95% confident that his average score is now between 112 and 118 (better than the 120 from earlier).

Chapter 17 practice questions

1. **(a)** Descriptive statistics. No inference are made from this sample.

(b) Inferential statistics. We use that sample to make an inference on the whole student population.

(c) Inferential statistics. We use that sample to make inferences to all vehicle of this specification.

(d) Descriptive statistics. What has been published is from the sales only.

2. To check the claim of "most-delicious" dessert, we will need a random sample of the school cafeteria users to ask them if it the most delicious dessert. As for the other claims a careful analysis of the amount of fat, carbohydrate and protein will need to be conducted and compared to the other dessert.

3. Although the proportion is indeed 40% (8/20), one classroom sample (even if random) could hardly be reliable to make an inference on the whole students population. As for the validity of the claim, we do not see how "skipping breakfast" could lead to the claim "not eating together".

4. **(a)** (Answers will vary). Retesting could increase our overall sample (and later increase our confidence in our inference). Retesting done at other times of the day, or days of the week, or location of the school could improve the representativeness of the sample.

(b) (Answers will vary). Parallel testing might help improve the quality of the survey in re-wording some part of it.

5. (Answers will vary). Use a random number generator to choose days of the week, times of the day and tally number of girls and boys. Redoing this over several weeks will increase reliability.

6. Two ways we could improve the reliability of the survey. First, prior to the blind test, ask if there is a preference (some students might recognise the taste). Second, alternate the order in which the sodas are given as one taste might affect the other.

7. Unbiased estimate of population mean is sample mean. So unbiased estimate of $\mu = 24.8$ (3 s.f.) years old.

For the unbiased estimate of the population variance, σ^2 , we use ${s_{n-1}}^2 = \frac{n}{n-1} \cdot {s_n}^2$

Hence, ${s_{n-1}}^2 = \frac{25}{24} \cdot {s_n}^2 = \frac{25}{24} \cdot 14.2 = 14.8$ (3 s.f.).

8. Unbiased estimate of population mean is sample mean. So unbiased estimate of $\mu = 145\,(3 \text{ s.f.})$ Km h^{-1} years old.

For the unbiased estimate of the population variance, σ^2 , we use $s_{n-1}{}^2 = \frac{n}{n-1} \cdot s_n{}^2$.

Hence, $s_{n-1}{}^2 = \frac{10}{9} \cdot s_n{}^2 = \frac{10}{9} \cdot 38.45 = 42.7$ (3 s.f.).

9. Unbiased estimate of population mean is sample mean. So unbiased estimate of $\mu = 1.10$ (3 s.f.).

For the unbiased estimate of the population variance, σ^2 , we use $s_{n-1}{}^2 = \frac{n}{n-1} \cdot s_n{}^2$.

Hence, $s_{n-1}{}^2 = \frac{10}{9} \cdot s_n{}^2 = \frac{10}{9} \cdot 0.00911 = 0.0101$ (3 s.f.).

10. As $s_{n-1}{}^2 = \frac{n}{n-1} \cdot s_n{}^2$,

we calculate $s_{n-1} = \sqrt{\frac{n}{n-1} \cdot s_{n-1}{}^2} = \sqrt{\frac{n}{n-1}} \cdot s_{n-1} = \sqrt{\frac{30}{39}} \cdot 4.3 = 4.37$ (3 s.f.).

11. Solve the equation for *n*. $2.66^2 = \frac{n}{n-1} \cdot 2.61^2$

$2.66^2 \cdot (n-1) = 2.61^2 \cdot n$ or $n = \frac{2.66^2}{2.66^2 - 2.61^2} \approx 26.85$. Rounded to the nearest integer, n=27.

12. As $\bar{x} = \mu$, population mean is 5.2 mm.

Using $\sigma_{\bar{X}}{}^2 = \frac{\sigma^2}{n}$ and $\sigma_{\bar{X}}{}^2 = 0.4$ mm. We calculate the population variance,

$\sigma^2 = \sigma_{\bar{X}}{}^2 \cdot n = 0.4 \cdot 20 = 8$ mm.

13. As $\bar{x} = \mu$, population mean is 38.5 sec.

Using $\sigma_{\bar{X}}{}^2 = \frac{\sigma^2}{n}$ and $\sigma_{\bar{X}}{}^2 = 5.3^2$ sec. We calculate the population variance,

$\sigma^2 = \sigma_{\bar{X}}{}^2 \cdot n = 5.3^2 \cdot 20 = 562$ mm.

14. As $\bar{x} = \mu$, population mean is 11.9 g.

Using $\sigma_{\bar{X}}{}^2 = \frac{\sigma^2}{n}$ and $\sigma_{\bar{X}}{}^2 = 5.3^2$ sec. We calculate the population variance,

$\sigma^2 = \sigma_{\bar{X}}{}^2 \cdot n = 5.3^2 \cdot 20 = 562$ mm.

15. The standard normal distribution has mean of 0 and standard deviation of 1.

(a) with a 80% CI, that means lower bound = 0.1 (10%)

and upper bound = 0.9 (90%).

Using GDC with inverse normal at 0.9, we get the interval (−1.28, 1.28)

(b) with a 90% CI, that means lower bound = 0.05 (5%)

and upper bound = 0.95 (95%).

Using GDC with inverse normal at 0.95, we get the interval (−1.64, 1.64)

(c) with a 97.5% CI, that means lower bound = 0.125 and upper bound = 0.985.

Using GDC with inverse normal at 0.985, we get the interval (−2.17, 2.17)

16. **(a)** $\hat{p} = \frac{18}{100} = 0.18$

(b) Using the GDC to find the inverse normal of a 90% confidence interval, gives us the value of 1.64 (3 s.f.). To find the 90% CI, we calculate

$$0.18 - 1.64 \cdot \sqrt{\frac{0.18 \cdot 0.82}{100}} \le p \le 0.18 + 1.64 \cdot \sqrt{\frac{0.18 \cdot 0.82}{100}}.$$

Hence, $0.117 \le p \le 0.243$

(c) The margin of error is calculated by $CI \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}$, in the case of a 95%

CI: $1.96 \cdot \sqrt{\frac{0.18 \cdot 0.82}{100}} = 0.0753$ (3 s.f.)

17. **(a)** Using the GDC to find the inverse normal of a 95% confidence interval, gives us the value of 1.96 (3 s.f.). To find the 95% CI, we calculate

$$0.58 - 1.96 \cdot \sqrt{\frac{0.58 \cdot 0.42}{200}} \le p \le 0.58 + 1.96 \cdot \sqrt{\frac{0.58 \cdot 0.42}{200}}$$

Hence, $0.512 \le p \le 0.648$

(b) The standard error for the proportion is calculated by $\sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}$

So, $SE_p = \sqrt{\frac{0.58 \cdot 0.42}{200}} = 0.0349$ (3 s.f.)

The margin of error is calculated by $CI \cdot \sqrt{\frac{\hat{p} \cdot (1-\hat{p})}{n}}$, in this case with a 95%

CI: $1.96 \cdot \sqrt{\frac{0.58 \cdot 0.42}{200}} = 0.0684$ (3 s.f.)

18. **(a)** In the case of a 95% confidence level, the margin of error is $1.96\cdot\sqrt{\frac{\hat{p}\cdot(1-\hat{p})}{100}}$

(b) To half the margin of error, we would need a denominator twice as big. As there is a square root, we will need it 4 times as big. Hence, we need a sample of 400 people.

19. CI on true population is estimated with $\bar{x}-CI\cdot\frac{\sigma}{\sqrt{n}}\le\mu\le\bar{x}+CI\cdot\frac{\sigma}{\sqrt{n}}$. In this case, with a 95% CI, $1240-1.96\cdot\frac{60}{\sqrt{20}}\le\mu\le1240+1.96\cdot\frac{60}{\sqrt{20}}$ or $1214\le\mu\le1266$. It does seem like the oven will need repair as the confidence interval for the mean temperature is below the 1300℃

Exercise 18.1

1. Degree of freedom υ is in this case $n-1$.

(a) $\upsilon = 20-1=19$

(b) With 12 randomly selected individuals, $\upsilon = 11$

(c) With 30 plants selected randomly, $\upsilon = 29$

2. (Using GDC and the *invT* function)

(a) $\upsilon = 12$, $\alpha = 0.05$ => $t = 1.78$

(b) $\upsilon = 18$, $\alpha = 0.01$ => $t = -2.55$

(c) $\upsilon = 24$, $\alpha = \dfrac{0.05}{2} = 0.025$ => $t = \pm 2.06$

(d) $\upsilon = 12$, $\alpha = \dfrac{0.05}{2} = 0.025$ => $t = \pm 2.18$

3. **(a)** Right-tailed test is a one tail-test and the *p*-value, 0.01 is less than α (0.05), the confidence level. Hence, the result is statistically significant.

(b) Left-tailed test is a one tail-test and the *p*-value, 0.001 is less than α (0.05), the confidence level. Hence, the result is statistically significant.

(c) With this two-tailed test, the *p*-value, 0.03 is more than $\dfrac{\alpha}{2}$ (0.025), the confidence level. Hence, the result is <u>not</u> statistically.

(d) With this two-tailed test, the *p*-value, 0.006 is more than $\dfrac{\alpha}{2}$ (0.005), the confidence level. Hence, the result is <u>not</u> statistically significant.

(e) As the t-statistic value is greater than the t critical value on a right-tailed test, the result is statistically significant.

(f) As the t-statistic value is greater than the t critical value on left-tailed test, the result is <u>not</u> statistically significant

4. We first find the *t*-value with $\upsilon = 17$, $\alpha = \dfrac{0.05}{2}$ => $t = 2.11$ (3 s.f.)

$120 - 2.11 \cdot \dfrac{15}{\sqrt{18}} \le \mu \le 120 + 2.11 \cdot \dfrac{15}{\sqrt{18}}$, or true mean is in the interval (112.5, 127.5)

5. We first find the *t*-value with $\upsilon = 17$, $\alpha = \dfrac{0.01}{2}$ => $t = 2.9$ (3 s.f.)

$120 - 2.9 \cdot \dfrac{15}{\sqrt{18}} \le \mu \le 120 + 2.9 \cdot \dfrac{15}{\sqrt{18}}$, or true mean is in the interval (109.8, 130.2)

6. We first find the *t*-value with $\upsilon = 25$, $\alpha = \frac{0.05}{2}$. $\Rightarrow t = 2.06$ (3 s.f.)

$540 - 2.06 \cdot \frac{80}{\sqrt{26}} \leq \mu \leq 540 + 2.06 \cdot \frac{80}{\sqrt{26}}$, or true mean is in the interval (507.7, 572.3).
Hence, the statistics refute the claim that the true population mean is $\mu = 500$ as it lies outside the true mean interval. This calculation could be achieved using the *TInterval* command of your GDC. Another way to solve this problem is to use the *T*-test command of your GDC which gives you a probability of 0.0173 of getting a *T*-value of 2.55.0.0173 being less than our 0.05 threshold, we refute the claim of the true mean being 500.

7. We first find the *t*-value with $\upsilon = 9$, $\alpha = \frac{0.05}{2}$. $\Rightarrow t = 2.26$ (3 s.f.)

$4.8 - 2.66 \cdot \frac{1.6}{\sqrt{10}} \leq \mu \leq 4.8 + 2.66 \cdot \frac{1.6}{\sqrt{10}}$, or true mean is in the interval (3.66, 5.94).

Hence, the statistics support the claim that the true population mean is $\mu = 4.5$ hrs as it lies inside the true mean interval. This calculation could be achieved using the *TInterval* command of your GDC. Another way to solve this problem is to use the *T*-test command of your GDC which gives you a probability of 0.568 of getting a *T*-value of 0.593. 0.568 being more than our 0.05 threshold, we do not refute the claim of the true mean being 4.5.

Exercise 18.2

1. The null hypothesis represents the status-quo, the no-difference and is represented by h_0; the alternative hypothesis represents the opposite and is represented by h_1.

(a) $h_0 : \mu \geq 128$ (the claim) and $h_1 : \mu < 128$

(b) $h_0 : \mu \leq 100$ (the claim) and $h_1 : \mu > 100$

(c) $h_0 : \mu \geq 42$ (the claim) and $h_1 : \mu < 42$

(d) $h_0 : \mu \leq 13$ (the claim) and $h_1 : \mu > 13$

2. The claim or null hypothesis is $h_0 : \mu \geq 48$; hence, $h_1 : \mu < 48$. This is a one-tailed test. Using GDC with *T*-test. We get a probability of 0.00322 to get the *T*-value of −3.16. Hence, we reject the null hypothesis and the claim as the probability is less than the threshold of 0.05 (one-tailed test).

3. The claim or null hypothesis is $h_0 : \mu \leq 0.1$ mm; hence, $h_1 : \mu > 0.1$ mm.

This is a one-tailed test. Using GDC with *T*-test. We get a probability of 0.663 to get the *T*-value of −0.433. Hence, we reject the null hypothesis and the claim as the probability is less than the threshold of 0.05 (one-tailed test).

4. We first find the *t*-value with $\upsilon = 7$ (as n=8), $\alpha = 0.05$. Using the invT command, we get $t = 2.36$ (3 s.f.). This is the *t*-value corresponding to a tail area of 0.025.

$9.5 - 2.36 \cdot \frac{3.1}{\sqrt{8}} \leq \mu \leq 9.5 + 2.36 \cdot \frac{3.1}{\sqrt{8}}$, or true mean is in the interval (6.91, 12.1).

Hence, we do not have enough evidence to reject the claim that there is between 7 to 9 pieces of salmon per package as this interval contains the claimed interval. This calculation could be achieved using the *TInterval* command of your GDC.

5. The claim or null hypothesis is $h_0 : \mu \leq 120$ min; hence, $h_1 : \mu > 120$ min.

This is a one-tailed test, using GDC with *T*-test. We get a probability of 0.00013 to get the *T*-value of 4.47. Hence, we reject the null hypothesis and the claim as the probability is less than the threshold of 0.05 (one-tailed test). Based on the random sample, we can reject the claim that it takes at most 2 hrs to climb the Grouse Grind trail.

6. The claim or null hypothesis is $h_0 : \mu \geq 23$ kg; hence, $h_1 : \mu < 23$ kg.

This is a one-tailed test. using GDC with *T*-test. We get a probability of 0.0521 to get the *T*-value of −1.81. Hence, we do not reject the null hypothesis and the claim as the probability is more than the threshold of 0.05 (one-tailed test). Based on the random sample, we do not reject the claim that at least 23 Kg of salmon and halibut are caught each day.

Exercise 18.3

1. **(a)** $h_0 : \mu = 158.2$ and□$h_1 : \mu = 167.2$

(b) Using GDC, keep all decimals for accuracy by storing values. First, find the t-value for $\alpha = 0.05$ level of significance using invT on your GDC. *t*-value for

$\alpha = 1.75305$

Second, find the critical value for μ_0.

$$\text{critical value} = 158.2 + 1.75305 \cdot \frac{50.4}{\sqrt{16}} = 180.288$$

Third, find the *t*-value for $\mu_1 : \dfrac{180.288 - 167.2}{\frac{50.4}{\sqrt{16}}} = 1.0387$

Lastly, find the probability to get that *t*-value, using the tcdf function and you get $\beta = 0.842$ (3 s.f.).

(c) Remember that the power of the test is the probability that we correctly reject a false null hypothesis and is found using $1 - \beta$.

Hence, power of the test is $1 - 0.842 = 0.158$

2. Given: $\mu_0 = 2.4Mil$, $\mu_1 = 3Mil$, $n = 40$, $\bar{x} = 2.3Mil$, $s_{n-1} = 0.325Mil$

(a) $h_0 : \mu = 2.4$ and□$h_1 : \mu = 3$

(b) Using GDC, keep all decimals for accuracy by storing values.

First, find the *t*-value for $\alpha = 0.05$ using invT. *t*-value_for_α = 1.6848

Second, find critical value for μ_0.

$$\text{critical value} = 3 + 1.6848 \cdot \frac{0.325}{\sqrt{40}} = 2.4865$$

Third, find the *t*-value for μ_1: $\dfrac{2.4865 - 3}{\frac{0.325}{\sqrt{40}}} = -9.91$.

This is a very big *t*-value.

We can expect a low probability to get that into our next step.

Finally, find the probability to get that *t*-value, using the tcdf function.
$\beta = 1.31 \cdot 10^{-12}$ (3 s.f.).

3. Given $\mu = 500ml$, $\sigma = 5ml$, $\bar{x} = 502.4ml$, $s_{n-1} = 0.325Mil$, $\alpha = 0.05$

(a) The chance of a type I error is α. Hence, type I error = 0.05

(b) Since we know the population standard deviation, we do not need to use the t-distribution. Instead, we'll use the z-distribution. First, find the critical value of Z for $\mu = 500ml$. Use invNorm of your GDC.

critical_value = 508.244

Second, find the probability to get that critical value, using the true mean (normalCdf on GDC). Hence, Type II error = 0.878 (3 s.f.)

4. Given: $\mu_0 = 230mg$, $\mu_1 = 241.6mg$, $n = 20$, $s_{n-1} = 20.2mg$, $\alpha = 0.05$

Using GDC, keep all decimals for accuracy by storing values.

First, find the *t*-value for $\alpha = 0.05$ using invT: t_value_for_α=1.7291

Second, find critical value for μ_0. $\text{critical value} = 230 + 1.7291 \cdot \dfrac{20.2}{\sqrt{20}} = 237.81$

Third, find the *t*-value for μ_1: $\dfrac{237.81 - 241.6}{\frac{20.1}{\sqrt{20}}} = -0.839$.

Finally, find the probability to get that *t*-value, using the tcdf function.

$\beta = 0.206$ (3 s.f.).

Exercise 18.4

1. The situation is a *t*-test with 9 observations. So *df* = 8 as $\nu = 9 - 1 = 8$.

(a) We are looking to test if the difference between the before and the after training is statistically significant. Hence, $h_0 : d \le 0$ and $h_1 : d > 0$, where *d* is the difference of time between 'before' and 'after'. This is a one tailed test. The null hypothesis is the training makes no difference in time.

(b) Using GDC and the spreadsheet we calculate the difference in times and run a *t*-test on the difference. Use *tTest* with $\mu_0 = 0$.

As seen below, we noticed that with $\bar{d} = 0.289$, $s_{n-1} = 0.322$, $\nu = 9 - 1 = 8$, we get a *t*-value of 2.69 with a probability of 0.0137 (3 s.f.).

As this is below our 0.05 threshold, we can reject the null hypothesis, that is we accept the claim that the training decrease their times at the $\alpha = 0.05$ level.

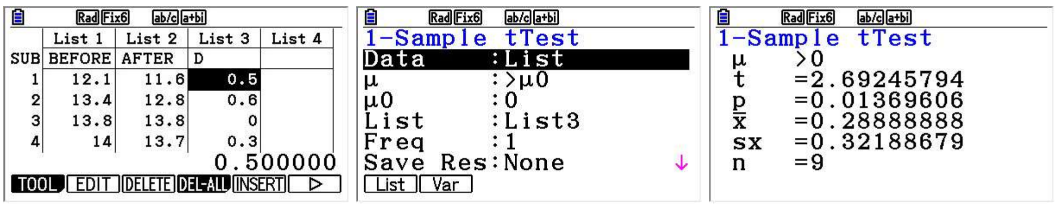
F =tTest(0,c[ ],1,1): CopyVar *Stat.*, *Stat3*.

	A	B	C differe...	D	E	F	G
=						=tTest(0,c	
1	12.1	11.6	0.5		Title	t Test	
2	13.4	12.8	0.6		Alternate...	μ > μ0	
3	13.8	13.8	0.		t	2.69246	
4	14	13.7	0.3		PVal	0.013696	
5	13.6	13.8	-0.2		df	8.	
6	12.8	12.9	-0.1		x̄	0.288889	
7	13.4	12.9	0.5		sx := sn-...	0.321887	
8	14.2	13.5	0.7		n	9.	
9	13.9	13.6	0.3				
10							

With a GDC, some may have the feature of matched – pairs, but some do not. Thus, we need to prepare the data for the test. As suggested above, we create a list of the differences, which gets tested. The hypotheses here are

$H_0 : \bar{d} \le 0$

$H_1 : \bar{d} > 0$



2. As we do not know the population σ, the situation is a *t*-test with 11 observations. So *df* = 10 as $\nu = 11 - 1 = 10$.

(a) We are looking to test the new mass-reducing program creates weight loss greater than 10 kg. Hence, $\text{h}_0 : d \geq 10\,\text{kg}$ and□$\text{h}_1 : d < 10\,\text{kg}$, where *d* is the difference of weight ('before' − 'after'). This is a one tailed test.

The null hypothesis is the mass-reducing program does not make a difference less than 10 kg.

(b) Using GDC and the spreadsheet we calculate the difference in weights and run a *t*-test on the difference. Use *tTest* with $\mu_0 = 10$. We noticed that with $\bar{d} = 6.77$, $s_{n-1} = 3.76$, $\nu = 10$, we get a *t*-value of 2.84 with a probability of 0.00871 (3 s.f.). As this is below our 0.05 threshold, we can reject the null hypothesis, $\text{h}_0 : d \geq 10$, and do not corroborate the claim that mass-reduction program help loose at least 10 kg.

3. As we do not know the population σ, the situation is a *t*-test with 10 observations. So *df* = 9 as $\nu = 10 - 1 = 9$

(a) We are looking to test if the tutoring service program makes a positive difference. That is the difference in scores between 'after' and 'before' training is greater that 0 (and is statistically significant). Hence, $\text{h}_0 : d \leq 0$ and □$\text{h}_1 : d > 0$, where *d* is the difference of scores ('after' - 'before'). This is a one tailed test. The null hypothesis is the tutoring program makes no (positive) difference in scores.

(b) Using GDC and the spreadsheet we calculate the difference in scores and run a *t*-test on the difference. Use *tTest* with $\mu_0 = 0$. As seen below, we noticed that with $\bar{d} = 2.1$, $s_{n-1} = 3.45$, $\nu = 9$, we get a *t*-value of 1.92 with a probability of 0.043 (3 s.f.). As this is below our 0.05 threshold, we can reject the null hypothesis, $\text{h}_0 : d \leq 0$, and do support the claim that the tutoring program help increased scores at the $\alpha = 0.05$ level.

F1 ="t Test"

	A	B	C difference	D	E	F	G
=						=tTest(0,c|	
1	62	63	1		Title	t Test	
2	58	64	6		Alternate Hyp	μ > μ0	
3	75	72	-3		t	1.92687	
4	53	57	4		PVal	0.043052	
5	81	85	4		df	9.	
6	46	51	5		x̄	2.1	
7	51	55	4		sx := sn-1x	3.44642	
8	91	87	-4		n	10.	
9	63	63	0				
10	45	49	4				

4. As we do not know the population σ, the situation is a *t*-test with 11 observations. So *df* = 10 as $\nu = 11 - 1 = 10$.

(a) We are looking to test if the speed reading program makes a positive difference. That is the score between 'after' and 'before' training is greater that 0 (and is statistically significant). Hence, $\text{h}_0 : d \le 0$ and $\text{h}_1 : d > 0$, where *d* is the difference of scores ('after' - 'before'). This is a one tailed test. The null hypothesis is the speed reading program makes no (positive) difference in scores.

(b) Using GDC and the spreadsheet we calculate the difference in scores and run a *t*-test on the difference. Use *tTest* with $\mu_0 = 0$. As seen below, we noticed that with $\bar{d} = 3.36$, $s_{n-1} = 5.22$, $\nu = 10$, we get a *t*-value of 2.14 with a probability of 0.0292 (3 s.f.). As this is below our 0.05 threshold, we reject the null hypothesis, $\text{h}_0 : d \le 0$, and support the claim that the speed reading program help increase scores at the $\alpha = 0.05$ level.

F =tTest(0,c[ ],1,1) : CopyVar *Stat.*, *Stat3*.

	A	B	C difference	D	E	F	G
=						=tTest(0,c	
1	94	100	6		Title	t Test	
2	102	113	11		Alternate Hyp	μ > μ0	
3	143	145	2		t	2.13691	
4	120	125	5		PVal	0.029169	
5	128	130	2		df	10.	
6	98	107	9		x̄	3.36364	
7	111	111	0		sx := sn-1x	5.22059	
8	131	133	2		n	11.	
9	150	145	-5				
10	134	130	-4				
11	106	115	9				

5. Because we are dealing with 12 boys and 16 girls randomly chosen, we are in the 2 sample *t*-test situation. We choose as null hypothesis $\text{h}_0 : \mu_{girls} \le \mu_{boys}$ and as alternative hypothesis $\text{h}_1 : \mu_{girls} > \mu_{boys}$. As the variances are considered equal we can pool the data and the degree of freedom $\nu = 16 + 12 - 2 = 26$. Using GDC and the 2-Sample *t*-test, we get a t value of 0.776 with a probability of 0.222 (3 s.f.). As this is above our 0.05 threshold, we do not reject the null hypothesis, $\text{h}_0 : \mu_{girls} \le \mu_{boys}$, and also do not support the claim that the mean score for girls is higher than the one for boys at the $\alpha = 0.05$ level.

6. Because we are dealing with a school with 26 candidates and another one with 23 candidates, we are in the 2 sample *t*-test situation. We choose as null hypothesis $h_0 : \mu_{school1} = \mu_{school2}$ and as alternative hypothesis $h_1 : \mu_{school1} \neq \mu_{school2}$. As the variances are considered equal we can pool the data and the degree of freedom $\nu = 23 + 26 - 2 = 47$ .Using GDC and the 2-Sample *t*-test, we get a t value of -2.22 with a probability of 0.0310 (3 s.f.). As this is below our 0.05 threshold, we reject the null hypothesis, $h_0 : \mu_{school1} = \mu_{school2}$, and also do not support the claim that the scores are equivalent at the $\alpha = 0.05$ level.

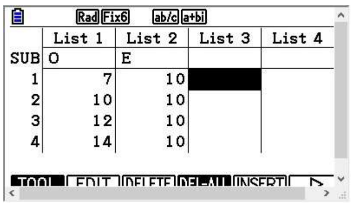
Exercise 18.5

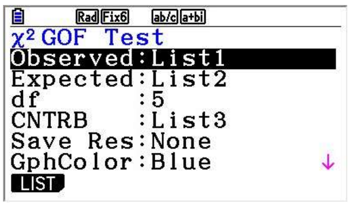
1. We first create our table of observed and expected outcomes

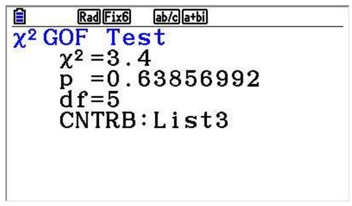
Outcomes	1	2	3	4	5	6	Total
Observed	7	10	12	14	8	9	60
Expected	10	10	10	10	10	10	60

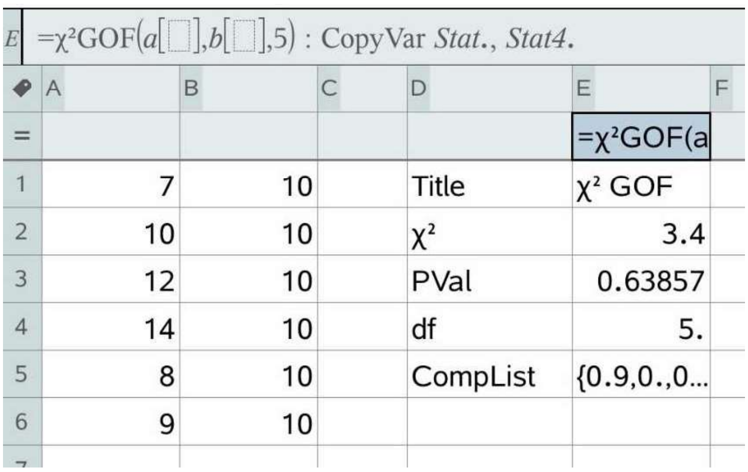
We set the null hypothesis H_0 : the observed outcome fits the indented model and as alternative hypothesis H_1 : the observed outcome does not fit the indented model.

Using a GDC, we calculate the Goodness of fit. Samples from 2 GDCs below







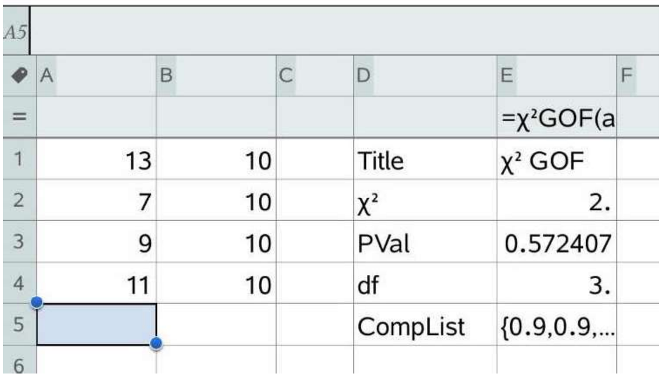


As seen above, the *p*-value is 0.639 (3 s.f.) which is above our 0.05 threshold. So we do not reject H_0 , and support the claim that the observed model fits the intended outcome (in this case, the dice is not biased).

2. We first create our table of observed and expected outcomes. As the regions are equal, each regions is expected to equally likely than the others.

Outcomes	**A**	**B**	**C**	**D**	**Total**
Observed	13	7	9	11	40
Expected	10	10	10	10	40

We set the null hypothesis H_0 : the observed outcome fits the indented model and as alternative hypothesis H_1 : the observed outcome does not fit the indented model. Using a GDC, we calculate the Goodness of fit.

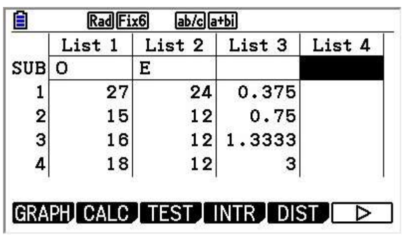


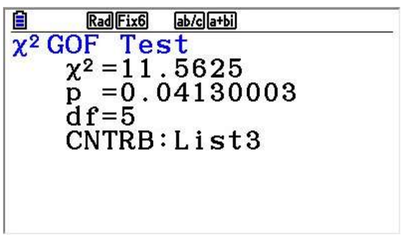
As seen above, the *p*-value is 0.572 (3 s.f.) which is above our 0.05 threshold. So we do not reject H_0 , and support the claim that the observed model fits the intended outcome (in this case, if the spinner has unbalanced weight it does not statistically affect the outcome of the spin.)

3. First step is to change the probabilities of the expected outcomes into frequency.

Outcomes	**1**	**2**	**3**	**4**	**5**	**6**	**Total**
Observed	27	15	16	18	13	31	120
Expected	24	12	12	12	12	48	120

We set the null hypothesis H_0 : the observed data fits the indented model of the biased dice and as alternative hypothesis H_1 : the observed outcome does not fit the indented model. Using GDC, we calculate the Goodness of fit.



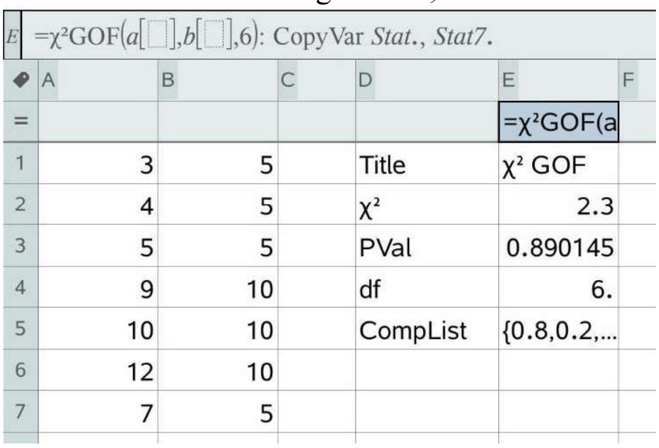


As seen above, the p-value is 0.0413 (3 s.f.) which is below our 0.05 threshold. So we reject H_0, and do not support the claim that the observed outcome fits the intended model (in this case, the dice cannot be considered biased at the 0.05 level.)

4. First step is to bring back the expected outcomes out of 50 to match the total number of observations. Notice that we do not violate the rule of having at least 5 expected values for each outcomes.

Outcomes	A	B+	B	B-	C	D	E	Total
Observed	3	4	5	9	10	12	7	50
Expected	5	5	5	10	10	10	5	50

We set the null hypothesis h_0 : the observed data fits the indented outcome of the scores set by the teacher and as alternative hypothesis h_1 : the observed outcome does not fit the indented model. Using a GDC, we calculate the Goodness of fit.


E =χ²GOF(a[],b[],6): CopyVar Stat., Stat7.

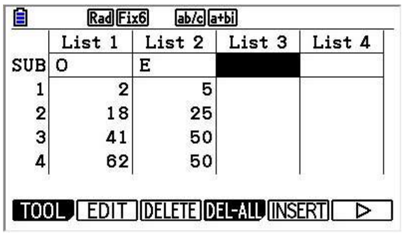
	A	B	C	D	E	F
=					=χ²GOF(a	
1	3	5		Title	χ² GOF	
2	4	5		χ²	2.3	
3	5	5		PVal	0.890145	
4	9	10		df	6.	
5	10	10		CompList	{0.8,0.2,...	
6	12	10				
7	7	5				

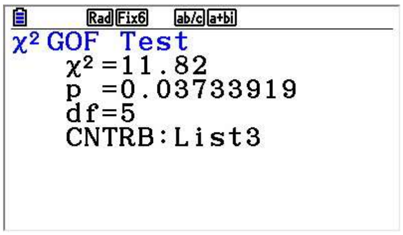
As seen above, the p-value is 0.890 (3 s.f.) which is above our 0.05 threshold. So we do not reject H_0, and do support the claim that the observed data fits the intended model at the 0.05 level (in this case, the predetermined grade distribution set by the teacher.)

5. The expected values are found using the binomial probability feature on your GDC:

$$160\times(0.03125, 0.15625, \cdots, 0.03125) = (5, 25, 50, 50, 25, 5)$$

We set the null hypothesis h_0 : the observed data fits the indented model of a fair coin. h_1 : the observed outcome does not fit the indented model. Using GDC, we calculate the Goodness of fit.





The p-value ≈ 0.0373. Since the p-value is less than α, reject H_0 at the level of significance of 0.05. (The observations do not fit the binomial model of a fair die tossed 5 times.)

6. **(a)** First, we calculate the expected outcomes. 90 days surveyed which is 18 weeks (90/5). We expect each day of the week to be equally likely to be an absent day.

Outcomes	**Monday**	**Tuesday**	**Wednesday**	**Thursday**	**Friday**	**Total**
Probability	0.2	0.2	0.2	0.2	0.2	1
Observed	24	14	12	14	26	90
Expected	18	18	18	18	18	90

H_0: The observed outcomes fit the intended model. H_1: The observed outcomes do not fit the intended model. The p-value ≈ 0.0533. Since the p-value is greater than α, do not reject H_0 at the level of significance of $\alpha = 0.05$.(There is insufficient evidence that absenteeism is dependent on the day of the week.)

(b) In this case, we need to combined Monday and Friday together.

Outcomes	**Monday / Friday**	**Tuesday**	**Wednesday**	**Thursday**	**Total**
Observed	50	14	12	14	90
Expected	36	18	18	18	90

H_0: The observed outcomes fit the intended model. H_1: The observed outcomes do not fit the intended model. The p-value ≈ 0.0265. Since the p-value is less than α, reject H_0 at the level of significance of $\alpha = 0.05$. (Friday and Monday absences combined are statistically higher than during the rest of the week.)

Exercise 18.6

1. **(a)** H_0 : there is no difference between the 10th and 9th graders in their music choices

H_1 : there is a difference that cannot be attributed to chance alone.

(b) Degree of freedom $\nu = (4-1)\cdot(2-1) = 3$

(c) Using GDC, we can store the observed data and pass it straight to the $\chi^2 2way$ function of the GDC.

As the *p*-value of the χ^2 test is 0.319 which is greater than the α value of 0.01. Hence, we do not reject the null hypothesis H_0 and conclude that the music taste and the grade level are independent.

2. **(a)** H_0 : there is no difference in marks between last year grade and this year grade

H_1 : there is a difference that cannot be attributed to chance alone.

(b) Degree of freedom $\nu = (6-1)\cdot(2-1) = 5$

(c) Using GDC, we can store the observed data and pass it straight to the $\chi^2 2way$ function of the GDC.

The χ^2 statistics is 3.55

(d) The *p*-value of the χ^2 test is 0.616 which is greater than the α value of 0.05. Hence, we do not reject the null hypothesis H_0 and conclude that the achievement level and the student's group are independent.

3. **(a)** H_0 : there is no difference in political preferences between city areas

H_1 : there is a difference that cannot be attributed to chance alone.

(b) Degree of freedom $\nu = (3-1)\cdot(3-1) = 4$

(c) Using GDC, we can store the observed data and pass it straight to the $\chi^2 2way$ function of the GDC.

The χ^2 statistics is 20.3 which gives a *p*-value of 0.00043 which is less than the α value of 0.05. Hence, we reject the null hypothesis H_0 and conclude that the political preferences and the city area are not independent.

4. There is a need first to recombine the first and second columns as their expected values would be less than 5.

So, we rewrite the table in this way.

Outcomes	**0–9**	**10–14**	**15–19**	**20–24**	**25–30**	**30+**
Students	18	20	23	40	35	12
Business Owners	17	16	18	16	8	4

(a) H_0 : there is no difference in postings between the 2 groups

H_1 : there is a difference that cannot be attributed to chance alone.

(b) Degree of freedom $\nu = (6-1)\cdot(2-1) = 5$

(c) Using GDC, we can store the observed data and pass it straight to the $\chi^2 2way$ function of the GDC.

The χ^2 statistics is 12.5 which gives a *p*-value of 0.0285 which is less than the α value of 0.05. Hence, we reject the null hypothesis H_0 and conclude that the number of postings and the 2 groups are not independent.

5. **(a)** H_0 : there is no difference in between color preferences between the groups
H_1 : there is a difference that cannot be attributed to chance alone.

(b) Degree of freedom $\nu = (4-1)\cdot(5-1) = 12$

(c) Using GDC, we can store the observed data and pass it straight to the $\chi^2 2way$ function of the GDC.

The χ^2 test gives a *p*-value of 0.9 which is more than the α value of 0.05. Hence, we do not reject the null hypothesis H_0 and conclude that the colour preferences are independent of the groups.

6. There is a need first to recombine the first and second columns as their expected values would be less than 5.

So we rewrite the table in this way.

IB awarded Mark	7–6	5	4	1–2–3	Total
Mathematics	10	3	5	7	25
Biology	6	7	9	7	29
Chemistry	9	10	7	5	31
Physics	10	9	10	6	35

H_0 : there is no difference in marks between the 4 groups

H_1 : there is a difference that cannot be attributed to chance alone.

The Degree of freedom $\nu = (4-1)\cdot(4-1) = 9$

The χ^2 test gives a *p*-value of 0.7 which is more than the α value of 0.05. Hence, we do not reject the null hypothesis H_0 and conclude that the marks preferences are independent of the subject groups.

7. H_0 : there is no difference outcomes between the 2 teams

H_1 : there is a difference that cannot be attributed to chance alone. The Degree of freedom $\nu = (2-1)\cdot(3-1) = 2$

The χ^2 test gives a *p*-value of 0.2 which is more than the α value of 0.05. Hence, we do not reject the null hypothesis H_0 and conclude that the league outcomes are independent of the teams.

8. First, we calculate the proportion of win for both team. Barcelona: $\frac{290}{380} = 0.58$ and Bayern Munich: $\frac{245}{340} = 0.54$. We use both value in the 2 prop-z Test.

The *p*-value of 0.6 is more than the α value of 0.05. Hence, we do not reject the null hypothesis H_0 and conclude that the league outcomes are independent of the teams.

Chapter 18 practice questions

1. **(a)** sample size n=18; hence $\nu = 17$

(b) 6 choices; hence $\nu = (6-1) = 5$

(c) 3 faculties, 5 universities; hence $\nu = (3-1)\cdot(5-1) = 8$

2. Using invT function on your GDC.

(a) *t*-value = 1.71. In this case, p = 0.05 as right-tailed and $\nu = 24$

(b) *t*-value = -1.71. In this case, p = 0.05 as left-tailed and $\nu = 24$

(c) *t*-value = ± 2.09. In this case, $p = \dfrac{0.05}{2} = 0.025$ as two-tailed and $\nu = 19$

(d) *t*-value = ± 2.90. In this case, $p = \dfrac{0.01}{2} = 0.005$ as two-tailed and $\nu = 17$

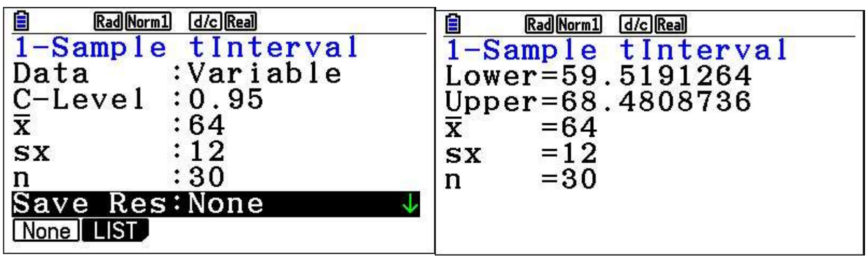
3. **(a)** p-value $< \alpha$; hence, results are not statistically significant

(b) p-value $< \alpha$; hence, results are statistically significant

(c) p-value $< \dfrac{\alpha}{2}$ (as it is a 2 tailed test); hence, results are statistically significant.

(d) The results are statistically significant if t-statistics > *t*-critical value (as it is right tailed); otherwise, it isn't. In this case, since the *t*-statistics is less than the critical value, then results are not statistically significant.

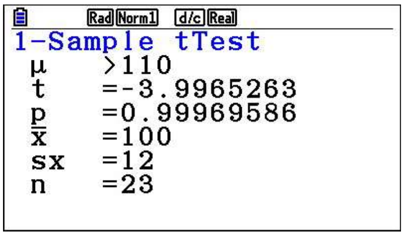
4. The population standard deviation, σ, is not given, so we use the t-statistics. We will perform the calculations manually for demonstration purposes. On exams, you are not expected to do that. Remember that $\bar{x} - t\cdot\dfrac{s_{n-1}}{\sqrt{n}} \le \mu \le \bar{x} + t\cdot\dfrac{s_{n-1}}{\sqrt{n}}$. To find the t-critical value, using invT on GDC with $p = 0.975$ and $\nu = (30-1) = 29$ gives a *t*-value of 2.0452. Finding the true mean interval: $64 - 2.04\cdot\dfrac{12}{\sqrt{30}} < \mu < 64 + 2.04\cdot\dfrac{12}{\sqrt{30}}$ or $59.5 < \mu < 68.5$. A sample GDC output:



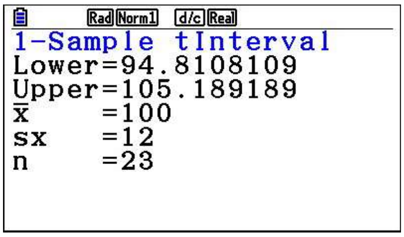
5. This can be done with a hypothesis test.

H_0: $\mu \le 110$, H_1: $\mu > 110$

The *p*-value = 0.99 > 0.05, we do not reject the null hypothesis. Thus we cannot support the claim of a mean larger than 110.



In fact, with a test statistics of 100, it is obvious, without using tests, that the mean is not more than 110. Moreover, your results can also be achieved with a 95% confidence interval:



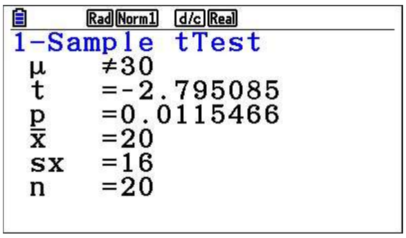
It is clear than any value beyond (94.8, 105.2) cannot support the claim.

Note here that the claim does not have to be the null hypothesis.

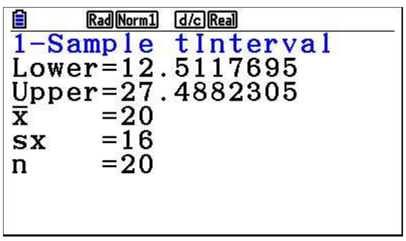
6. This can be done with a hypothesis test or a confidence interval.

H_0: $\mu = 30$, H_1: $\mu \neq$`30

The *p*-value = 0.011< 0.05, we reject the null hypothesis. Thus we cannot support the claim of a mean of 30 minutes.



A confidence interval willalso confirm the above result:



A confidence interval of (12.5, 27.5) does not contain the 30 minutes claim.

7. **(a)** $H_0 : \mu = 62.4$ (the claim) and $H_1 : \mu \neq 62.4$

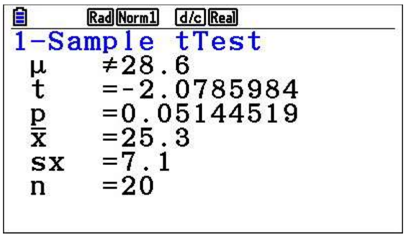
(b) $H_0 : \mu \geq 13$ (the claim) and $H_1 : \mu < 13$

(c) $H_0 : \mu \leq 72.3$ and $H_1 : \mu > 72.3$ (the claim)

(d) $H_0 : \mu \geq 102$ and $H_1 : \mu < 102$ (the claim)

8. $H_0 : \mu = 28.6$ (the claim) and $H_1 : \mu \neq 28.6$

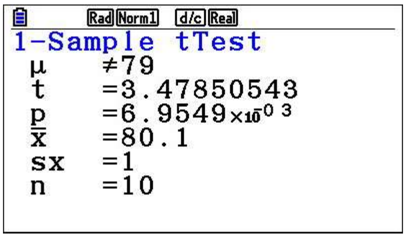
Using GDC and *t*-test



We see that *p*-value is 0.0514 which is greater than $\alpha = 0.05$, so we do not have evidence against the claim that American BMI = 28.6.

9. $h_0 : \mu = 79\,°C$ (the claim) and $h_1 : \mu \neq 79\,°C$

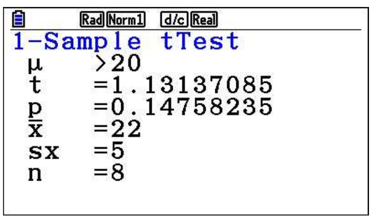
Using GDC and *t*-test



With *t*-value of 3.48 we get a *p*-value of 0.00696 which is less than $\alpha = 0.05$, so we reject the null hypothesis and reject the claim that the coffee is served at 79 °C.

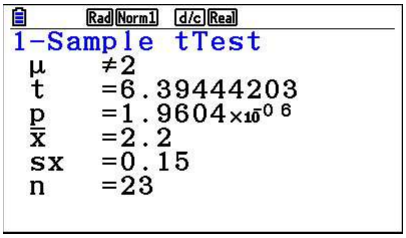
10. $H_0 : \mu \leq 20\,\text{min}$ (the claim) and $H_1 : \mu > 20\,\text{min}$

Using GDC and *t*-test



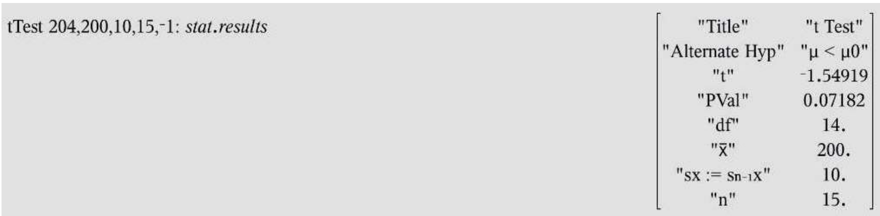
We see that *p*-value is 0.148 which is greater than $\alpha = 0.05$, so we do not have enough evidence to reject the null hypothesis. We cannot support the claim that the mean wait time is in excess of 20 minutes

11. $H_0 : \mu = 2$ A (the claim) and $H_1 : \mu \neq 2$ A. Using GDC and *t*-test



With *t*-value of 6.39 we get a *p*-value of 0.000002 which is less than $\alpha = 0.05$, so we reject the null hypothesis and reject the claim that the fuses are manufactured to specifications (2A).

12. $H_0 : \mu = 204$ ℃ (the claim) and $H_1 : \mu < 204$ °C. Using GDC and *t*-test

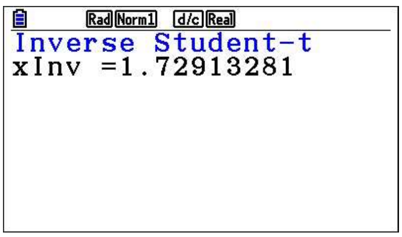


With *t*-value of -1.56 we get a *p*-value of 0.0718 which is more than $\alpha = 0.05$, so we do not reject the null hypothesis. We cannot support the magazine's claim that the ovens are not reaching the indicated temperature.

13. **(a)** $H_0 : \mu = 1.2$ (the claim) and $H_1 : \mu = 1.6$. Notice that in this case, we are not testing a general alternative hypothesis. In order to calculate the probability of type II error, we need a specific alternative one. In this case it is $H_1 : \mu = 1.6$.

(b) The probability of type I error is α . Hence, P(type I error) = 0.05

Assuming the mean to be 1.2, the critical value corresponding to an $\alpha = 0.05$ level of $H_0 : \mu = 1.2$ using invT on your GDC:



t-value (for α = 0.05) = 1.72913, thus the critical value is

critical value $\bar{x}^* = 1.2 + 1.72913 \cdot \dfrac{1.8}{\sqrt{20}} = 1.89596$

Now, find the t-value corresponding to 1.89596, assuming $H_1 : \mu = 1.6$ to be true:

$$t = \frac{1.89596 - 1.6}{\frac{1.8}{\sqrt{20}}} = 0.73532 \text{, therefore}$$

$$\beta = P(\bar{x} < 1.89596) = P(t < 0.73532) = 0.764$$

(c) Remember that the power of the test is the probability that we correctly reject a false null hypothesis and is found using $1 - \beta$.

Hence, power of the test is $1 - 0.764 = 0.236$

14. (a) $H_0 : \mu = 0.045$ (the claim) and $H_1 : \mu = 0.054$. Notice that in this case, we are not testing a general alternative hypothesis. In order to calculate the probability of type II error, we need a specific alternative one. In this case it is $H_1 : \mu = 0.054$

(b) The probability of type I error is α . Hence, P(type I error) = 0.05

Assuming the mean to be 0.45, the critical value corresponding to an $\alpha = 0.05$ level of $H_0 : \mu = 0.045$ using invT on your GDC:

t-value (for α = 0.05) = 1.796, thus the critical value is

$$\text{critical value } \bar{x}^* = 0.045 + 1.796 \cdot \frac{0.003}{\sqrt{12}} = 0.0466$$

Now, find the t-value corresponding to 0.0466, assuming $H_1 : \mu = 0.054$ to be true:

$$t = \frac{0.0466 - 0.054}{\frac{0.003}{\sqrt{12}}} = -8.54 \text{, therefore}$$

$$\beta = P(\bar{x} < 0.0466) = P(t < -8.54) = 0.00000205$$

(c) Remember that the power of the test is the probability that we correctly reject a false null hypothesis and is found using $1 - \beta$.

Hence, power of the test is $1 - 0.00000205 = 0.9998 \approx 1$

15. First, let's calculate the difference. The table becomes:

Before training	2.1	1.6	1.3	1.2	2.3	1.6	1.2	1.5
After training	1.9	1.4	1.2	1.8	2.2	1.7	1.3	1.1
Difference, *d*	0.2	0.2	0.1	–0.6	0.1	–0.1	–0.1	0.4

(a) Null hypothesis is the training makes no difference in times.

We define "d" as before – after

$H_0 : d \leq 0$ and $H_1 : d > 0$ (the claim).

(b) Use your GDC with 1-sample *t*-test. We get a *p*-value of 0.589 (3 s.f.). Hence, we do not reject the null-hypothesis, and so, claim that we do not have enough evidence to say that the program reduces reaction time.

16. First, let's calculate the difference. The table becomes:

Before	142	150	140	145	152	140	146	148
After drug use	131	140	144	147	142	130	150	142
Difference, *d*	11	10	–4	–2	10	10	–4	6

(a) Null hypothesis is the experimental drug makes no difference in times.

We define "*d*" as before – after $H_0 : d \leq 6$ and $H_1 : d > 6$ (the claim).

(b) Use GDC and run a 1-sample *t*-test. We get a *p*-value of 0.708 (3 s.f.).

Hence, we do not reject the null-hypothesis. We do not have enough evidence to claim that the experimental drug reduces the systolic blood pressure by at least 6 mm Hg.

17. First, let's calculate the difference. The table becomes:

Without music	42	35	53.4	49.2	31.2	26.4	50.2	35	40.2	45.2
With music	41.4	39.2	55.2	51	33.2	26	48	27.2	38.4	47
Difference, *d*	–0.6	4.2	1.8	1.8	2	–0.4	–2.2	–7.8	–1.8	1.8

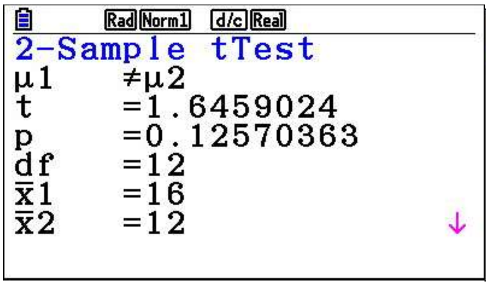
(a) Null hypothesis is the experimental drug makes no difference in times. We define "*d*" as before – after

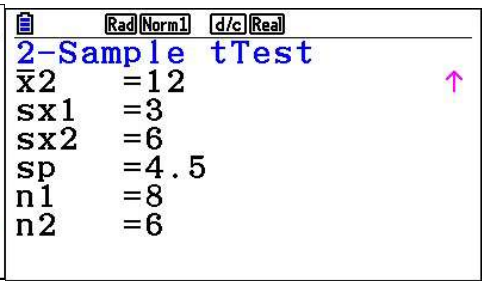
$H_0 : d \geq 0$ and $h_1 : d < 0$ (the claim).

(b) Run a 1-sample *t*-test. We get a *p*-value of 0.544 (3 s.f.). Hence, we do not reject the null-hypothesis. We do not have enough evidence to claim that music in the background improve typing speed.

18. $H_0: \mu_1 - \mu_2 = 0, H_1: \mu_1 - \mu_2 \neq 0$

As we have two different samples, we will use a 2-sample *t*-test with pooled data as the variances can be considered equal. On a GDC, we use a 2-Sample *t*-test function with the statistics given.

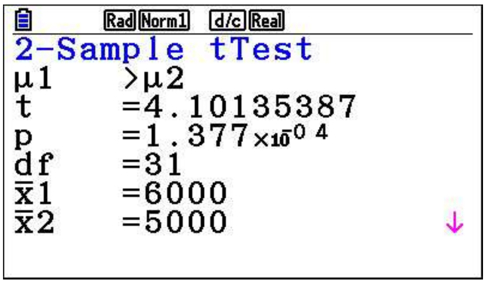




As the *p*-value = 0.126 (more than $\alpha = 0.05$), we do not reject the null-hypothesis and conclude that we do not have sufficient evidence that there is a difference in the battery life of the 2 brands.

19. $H_0: \mu_{\text{men}} \leq \mu_{\text{women}}$ and $H_0: \mu_{\text{men}} > \mu_{\text{women}}$ (the claim).

As we have two different samples, we will use a 2 sample *t*-test with pooled data as the variances can be considered equal. On a GCD, we use the 2-Sample t Test function with the statistics given.



The *p*-value is 0.000377 which is smaller than 0.05. We reject the null hypothesis and claim that we have statistical evidence that men's salaries are higher than women's salaries.

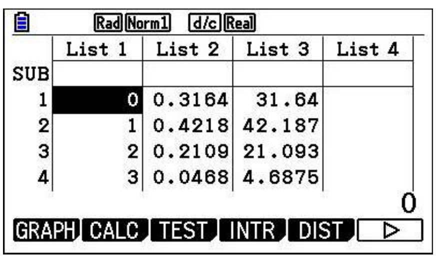
20. This is a GOF test. H_0: the observed model fits the expected binomial distribution

H_1: the observed model does not fit the indented model.

To calculate the expected frequencies, use a GDC.

Put your possible number of cards – 0 – 5 into a list, then use the binomial pdf command to multiply this list by 100 and store into a third list.

A sample is shown here.



SUB	List 1	List 2	List 3	List 4
1	0	0.3164	31.64	
2	1	0.4218	42.187	
3	2	0.2109	21.093	
4	3	0.0468	4.6875	

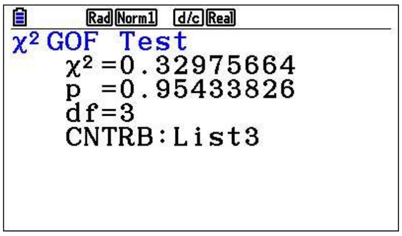
A table of expected and observed frequencies is

Outcomes	0	1	2	3	4	**Total**
Observed	30	45	20	4	1	**100**
Expected	31.64	42.19	21.09	4.69	0.39	**100**

As we cannot have expected values less than 5, we combine the last 2 columns

Outcomes	**0**	**1**	**2**	**3-4**	**Total**
Observed	30	45	20	5	100
Expected	31.64	42.19	21.09	5.08	100

Use GDC GOF test



The p-value is 0.954 which is obviously larger than 0.05. We fail to reject the null hypothesis.Thus, we support the claim that the observed data fits the intended outcome at the 0.05 level (in this case, the binomial distribution.)

21. Similar to the previous exercise, this is a GOF test. Use a GDC to run the test.

H_0: data fits a uniform distribution

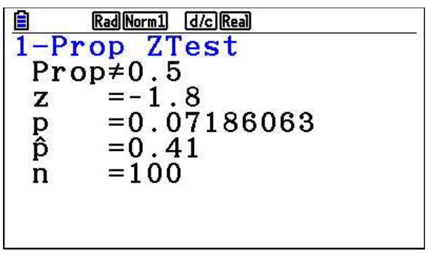
H_1: data does not fit a uniform distribution. If the distribution were uniform, the expected values would each be equal to 16. The p-value ≈ 0.328 with $df = 4$, so we do not reject the null hypothesis and conclude that there is no difference between the observed and expected values, and the distribution can be considered to be uniform at the level of significance of 0.05.

22. H_0: data fits the distribution suggested by the station

H_1: data does not fit the expected distribution.

The intended ratio 2 : 8 : 2 : 3 : 1 means the expected values are {10, 40, 10, 15, 5}.The p-value ≈ 0.768 with d.f. = 4, so there is no difference between the observed and expected values, and the distribution can be considered to match the intended ratio at the level of significance of 0.05.

23. One method is to use a 1-proportion test. For a coin to be fair the proportion of heads or tails must be equal to 0.5. $H_0 : p = 0.5$; $H_1 : p \neq 0.5$



With p-value of 0.0719, we fail to reject the null hypothesis. Thus, we can support the claim that the coin is a fair one.

24. This is a Chi-squared independence test.

H_0: There is no difference between genders in the amount of TV watched. (claim)

H1: There is a difference that cannot be attributed to chance alone. The p-value ≈ 0.109, so we do not reject the null hypothesis at the level of significance of 0.05. Gender and hours of TV watched are independent.

25. **(a)** H_0 : there is no difference between the groups and the choice of uniform

H_1 : there is a difference that cannot be attributed to chance alone. Degree of freedom $\nu = (4-1)\cdot(2-1) = 3$

Enter the observed data into a matrix and run test of independence using your GDC. p-value of the χ^2 test of independence is 0.0844 (3 s.f.), we do not reject the null-hypothesis and consider the various groups and the choice of uniform as independent.

(b) There is a notable difference in the response from MYP parents compared to the rest. Isolating that group from the others may provide a good statistical analysis of the difference. However, dealing with such a situation may require procedures beyond your present syllabus and will not be included in exam questions.

26. **(a)** The hypotheses: H_0: here is no difference in opinion regarding tax reform between voting regions. (claim)

H_1: There is a difference that cannot be attributed to chance alone. The χ^2 statistics is approximately 12.3 and the p-value ≈ 0.0547, so we do not reject the null hypothesis at the level of significance of 0.05. Opinions on tax reform are independent of the voting regions.

(b) When the Metro (core) region is compared to the other regions combined, the observed matrix becomes

$$\begin{bmatrix} 71 & 100 & 50 \\ 86 & 62 & 34 \end{bmatrix}.$$

The χ^2 statistic is approximately 9.71 and the value ≈ 0.00779, so we can reject the null hypothesis that there is no difference between the Metro (core) region's opinions compared to the others combined at the level of significance of 0.05.

27. **(a)** The hypotheses are

H1: There is no difference between the brands in their effectiveness. (claim)

H1: There is a difference that cannot be attributed to chance alone. The p-value ≈ 0.498, so we do not reject the null hypothesis at the level of significance of 0.05. There is no difference between brands and their effectiveness.

(b) When the other two are combined, the observed matrix becomes

$$\begin{bmatrix} 25 & 49 & 15 \\ 7 & 22 & 11 \end{bmatrix}$$

The p-value ≈ 0.247 so we do not reject the null hypothesis that the distributions are same.

28. **(a)** From GDC:

Unbiased estimate of μ is $\bar{x} = 2.36$, and unbiased estimate of

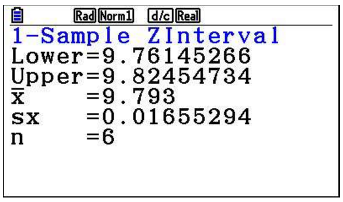
σ^2 is $s_{n-1}^2 = 33.65$

(b) **(i)** $\text{H}_0 : \mu \le 0; \quad \text{H}_1 : \mu > 0$

(ii) This is a 1-sample t test for the mean. From GDC: p-value = 0.103

(iii) Since 0.103 > 0.05. We fail to reject the null hypothesis - There is insufficient evidence at the 5% level to support the claim that extra tuition improves examination marks.

29. **(a)** Use a GDC to create a 99% confidence interval. Here is a sample. Also, since $\sigma = 0.03$ is given, we will use a Z-interval instead of a t-interval. (on exams, you will be using t-intervals)



A 99% confidence interval estimate of the mean is (9.761, 9.825).

(b) If this process is carried out a large number of times, approximately 99% of the intervals will contain the real mean μ of the population.

(c) The width of a confidence interval is given by $z_{\frac{\alpha}{2}} \cdot \dfrac{\sigma}{\sqrt{n}}$.

As requested, to half this interval, we need to find the new level, α_{new} such that

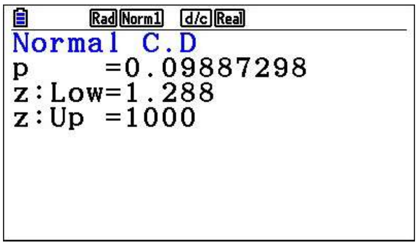
$z_{\frac{\alpha_{new}}{2}} = \dfrac{1}{2} z_{\frac{\alpha}{2}}$.

This is so, because all other parameters in the formula are still the same.

For a 99% confidence, $z_{\frac{\alpha}{2}} = 2.576$. Thus $z_{\frac{\alpha_{new}}{2}} = 1.288$.

In order to find $\frac{\alpha_{new}}{2}$, we find the area under the standard normal distribution to the right of 1.288, multiply it by 2, and subtract the result from 1. Alternatively, you can use the “solver”.

A sample output:



Thus, the new level is $1-2\times 0.09887298 \approx 0.80$, and therefore the new confidence level is 80%.

Exercise 19.1

1. **(a)** Precipitation is the explanatory variable; autism prevalence rate is the response variable.

(b) No; no matter how strong the association is, it does not prove that precipitation causes autism, only that they are associated.

2. **(a)** No association.

(b) Strong nonlinear association, possibly quadratic.

(c) Nearly perfect negative linear association.

(d) Strong positive linear association.

(e) Moderate negative linear association.

(f) No association.

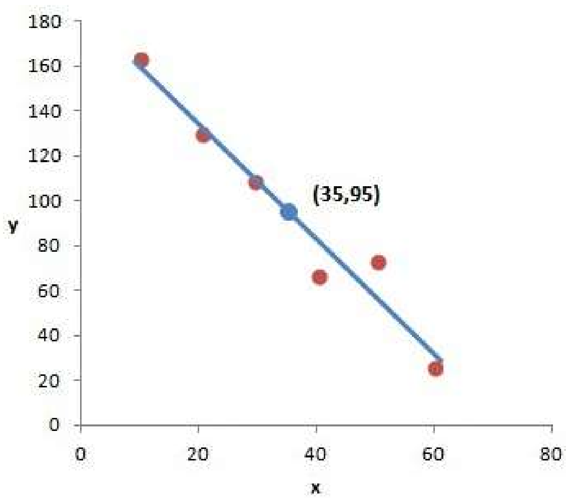
(g) Strong positive nonlinear association, possibly exponential.

(h) Mostly strong positive linear association, but a cluster of outliers is a departure from the major pattern.

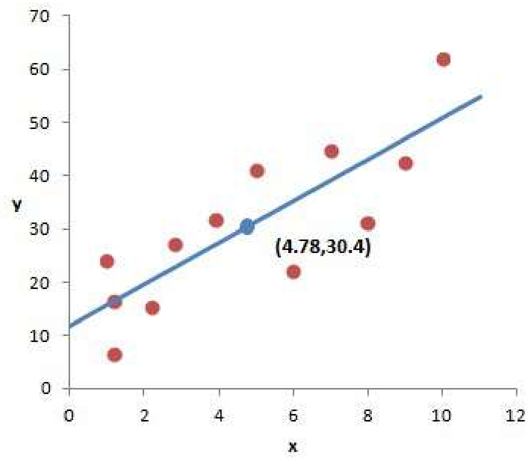
(i) Very strong negative linear association with one outlier.

3. First, we find the mean of x and the mean of y, and make sure that any line we find must contain the point $(\bar{x}, \bar{y})$. Remember that all numbers listed are approximate and may contain some rounding.

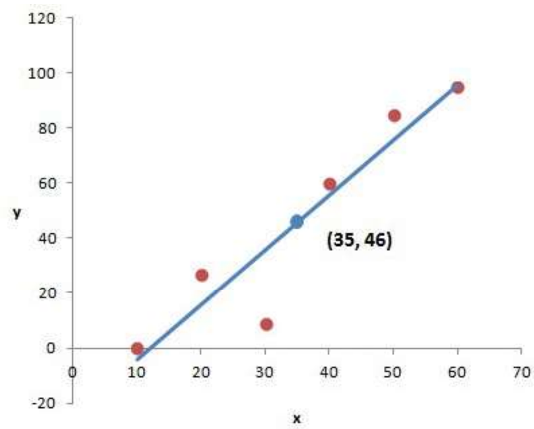
(a) The best fit line is approximately $y = -2.6x + 185$



(b) The best fit line is approximately $y = 4x + 12$



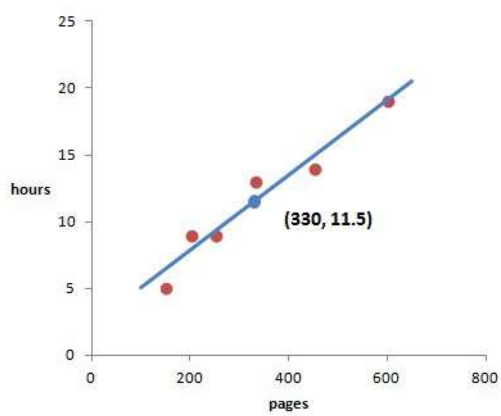
(c) The best fit line is approximately $y = 2x - 24$



4. **(a)** Number of pages is the explanatory variable, hours to finish is the response variable.

(b) Scatter diagram is below. The association is strong, positive, and approximately linear.

(c) The best fit line is approximately $t = 0.03n + 2.22$

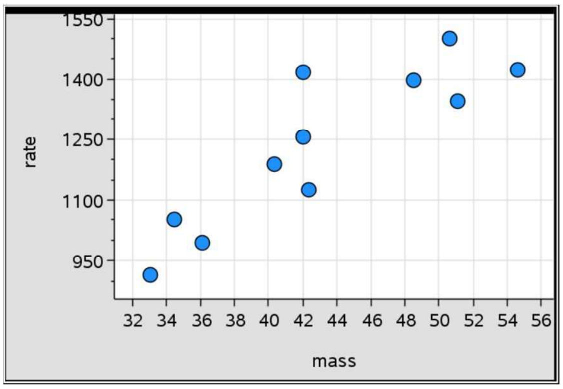


(d) The gradient of 0.03 indicates that each additional page adds about 0.03 hours to the reading time, or an additional 100 pages adds 3 hours. The *t*-intercept of 2.22 indicates that a book with zero pages will take 2.22 hours to read:

In general, and since our closest number to zero is 150, trying to interpret the intercept is not an appropriate practice. In cases, where we have included in our data zero or being close to it, then this may indicate the extra time it takes to get the book and find the last page and any other task that is not related to the number of pages.

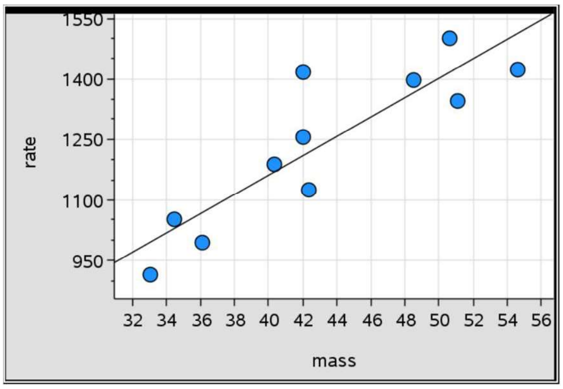
(e) $t = 0.03 \times 500 + 2.22 \approx 17.2$. About 17 hours.

5. **(a)** The explanatory variable is mass.



(b) Using a GDC: $\bar{M} = 43.2$, $\bar{R} = 1240$

(c) A possible best-fit line is shown below. (Using a GDC)



(d) There appears to be a strong, positive, linear correlation. The gradient of the line is about 24, which indicates that metaboloic rate increases by about 24 units for each additional kg in mass.

(e) Estimate from the graph: about 1160.

(f) Kevin's mass is beyond the range of the observed data. If we predict his metabolic rate using this model, it would be an extrapolation, whose results may not be reliable.

6. **(a)** may be correct because the association appears to be strongly positive.

(b) is not since the data do not lie exactly on a line.

(c) is correct because if we sketch the line, the points will be tightly spread around it.

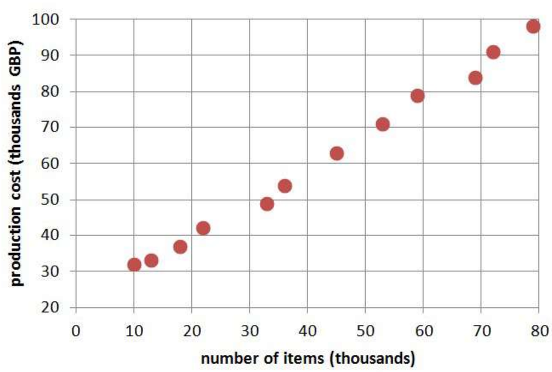
(d) is obviously false since there is a positive association.

(e) is false because with correlation close to 1, the points will fall on the line of regression almost exactly, which is not the case here.

(f) is obviously false as it deals with data way beyond the present data.

7. **(a)** Number of items produced is the explanatory variable; production cost is the response variable.

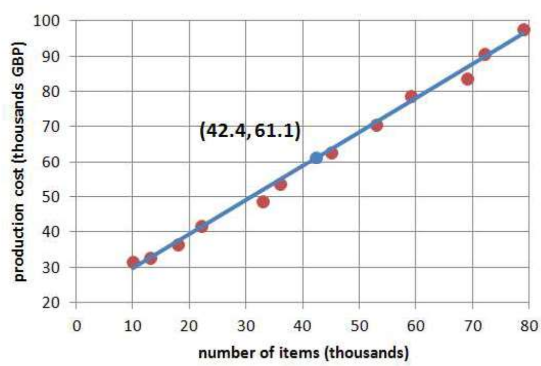
(b) A scatter diagram is shown below.



(c) Strong, positive, approximately linear, no outliers visible.

(d) Using a GDC: $(\bar{x}, \bar{y}) = (42.4, 61.1)$

(e) A line of best fit must contain the "mean-point" shown below.



(f) Draw a vertical line at number of items 70. It intersects the line of best fit at approximately 88 thousand GBP.

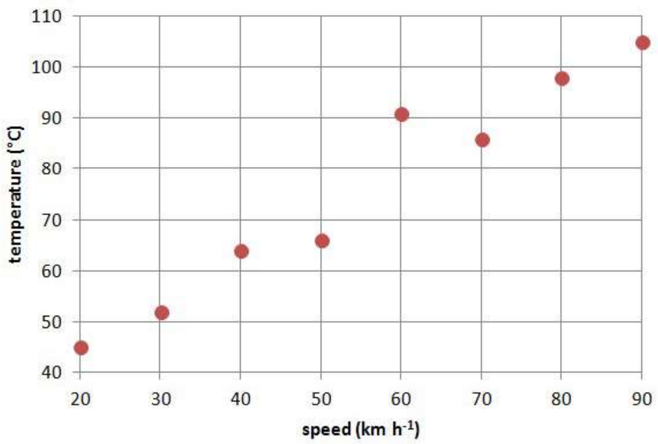
(g) We notice on the best fit line that for every 10 units of horizontal change there is about 10 units of vertical change, thus, approximate gradient is 1.0 (GDC best fit is 0.966.) This suggests that that, on average, production cost increases by about 1000 GBP for each additional 1000 items produced.

(h) 20 (approximate); this could be interpreted as the fixed costs to keep the factory operating. This is an extrapolation since out lowest number of items is 18000, which is very far from zero. Thus, in principle such an interpretation is not appropriate.

(i) Predicting production cost for 100 thousand items would be an extrapolation. It is about 20 thousand items beyond our highest production. Such prediction is not reliable.

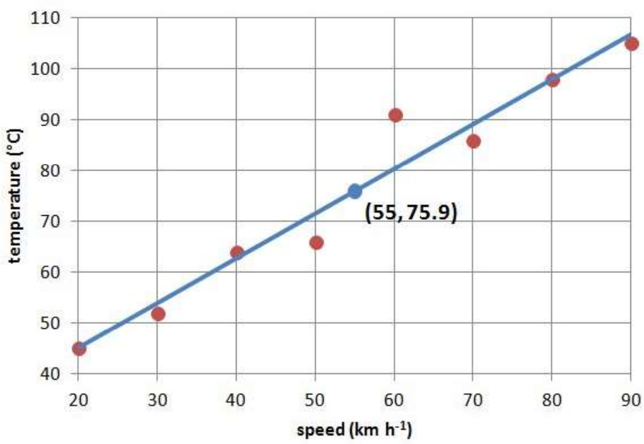
8. **(a)** Speed is the explanatory variable; temperature is the response variable.

(b) The association is strong, positive, and approximately linear.



(c) Using a GDC: $(\bar{x}, \bar{y}) = (55, 75.9)$

(d) Shown below.



(e) At 60 km h^{-1}, a vertical line intersects the line of best fit at 80 degrees. Thus, Temperature is about 80 degrees.

(f) About 0.9 (best fit is 0.882). This indicates that tire temperatures increase, on average, by about 0.9 °C for each additional km h^{-1}.

(g) about 27 (best fit is 27.4). This suggests that the initial tire temperature before being driven was about 27 °C. This is an extrapolation since the lowest speed recorded is 20 km h^{-1}, which is far from zero.

(h) A prediction from a speed of 150 km h^{-1} would be an extrapolation since the highest speed recorded is 90 km h^{-1}. 150 is way too far above the maximum.

Exercise 19.2

1. Spearman's r_s measures the strength of a **monotonic** (continually increasing or decreasing) relationship. It is not sensitive to outliers. Pearson's r measures the strength of a **linear** relationship and is sensitive to outliers. r_s uses the ranks while r uses the raw data values.

2. A: The data demonstrate a negative relationship. It is relatively weak but not the weakest, and hence it is $r = -0.40$

B: The data demonstrate an almost perfect negative association, and hence, it is $r = -0.99$

C: A strong positive association, $r = 0.90$

D: Almost no association can be observed, $r = -0.03$

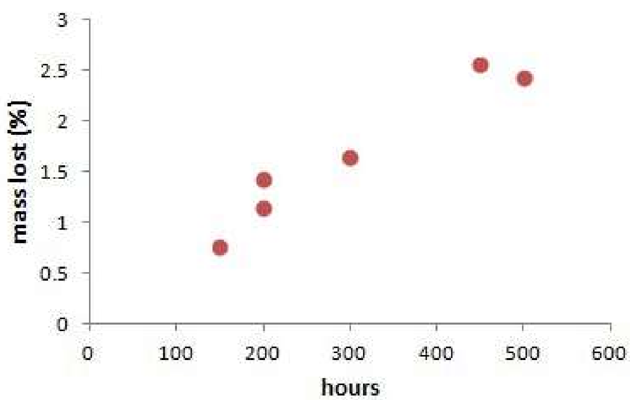
E: A positive association. Data are not very tightly spread, $r = 0.51$

F: A negative association. Data are not very tightly spread, $r = -0.58$

G: A strong negative association $r = -0.95$

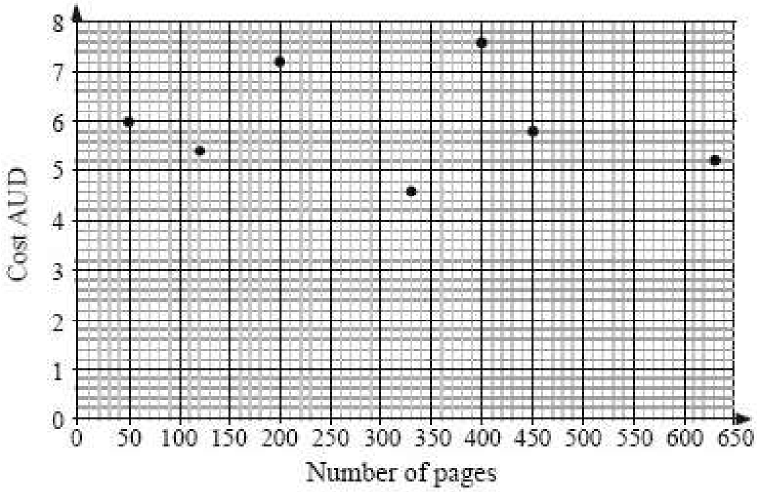
H: relatively strong positive association, $r = 0.74$

3. **(a)** Using a GDC: $r = 0.967$



(b) There is a strong positive correlation between hours in the acid bath and mass of metal lost.

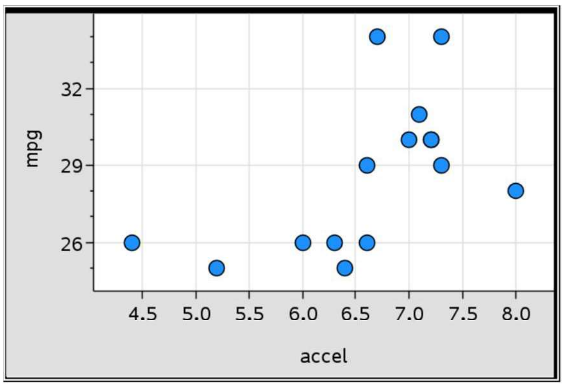
4. **(a)**



(b) Using a GDC: $r = -0.141$

(c) No, the linear correlation is too weak for reliable predictions, as seen on the scatter plot and shown by the value of r.

5. **(a)** Using a GDC: $r = 0.561$



(b) There is a moderate positive correlation between acceleration and fuel efficiency; as cars get slower the efficiency appears to increase.
It appears to be only somewhat linear, which is supported by the value of r.

(c) The ranked data is shown below.

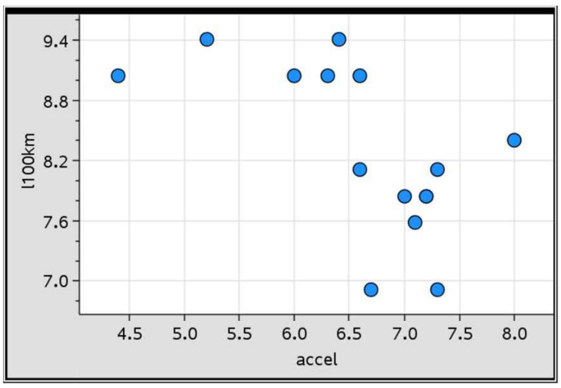
SPORTY CARS/ROADSTERS	time to 60mph rank	MPG rank
Mazda MX-5 Miata Club	8	1.5
Honda Civic Si	2	1.5
Fiat 124 Spider Lusso	6	3
Mini Cooper S	7	5
Subaru BRZ Premium	4	5
Toyota 86	4	5
Volkswagen GTI Autobahn	9	7.5
Ford Fiesta ST	2	7.5
Fiat 500 Abarth	1	9
Porsche 718 Boxster (base)	15	11.5
Subaru Impreza WRX Premium	13	11.5
Audi TT 2.0T (AT)	12	11.5
Ford Focus ST	9	11.5
BMW M235i	14	14.5
Ford Mustang Premium (2.3T, AT)	11	14.5

$r_s = 0.670$; in general, there is a moderate positive rank correlation between acceleration and fuel efficiency in MPG; as acceleration times increase MPG increases as well. Cars that accelerate slower are more efficient.

(d) Pearson's r measures the strength of a linear correlation.

Since this data appears to be somewhat nonlinear, Spearman's r_s is a more appropriate measure since it measures the strength of the monotonic association. It is also not influenced by some of the extreme values that appear to be somewhat far from the trend such as Fiat 500 or the Porsche.

6. **(a)** Using a GDC: $r = -0.598$



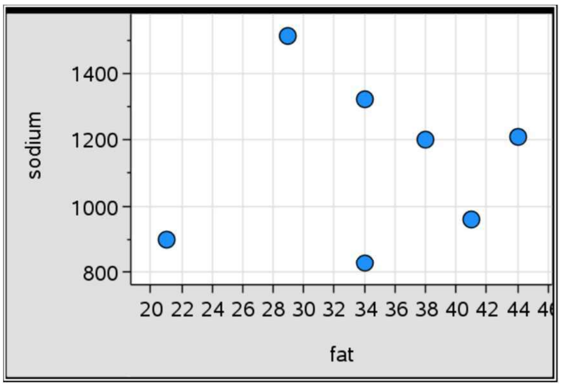
(b) There is a moderate negative linear correlation. As cars get slower, the L per 100 km required decreases. The linear correlation is slightly stronger than in the previous exercise. By converting to L per 100km, we had to find the reciprocal of the units which may have made the association slightly more linear.

(c) The ranked data is shown below.

SPORTY CARS/ROADSTERS	time to 60mph rank	L per 100 km rank
Mazda MX-5 Miata Club	8	14.5
Honda Civic Si	2	14.5
Fiat 124 Spider Lusso	6	13
Mini Cooper S	7	11
Subaru BRZ Premium	4	11
Toyota 86	4	11
Volkswagen GTI Autobahn	9	8.5
Ford Fiesta ST	2	8.5
Fiat 500 Abarth	1	7
Porsche 718 Boxster (base)	15	4.5
Subaru Impreza WRX Premium	13	4.5
Audi TT 2.0T (AT)	12	4.5
Ford Focus ST	9	4.5
BMW M235i	14	1.5
Ford Mustang Premium (2.3T, AT)	11	1.5

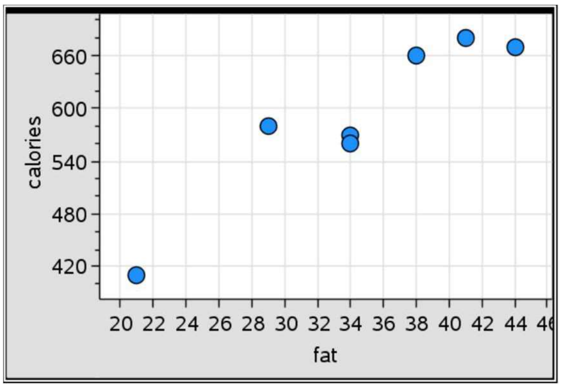
(d) $r_s = -0.670$; in general, there is a moderate negative rank correlation between acceleration and fuel efficiency in L per 100 km; as acceleration times increase, L per 100 km decreases. Cars that accelerate slower are more efficient in general. The value of r_s has the opposite sign from the previous exercises as the direction of correlation has reversed.

7. **(a)** Using a GDC: $r = 0.0852$



(b) There is almost no association between fat and sodium for these menu items.

8. **(a)** Using a GDC: $r = 0.940$



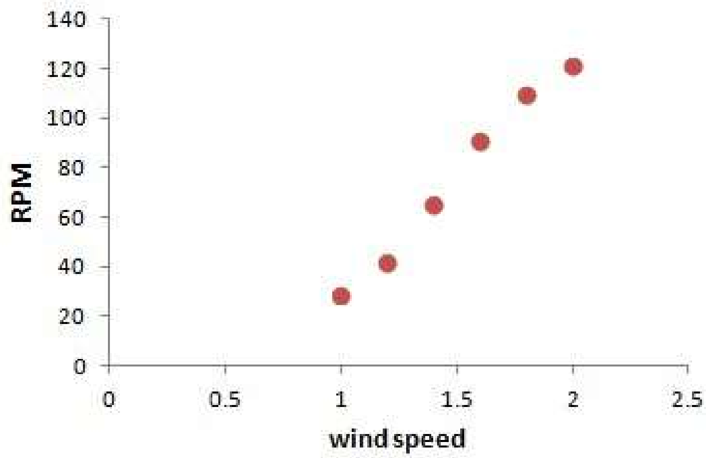
(b) There is a very strong positive linear correlation between fat and calorie content for these menu items.

9. **(a)** While a has the strongest linear correlation, we should not make any conclusions without looking at a scatter diagram. It could be that the design of the experiment causes outliers in variable a which are causing the relatively strong linear correlation.

(b) Without knowing more about the nature and design of the experiment, we cannot conclude that changes in a **cause** changes in S. However, if Ian deliberately manipulated the values of a and measured S, we may have strong evidence for causality (we would also need to suspect an underlying relationship between the two variables).

10. Using a GDC

(a) $r = 0.993$

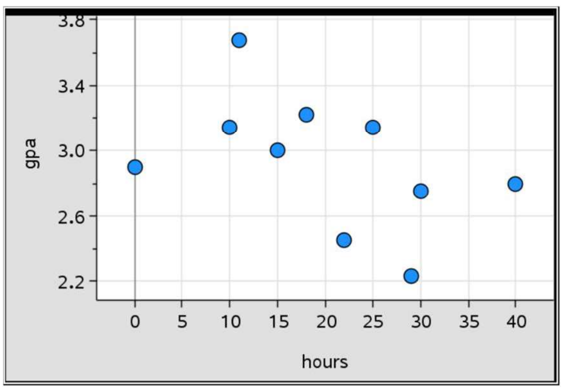


(b) There is a strong positive linear correlation between wind speed and RPM.

(c) The scatter diagram appears to have a slight curve to it and it is reasonable to suspect that the force applied by wind may be nonlinear. A logistic or power model may be more appropriate.

11.

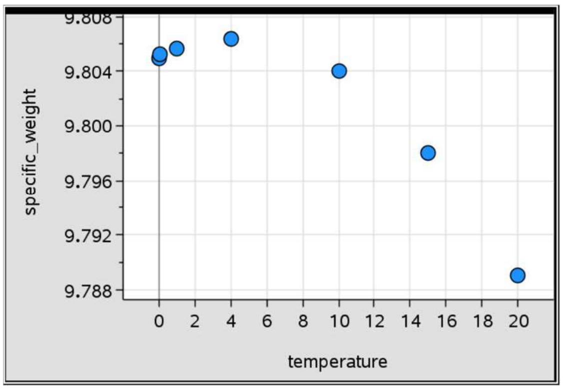
(a) The association appears moderate, negative, and approximately linear.



(b) $r_s = -0.533$; there is a moderate negative rank correlation between GPA and hours worked.

(c) $r = -0.461$; since the form of the association is approximately linear with no strong outliers, r is relatively close to r_s.

12. **(a)** The association is generally negative but is nonlinear with a possible quadratic form.



(b) $r_s = -0.643$; there is a moderate negative rank correlation between temperature and the specific weight of water.

(c) The form is non-monotonic.

Exercise 19.3

1. $R^2 = 0.36$

2. $R^2 = 0.64$

3. $r = \pm 0.9$

4. $r = -0.7$

5. Model B since the spread of data is less than in model A.

6. **(a)** True

(b) False: although the models both fit the data equally, we would need to examine the data visually and in context to determine which model is more appropriate.

7. (a) Using a GDC: $L = 0.00475h + 0.242$. On average, each additional hour increases mass loss by 0.00475%. The L intercept of 0.242 is not meaningful in this context (it is an extrapolation, and it suggests a mass loss of 0.242% for 0 hours).

(b) $0.00475 \times 100 = 0.475\%$

(c) $R^2 = 0.935$. The LSRL predicts 93.5% of the *variation* in mass loss using time in bath as the explanatory variable. Alternatively, 9305% of the *variation* in mass loss can be explained by variation in time in bath.

(d) $L = 0.00475 \times 400 + 0.242 = 2.14\%$

(e) Although the value of R^2 is relatively close to 1, the LSRL we generated should only be used to predict mass loss from hours in the acid bath. The model was developed using the time in bath as an independent variable. Using the model 'in reverse' makes the mass lost independent and so, lowers our confidence in the prediction significantly.

8. **(a)** Using a GDC: $E = 1.85t + 16.4$. On average, for each additional second in 0-60 mph time, fuel efficiency increases my 1.85 miles gal^{-1}. The E-intercept of 16.4 is not meaningful in this context; it is an extrapolation and a car would have to have instantaneous acceleration to 60 mph^{-2}! That is, it has to accelerate from 0 to 60 in 0 seconds!

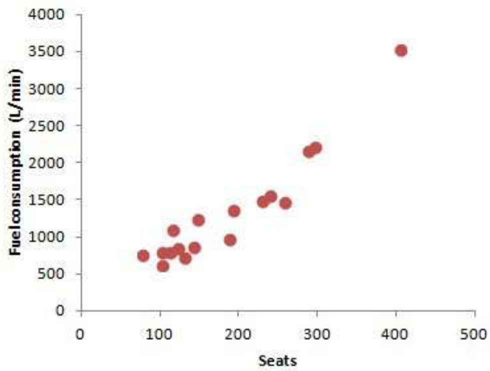
(b) $R^2 = 0.315$. The LSRL predicts 31.5% of the variation in efficiency using 0-60 mph time as the explanatory variable. Alternatively, 31.5% of the variation in efficiency can be explained by variation in time.

(c) $E = 1.85 \times 5.7 + 16.4 \approx 26.9$ miles gal^{-1}. Since the LSRL predicts only 31.5% of the variation in efficiency, there are major other factors affecting efficiency.

(d) 9 seconds is beyond the range of the observed value of the explanatory variable; any prediction would be an extrapolation.

9.

(a) Using a GDC:



The association between the number of seats and fuel consumption is strong, positive, and approximately linear. There is one possible outlier but it appears to fit the general trend.

(b) $F = 7.87n - 126$. The fuel consumption increases by 7.87 L min^{-1} for each additional seat. The F-intercept of -126 is not meaningful in this context since a plane with 0 seats is not realistic.

(c) $R^2 = 0.898$. The LSRL predicts 89.8% of the variation in fuel consumption using the number of seat as the explanatory variable. That is, 89.8% of the variation in fuel consumption can be explained by variation in the number of seats.

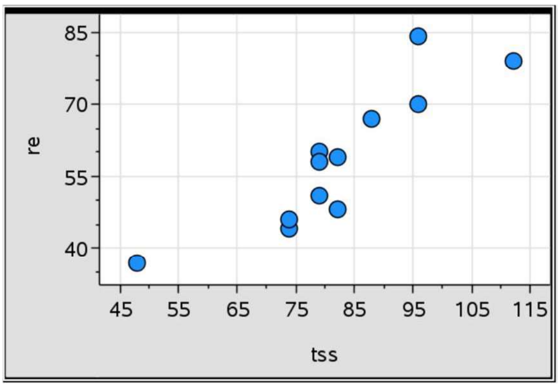
(d) $F = 7.87 \times 350 - 126 = 2628.5 \approx 2630$ L min^{-1}

(e) The data appears to have a slight nonlinear form in the scatter diagram. It would make sense that fuel consumption would increase in a faster-than-linear rate as the limits of current technology are reached, so a nonlinear model may not be the most appropriate.

10. (a) Using a GDC.

The scatter diagram shows a strong, positive, linear association.

There is a possible outlier at (48, 37).



(b) $E = 0.815(S) - 8.62$. For each additional unit increase in Training Stress Score, Relative Effort increases by 0.815 units. The E-intercept of -8.62 is not meaningful in this context since 0 stress score was not included in the collected data.

(c) $R^2 = 0.771$. The LSRL predicts 77.1% of the variation in Relative Effort using the Training Stress Score as the explanatory variable. That is 77.1% of the variation in Relative Effort can be explained by variation in Training Stress Score.

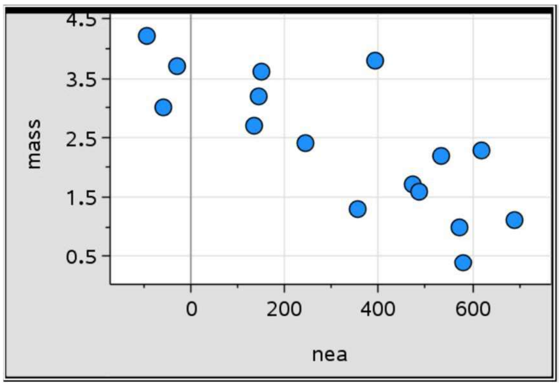
(d) $E = 0.815(60) - 8.62 \approx 40.3$

(e) This is equivalent to the range of data used in setting up the model: $48 \le S \le 112$

(f) Since the low outlier is the minimum value in the domain, if we remove the outlier, we would need to adjust the domain to $74 \le S \le 112$ and a prediction based on a Relative Effort of 60 would then become an extrapolation.

(g) Without (48,37), the LSRL is $E = 0.996(S) - 24.7$ with $R^2 = 0.756$. The LSRL has changed significantly but the value of R^2 is only slightly different, but it did not improve. In fact, there is another potential outlier at (96, 84), which, if removed may improve the model. The LSRL is $E = 0.874(S) - 15.6$ with $R^2 = 0.806$

11. **(a)** The scatter diagram shows a strong negative approximately linear association between change in NEA and change in mass.



(b) $M = -0.00344N + 3.51$. On average, for each additional 100 calories in change of non-exercise activity, change in mass decreases by 0.344 kg.

The M-intercept suggests that when non-exercise activity does not change, mass will increase by 3.51 kg.

(c) $R^2 = 0.606$. The LSRL predicts 60.6% of the variation in change in mass using change in non-exercise activity as the explanatory variable.

(d) $M = -0.00344 \times 200 + 3.51 \approx 4.20$ kg

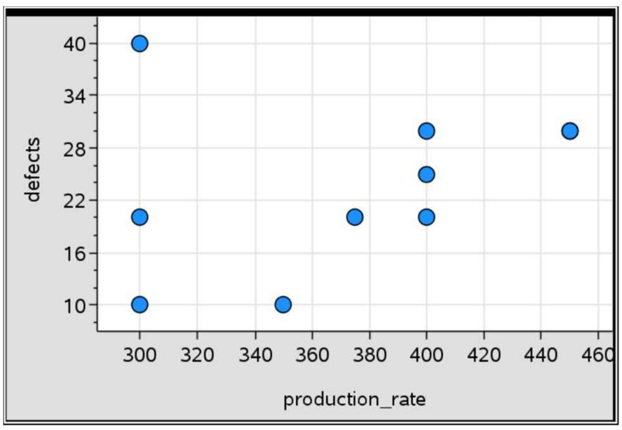
(e) A negative change in NEA suggests that the individual did less non-exercise activity after being overfed.

(f) $-94 \le N \le 690$

(g) The change in mass for $N = 1100$ would be negative, suggesting that the person was able to lose weight by eating more!

12. **(a)** Production Rate is the explanatory variable; Defects is the response variable.

(b) There appears to be a weak positive association between production rate and defects. However, an outlier at (300,40) may be affecting the strength of the association.

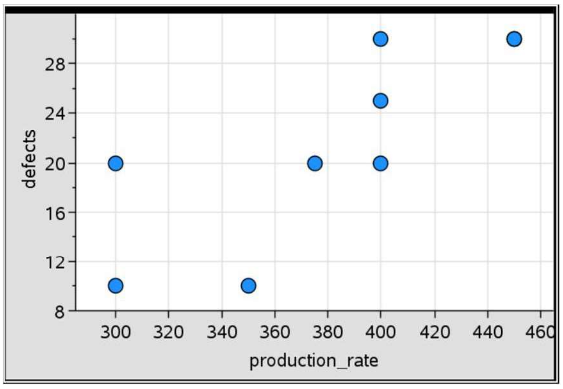


(c) $D = 0.0479R + 5.67$; The number of defects increases by about 5 for each 100 unit increase in production rate. The D-intercept of 5.67 is not meaningful in this context.

(d) $R^2 = 0.0872$; the LSRL predicts 8.72% of the variation in the number of defects using production rate as the explanatory variable.

(e) The low value of R^2 suggests that there are more significant factors determining the variation in the number of defects. It is better to try to find other factors instead.

(f) There appears to be a moderate positive approximately linear association between production rate and defects .



(g) $D = 0.113R - 21.3$; The number of defects increases by about 11 for each 100 unit increase in production rate. The D-intercept of -21.3 is not meaningful in this context.

(h) $R^2 = 0.630$; the LSRL predicts 63.0% of the variation in the number of defects using production rate as the explanatory variable.

(i) The production rate appears to be a significant predictor of the number of defects. Lower production rates should be attempted and further data can then be recorded.

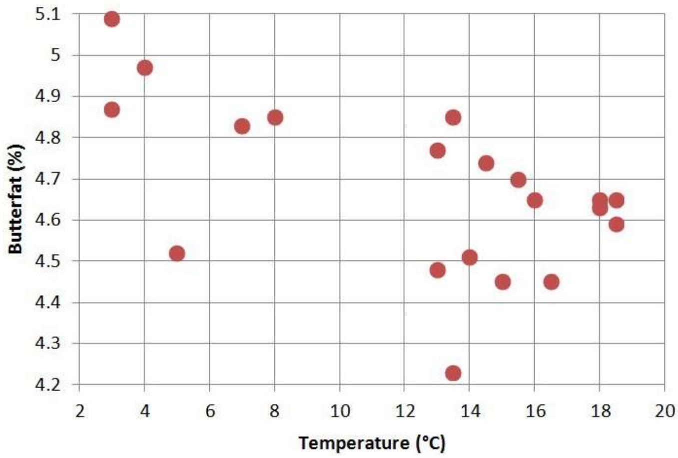
(j) It is OK to remove the outlier to investigate how influential it is. However, we must be careful to not put too much confidence in our predictions and recommendations using the revised data. Instead, we must investigate the outlier and see what caused it: was it a particularly unskilled worker? Mis-entered data? A power failure during production or other failure? Something else? If there are no unusual causes for the outlier, we should not remove it in our final analysis.

13. **(a)** There is a strong positive association between year and millions of active monthly users. The form appears nonlinear or piecewise linear.

(b) Suitable domains would be $0 \le y < 5$ and $5 \le y \le 8$

14. **(a)** Temperature is the explanatory variable, butterfat is the response variable.

(b) The association appears to be negative and moderate and approximately linear.



There is a gap in the data; why are there no data values for temperatures in the interval $8 < T < 13$? We should proceed with caution.

(c) The LSRL is $F = -0.0216T + 4.94$. The butterfat content decreases by 0.02% for each increase of 1 °C. The F-intercept of 4.94 indicates that at a temperature of 0 °C, we predict that the butterfat content would be 4.94% ; beware, this is an extrapolation.

(d) $R^2 = 0.318$; 31.8% of the variation in butterfat content is predicted by the LSRL using temperature as the explanatory variable.

(e) Correlation is not causation: while temperature and butterfat may have a moderate correlation, we cannot claim that lowering the temperature will cause increased butterfat content. Further research is needed; an investment in a climate-controlled barn may be an expensive experiment.

Exercise 19.5

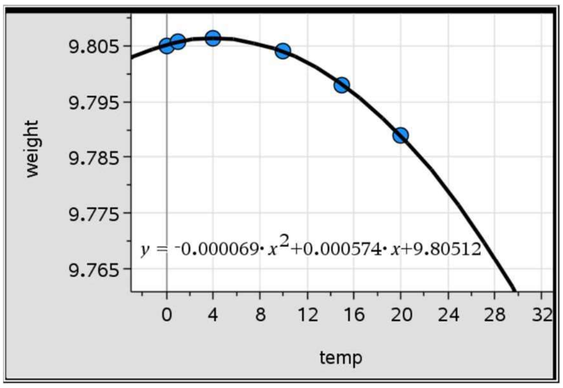
1. **(a)** log(y) vs x; exponential model

(b) y vs log(x); logarithmic model

(c) logarithmic linearisation should not be used since data appears to have a turning point

(d) log(y) vs log(x); power model

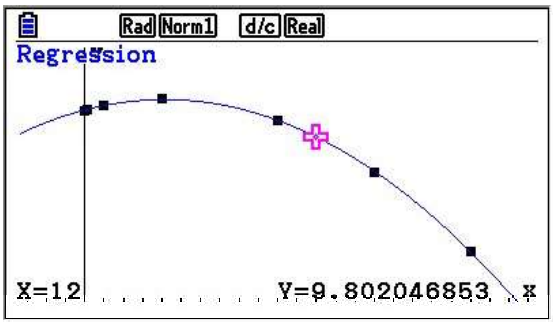
2. **(a)** The relationship is not linear. Apparently, the weight increases at first and then around 4 it starts decreasing. Apparently, the data can be modelled by a polynomial. We used a quadratic model. A scatter diagram with an appropriate model is shown below. A cubic model may also be used.



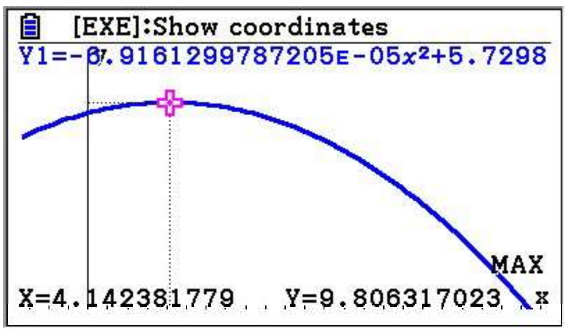
(b) The data does not appear to fit an exponential, logarithmic or power model; the data has a turning point, so linearisation with logarithms cannot be used.

(c) The data appears to have a quadratic form, so we will use quadratic regression. The quadratic model is $w = -0.000069T^2 + 0.000574T + 9.80512$. $R^2 = 0.9997$, which is a good indication of the appropriateness of the model.

(d) $w = -0.000069(12)^2 + 0.000574(12) + 9.80512 \approx 9.802$. Alternatively, we can find the predicted using our GDC

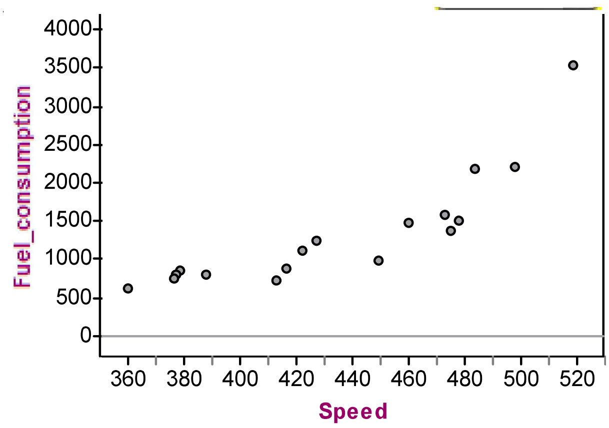


(e) According to our model, the maximum specific weight is 9.81 at 4.15 °C.



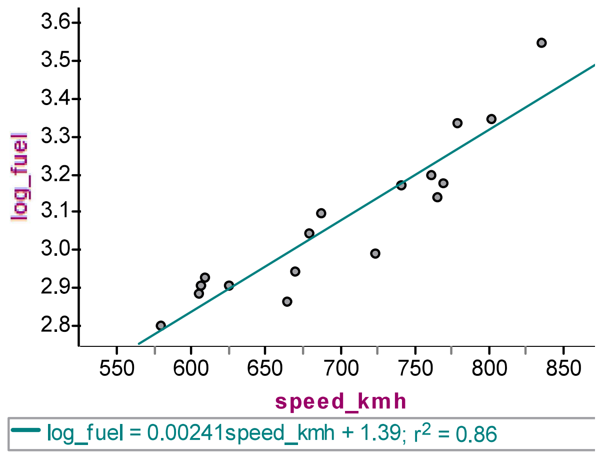
3. **(a)** The association is strong, positive, and nonlinear.

There are no dramatic outliers.



(b) The relationship appears to be of exponential nature. log(fuel) vs speed produces the best linearisation, with $\log(F) = 0.00241v + 0.192$

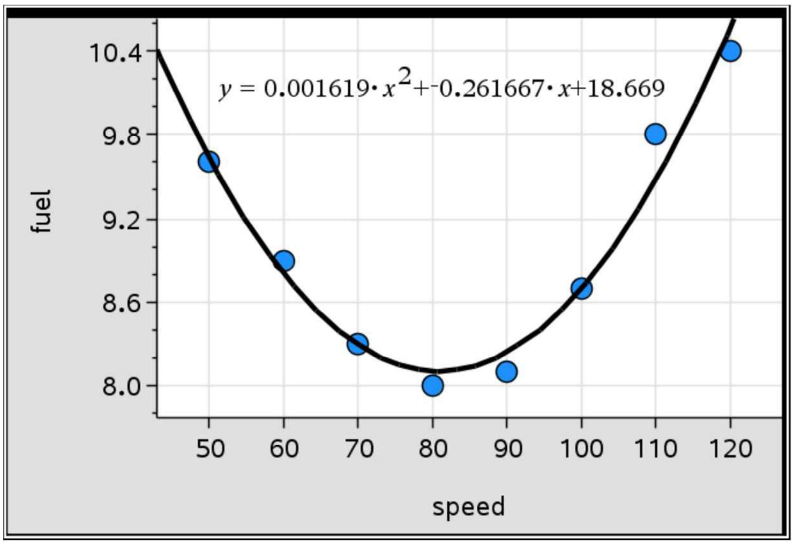
$$\Rightarrow 10^{\log(F)} = 10^{0.00241v+0.192} \Rightarrow F = 10^{0.192}10^{0.00241v} = 1.55(1.00556)^{v}.$$



(c) The model suggests that fuel consumption increases by 0.556% [1.00556-1]for each additional km h^{-1} increase in speed. The initial value (F-intercept) of 24.5 is not meaningful in this context; it is an extrapolation and it does not make sense to have fuel consumption at a speed of 0. $R^2 = .862$; 86.2% of the variation in fuel consumption is predicted by the exponential model with speed as the explanatory variable.

(d) $F = 1.55(1.00556)^{700} \approx 75.3$ L min^{-1} (rounding in intermediate steps may result in slightly different answers.)

4. **(a)** A scatter diagram with an appropriate model is shown below.



(b) The data appears to have a quadratic form. the data has a turning point, so linearisation with logarithms cannot be used. So, we will use quadratic regression.

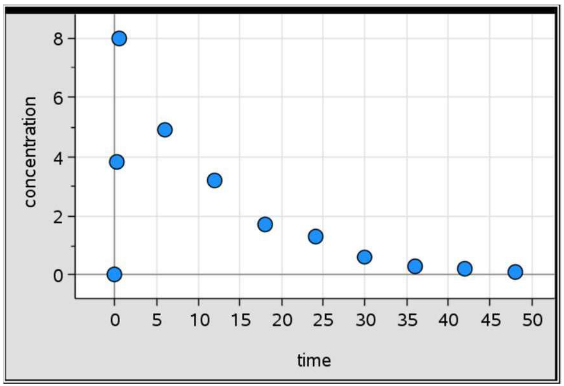
(c) The quadratic model is $C = 0.001619v^2 - 0.261667v + 18.669$.

(d) $C = 0.001619(95)^2 - 0.261667(95) + 18.669 \approx 8.42$ L per 100 km

(e) 140 km h^{-1} is outside the range of observed values, hence the prediction would be an extrapolation.

(f) According to our model, maximum fuel efficiency (minimum fuel consumption) of 8.1 L per 100 km occurs at 80.8 km h^{-1}. This seems reasonable, although it should be noted that the observed data includes a value of 8.0 L per 100 km, which is less than the theoretical minimum fuel consumption.

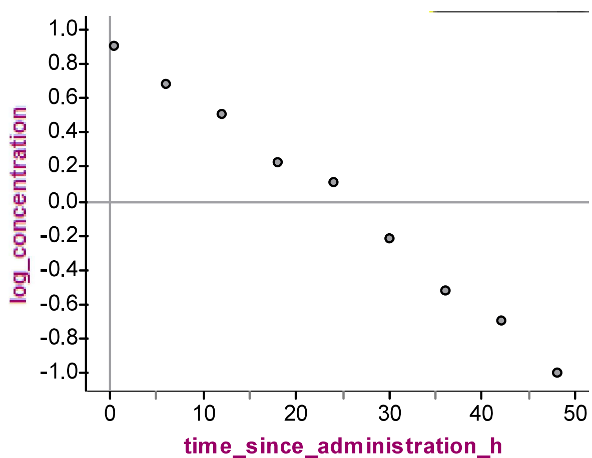
5. **(a)** The data appears to have a strong negative nonlinear form, except for the first two data points. This is probably because this is the time immediately after the drug was given, so blood concentration of the drug is increasing during this time.



The data has a clear turning point at 0.5 hours; it is increasing up until that time and decreasing after. An exponential model (decay) appears appropriate.

(b) We will use the data with $t \geq 0.5$. An exponential model appears appropriate so we will use a $\log(y)$ vs x linearisation.

(c) The linearised scatter diagram is shown below.



The linear model is $\log(C) = -0.0399t + 0.960$

$\Rightarrow 10^{\log(C)} = 10^{-0.0399t+0.960} \Rightarrow C = 9.12(0.912)^t$.

(d) $t \geq 0.5$

(e) For each additional hour, the blood concentration of this drug decreases by 8.8%.

(f) $R^2 = 0.994$; 99.4% of the variation in the blood concentration is predicted by the exponential model using time as the explanatory variable.

(g) $C = 9.12(0.912)^4 \approx 6.31$ mmol L^{-1}

(h) According to the model, at time t = 0 the amount in blood is 9.12 mmol L^{-1}

To find the half-life we solve the equation

$$\frac{9.12}{2} = 9.12(0.912)^t \Rightarrow 0.912^t = \frac{1}{2} \Rightarrow t = \frac{-\ln 2}{\ln 0.912} \approx 7.52 \text{ hours}$$

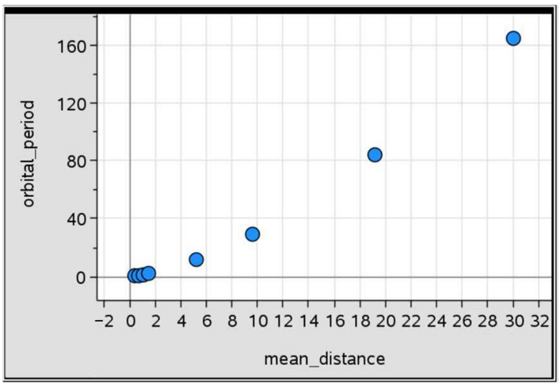
6. Estimating distances using the graph can be achieved by drawing vertical and horizontal lines from the point in question and reading where these lines meet the respective axes.

(a) Average distance from sun ≈ 800 million km, length of year ≈ 4000 days

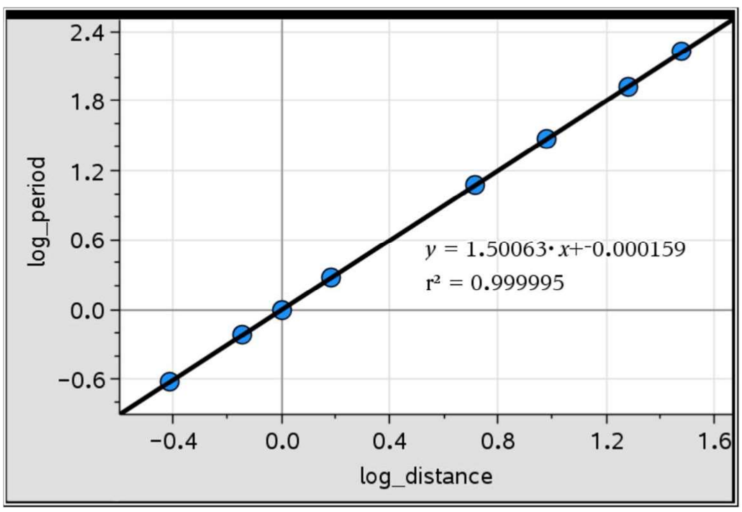
(b) Average distance from sun ≈ 60 million km, length of year ≈ 90 days

(c) A power model appears to be appropriate since the data appears linear on a log-log graph.

7. **(a)** There is a strong, positive nonlinear association between average distance from the Sun and the orbital period.



(b) The best linearisation is given by the log-log scatter diagram:



The model is $\log P = 1.50063 \log D + 0.000159 \Rightarrow 10^{\log P} = 10^{\log D^{1.500636} + 0.000159}$

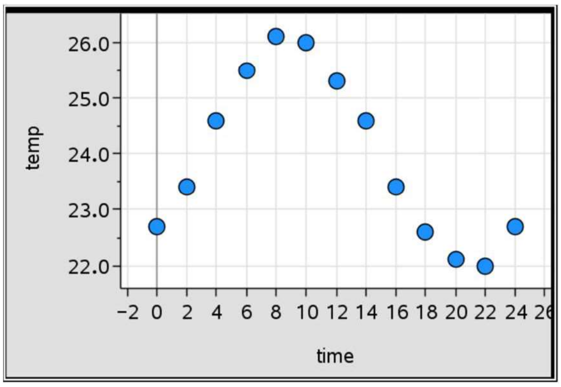
$$\Rightarrow P = 10^{0.000159} D^{1.500636} = 1.00037 D^{1.500636} \Rightarrow P \approx D^{1.5}$$

(c) The orbital period of equal to the cube of the square root ($\frac{3}{2}$ power) of the distance from the sun. With $R^2 = 1.00$ (3 s.f.), this model predict 100% of the variation in orbital period using distance from the sun as the explanatory variable.

(d) $P \approx 2.77^{1.5} \approx 4.61$ years, notice that if you use the more accurate model $P = 1.00037(2.77)^{1.500636} \approx 4.614$

(e) $\frac{4.61 - 4.60}{4.60} = 0.00217 = 0.217\%$

8. **(a)** There appears to be a strong sinusoidal association between temperature and time.



(b) The pattern appears sinusoidal/there are turning points.

Logarithmic linearisation is not appropriate in such cases.

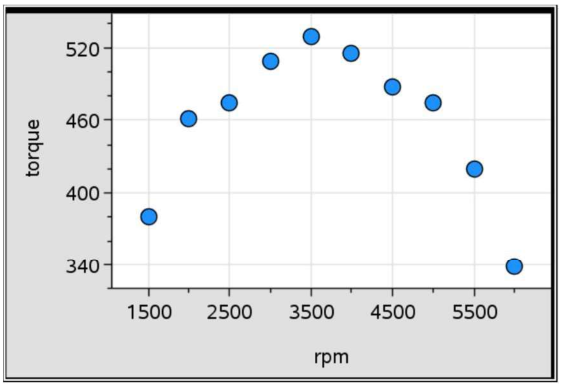
(c) Using GDC/software: $T = 2.03\sin(0.263x - 0.782) + 24.0$

(d) The average in a sinusoidal model is the main axis of the sine wave.

In this case it is 24 °C

(e) $T = 2.03\sin(0.263(17) - 0.782) + 24.0 \approx 22.9$ °C

9. **(a)** There appears to be a strong nonlinear relationship between RPM and torque.



(b) There is a turning point in the data.

(c) $N = -0.0000309r^2 + 0.223r + 122$

(d) Maximum torque is 523 Nm at 3610 RPM.

Chapter 19 practice questions

1. **(a)** The value of r in the interval $-1 \le r \le 1$,

(b) If the association is negative, the value of r must be negative.

(c) If life expectancy increases as body mass increases, then r_s must be positive.

(d) The gradient in the LSRL is negative, but r is positive.

2. **(a)** There is no logical connection between the variables: 0

(b) The higher the taxes, the less people would like to consume: –1

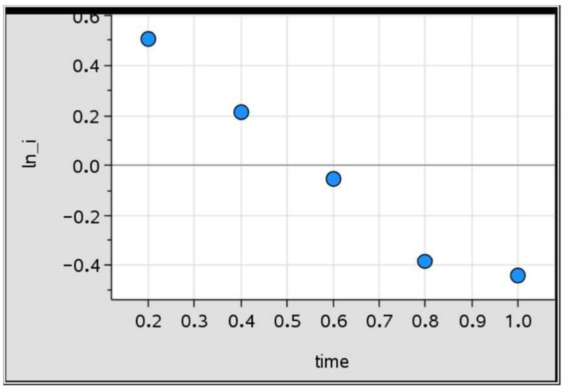
(c) As children grow, their weight would increase +1

(d) The faster the car goes, the more difficult it becomes to stop! +1

(e) No real connection: 0

(f) The lower the temperature, the more we need to heat: –1

3. **(a)** The scatter diagram shows a strong, negative, approximately linear association, with $r = -0.984$.

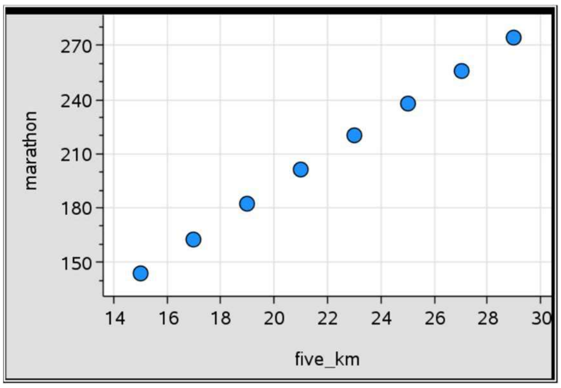


(b) GDC: $\ln(I) = -1.25t + 0.719$

(c) $R^2 = 0.968$; 96.8% of the variation in $\ln(I)$ is predicted by the LSRL using time as the explanatory variable.

(d) $\ln(I) = -1.25t + 0.719 \Rightarrow e^{\ln(I)} = e^{-1.25t+0.719} \Rightarrow I = 2.05e^{-1.25t}$; $I_0 = 2.05$, $k = -1.25$

4. **(a)** The scatter diagram shows a nearly perfect, positive linear correlation with $r = 1.00$



(b) $M = 9.30T + 5.20$. For each additional minute in 5 km time, we estimate an additional 9.3 minutes in marathon time. The M intercept of 5.20 is not meaningful in this context.

(c) $M = 9.30(20) + 5.20 = 191$ minutes

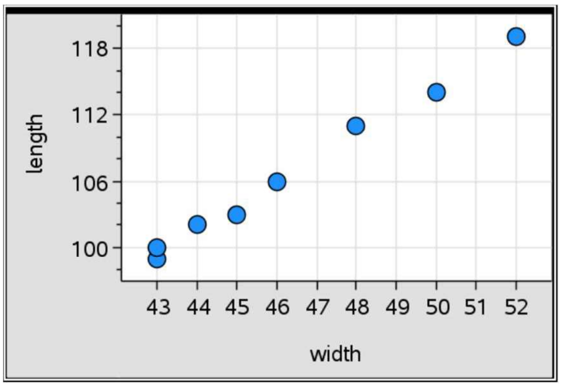
(d) **(i)** The slope gradient be $\frac{42.195}{5} = 8.44$ min min^{-1}.

(ii) The model gradient from part **(b)** is 10% more than this theoretical gradient.

(iii) We see that, on average, runners' pace is about 10% slower on a marathon than on a 5 km run.

5. Using a GDC

(a) There is a very strong, positive, linear correlation between leaf width and length, with $r = 0.997$.



(b) $L = 2.15W + 6.85$. For each additional mm in width, we estimate an additional 2.15 mm in length. The L intercept of 6.85 is not meaningful in this context.

(c) $43 \le W \le 52$

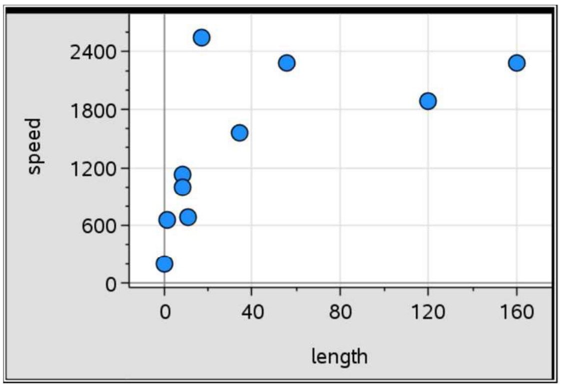
(d) $L = 2.15(47) + 6.85 \approx 108$ mm

(e) 60 mm is outside the domain for our model/it would be an extrapolation.

6. No, the study was observational and so did not control alcohol intake and then observe responses. There may be other factors that are linked with alcohol intake that reduce the risk of cardiovascular disease.

7. **(a)** There is a positive association. It appears somewhat nonlinear. It increases first and tends to level for larger numbers.

There appears to be an outlier at (17, 2550) – the Common swift.



(b) There is one outlier (Common swift). Assuming the data is correct, we cannot justify removing this data point simply because it doesn't 'fit.'

(c) $r = 0.620$. There is a moderate positive linear correlation between body length and flying speed.

(d) $r_s = 0.820$. There is a strong positive rank correlation between body length and flying speed.

(e) The data appears to have a slight nonlinear curve, so Spearman's r_s is more appropriate. However, the data is approximately linear so tis is also acceptable to measure the strength of the linear association with Pearson's *r*.

(f) A log-log linearisation appears to produce the most linear pattern; the model is

$\log v = 0.354 \log L + 2.666$

$\Rightarrow 10^{\log v} = 10^{0.354 \log L + 2.666} = 10^{2.666} L^{0.354}$

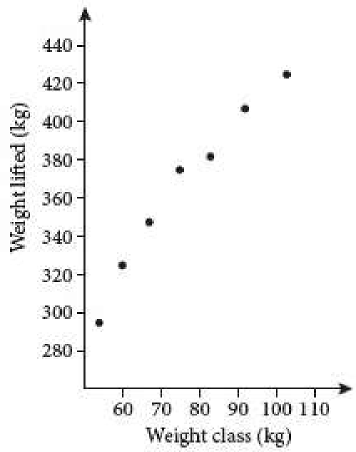
$\Rightarrow v \approx 463 L^{0.354}$

(g) $R^2 = 0.818$; About 81.8% of the variation in flying speed is accounted for by the power model with body length as the explanatory variable.

(h) $v = 463(130)^{0.354} \approx 2594$ cm s^{-1}

8. Using a GDC

(a) There appears to be a strong, positive association between weight class and weight lifted. The association is linear or slightly nonlinear.



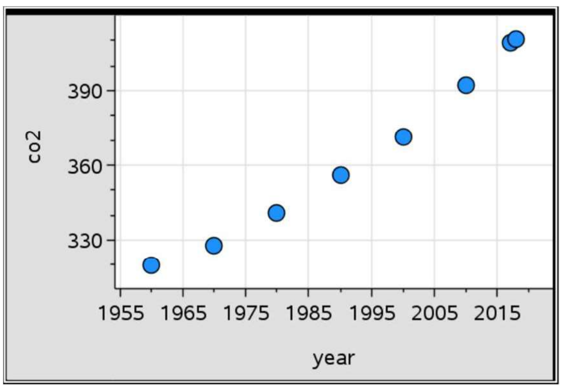
(b) $m = 464\log(w) - 510$

(c) $R^2 = 0.986$; 98.6 of the variation in weight lifted is predicted by the model using weight class as the explanatory variable.

(d) **(i)** Taner Sagir performed better than the model would predict; he lifted 375 kg but the model predicts he would lift 366 kg for a difference of +9 kg.

(ii) Halil Mutlu performed worse than the model would predict; he lifted 295 kg but the model predicts he would lift 302 kg for difference of – 7 kg.

9. **(a)** There appears to be a strong, positive association, approximately linear but with some evidence of an nonlinear association, with no outliers.



(b) $r = 0.991$, there is a strong positive linear correlation between year and CO_2 concentration.

(c) (i)

Year	0	10	20	30	40	50	57	58
CO_2 concentration (ppm)	320	328	341	356	372	392	409	411

(ii) $r = 0.991$, there is a strong positive linear correlation between years since 1960 and CO_2 concentration. The correlation coefficient has not changed by subtracting 1960 from each date value.

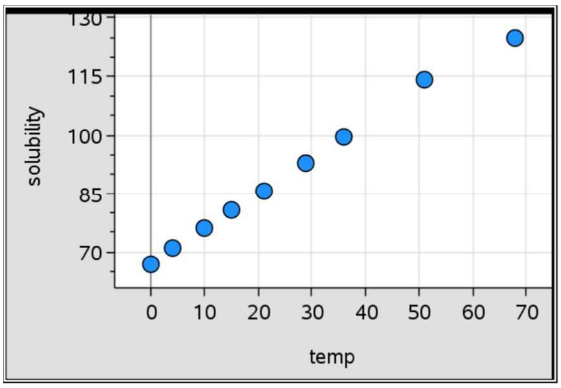
(iii) $C = 1.62y + 313$. The slope of 1.62 ppm year^{-1} indicates that CO_2 concentration is increasing by 1.62 ppm for each additional year. The intercept of 313 indicates the predicted CO_2 concentration in 1960.

(iv) years from 1960 to 2018

(v) $C = 1.62(45) + 313 = 386$ ppm

(vi) The year 2050 is beyond the range of the observed data; it would be an extrapolation.

10. (a) There appears to be a nearly perfect positive linear association between temperature and solubility of $NaNO_3$.



(b) $r = 0.999$, There is a nearly perfect positive linear association between temperature and solubility of $NaNO_3$.

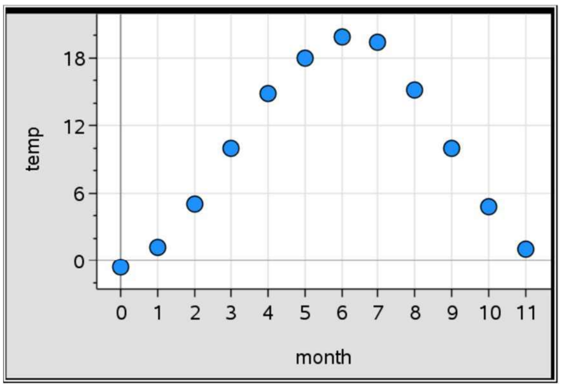
(c) $S = 0.872T + 67.5$. The gradient of 0.872 g per 100 ml per °C indicates that for each additional °C, we expect an additional 0.872 g per 100 ml can be dissolved. The *S*-intercept of 67.5 indicates that at 0 °C, we expect solubility of 67.5 g per 100 ml.

(d) At $T = 25$ °C, we expect solubility of $S = 0.872(25) + 67.5 = 89.3$ g per 100 ml.

(e) 95 °C is beyond the range of observed data; it would be an extrapolation.

(f) We should not use this model to predict temperature from solubility. (Also we do not know if the solution given has the maximum amount of $NaNO_3$ dissolved for that temperature.)

11. **(a)**



(b) We know the temperature will be periodic by year and it appears to have a sinusoidal form.

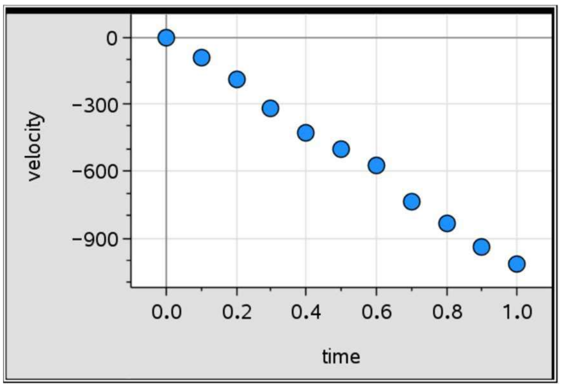
(c) $T = 10.3\sin(0.515M - 1.53) + 9.72$

(d) The average of a wave model is the axis of the graph. In this case it is 9.72 °C

(e) Remember that the period of $\sin bx$ is $\frac{2\pi}{|b|} = \frac{2\pi}{0.515} \approx 12.2$ months

(f) We know that the temperature model should have a period of 12 months.

12. **(a)** The appears to be a very strong negative linear correlation between time and velocity.



(b) $r = -0.998$. There is an almost perfect negative linear correlation between time and velocity.

(c) GDC output: $v(t) = -1040t + 7.87$

(d) The gradient of –1040 cm sec^{-2} indicates that for each additional second, velocity changes by –1040 cm sec^{-1}. The intercept of 7.87 suggests that the object was moving upward at 7.87 cm sec^{-1} at the start of the experiment.

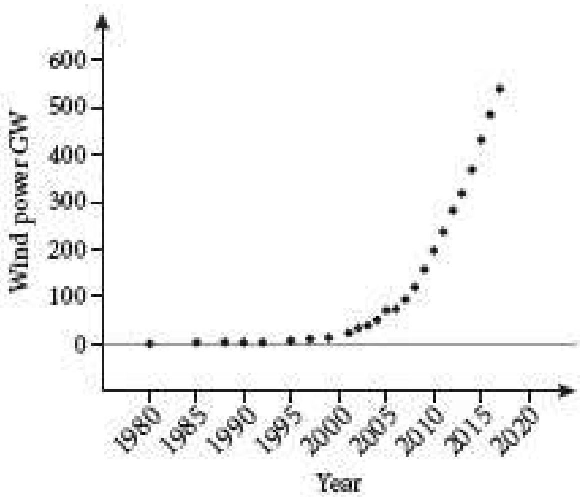
(e) $v(0.15) = -1040(0.15) + 7.87 = -148.13$ cm sec^{-1}

(f) The rate of change of velocity predicted by this experiment is –1040 cm sec^{-2}. This is greater that the possible acceleration due to gravity; most likely there are errors in measurement.

(g) $s(t) = \int v(t)\,\mathrm{d}t = -520t^2 + 7.87t + c; s(0) = 0 \Rightarrow s(t) = -520t^2 + 7.87t$

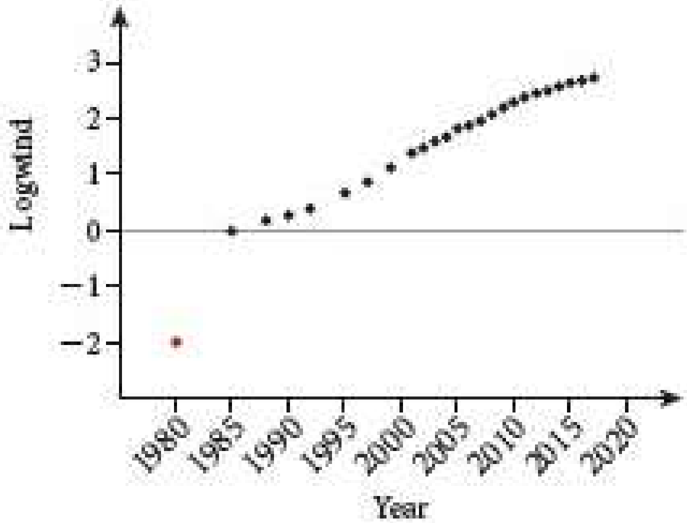
(h) $s(1) - s(0) = -520 + 7.87 \approx 512$ cm

13. **(a)** There is a strong, positive, nonlinear associate between year and global wind energy production.



(b) The data has a clear exponential pattern.

(c)



(d) Wind power production in 1980 is an outlier; wind energy increased significantly more from 1980 to 1985 than any other comparable period.

(e) $\log w = 0.2014 + 0.1004t$

$\Rightarrow 10^{\log w} = 10^{0.2014} \cdot 10^{0.1004t} = 1.59\left(10^{0.1004}\right)^t = 0.159\left(1.26\right)^t$

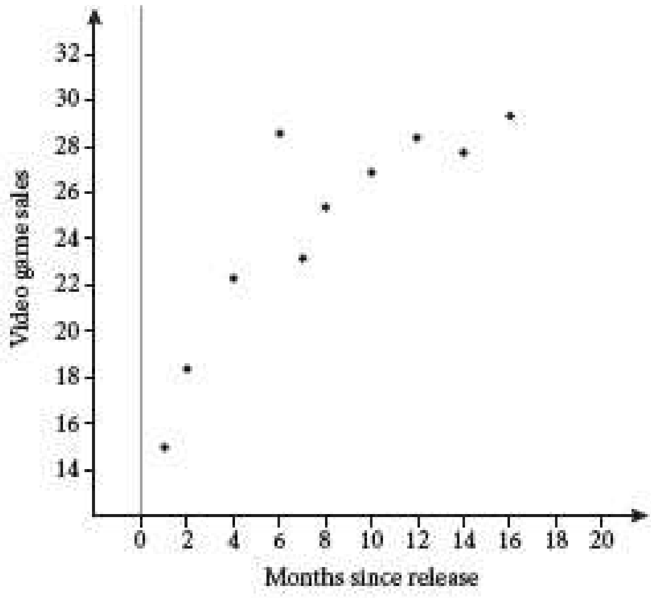
(f) Global wind energy production has increased an average of by 26% per year since 1985.

(g) $R^2 = 0.959$; 95.9% of the variation in global wind power production is predicted by the model using year as the explanatory variable.

(h) Considering 0.159 as the initial production because it corresponds to $t = 0$, then double this is 0.318.

$$0.318 = 0.159 \cdot 1.26^t \Rightarrow 2 = 1.26^t \Rightarrow t = \frac{\log 2}{\log 1.26} = 2.999 \approx 3 \text{ years}$$

14. **(a)**



(i) At month 6 (December), there appears to be an outlier. Since it is greater than the surrounding months, it may be increased sales during the holidays.

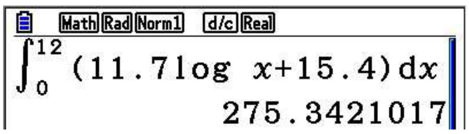
(ii) There appears to be a logarithmic pattern to the data.

(b) $S = 11.7\log(m) + 15.4$. Logarithmic growth decreases over time, which makes sense in this context: we expect sales to increase rapidly and then the rate of growth should slow.

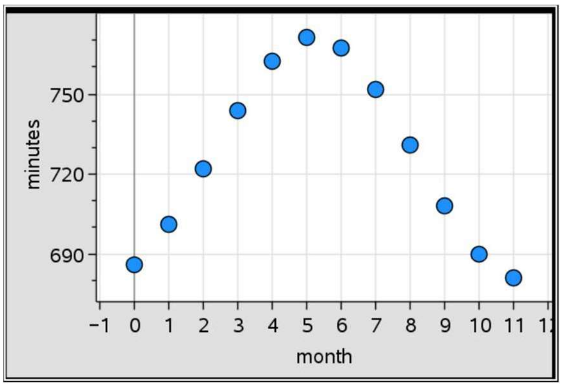
(c) $S = 11.7\log(5) + 15.4 \approx 23.6$ thousand sales

(d) **(i)** $a = 0,\ b = 12$

(ii) $S = \int_0^{12} (11.7\log(m) + 15.4)\,dm \approx 275$ thousand sales



15. **(a)** Using a GDC



(b) We know that the length of the day is cyclical.

(c) $L = 44.8\sin(0.519M - 1.12) + 726$

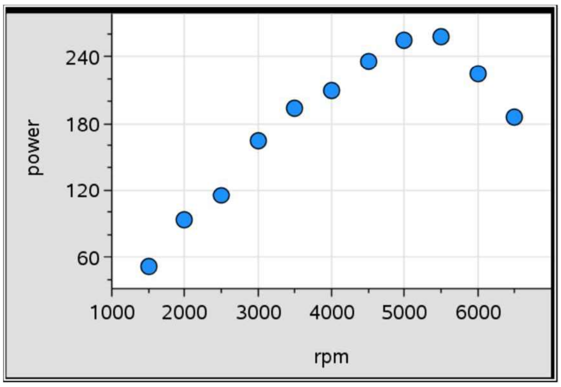
(d) The average in a sinusoidal model is the axis: 726 minutes

(e) Remember that the period of $\sin bx$ is $\frac{2\pi}{|b|} = \frac{2\pi}{0.519} \approx 12.1$ months.

This value is reasonable as we expect the period of to be near 12 months.

(f) $N = -44.8\sin(0.519M - 1.12) + 726$

16. **(a)** There appears to be a strong association between power and RPM. A turning point suggests the association is nonlinear.



(b) There is a turning point in the pattern.

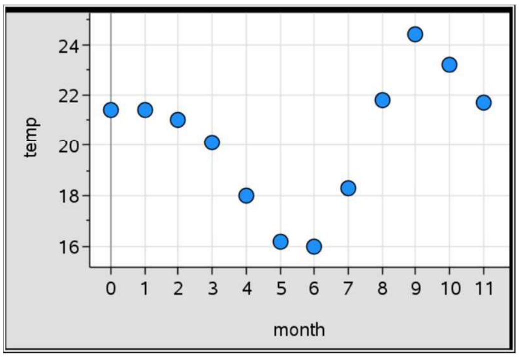
(c) We use a quadratic model: $P = -0.0000157s^2 + 0.159s - 165$.

Other models are also possible.

(d) We find the coordinates of the vertex of the parabola representing the model. Maximum power is 237 kW at 5060 RPM.

(e) The model does not necessarily pass through every point. As a best fit, it will miss some points. It appears that the turning point in the model tends to under-predict maximum power since 257 kW was observed at 5500 RPM.

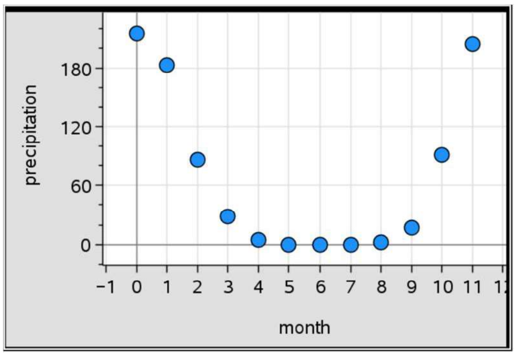
17. (a)



(b) The pattern is not sinusoidal because there appears to be unequal local maxima.

(c) There are 3 turning points; a cubic model requires at most 2 turning points; a quadratic model requires at most 1 turning point.

18. (a)



(b) $P = 7.34\,\text{m}^2 - 86.2\,\text{m} + 234$

(c) $R^2 = 0.944$; the model accounts for 94.4% of the variation in precipitation using month as the explanatory variable.

(d) Since the model has a parabola shape, and this parabola opens upwards, the minimum is the y-coordinate of the vertex, which is −19.2 mm!

(e) The model gives inappropriate predictions: it is not possible to have negative precipitation. In fact, by looking at the shape, another model using a polynomial of 4th degree may be a better choice.

Exercise 20.1

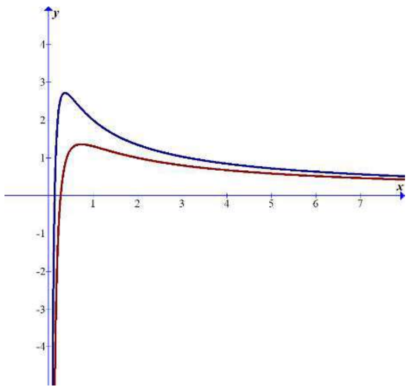
Note: conventions for this chapter**:** DE = Differential equation, LHS = left-hand side, RHS = right-hand side.

1. **(a)** We need to show that $f(x)=\frac{1}{x}(\ln x+k)$ satisfies the DE $x^2\frac{\mathrm{d}y}{\mathrm{d}x}+xy-1=0$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-k-\ln x}{x^2}$$

$$\Rightarrow x^2\times\frac{1-k-\ln x}{x^2}+x\left(\frac{1}{x}(\ln x+k)\right)-1=1-k-\ln x+\ln x+k-1=0$$

(b) $x^2\frac{\mathrm{d}y}{\mathrm{d}x}+xy-1=0\Rightarrow x^2\frac{\mathrm{d}y}{\mathrm{d}x}=1-xy\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-xy}{x^2}$



(c) $f(1)=2\Rightarrow 2=\frac{1}{1}(\ln 1+k)\Rightarrow k=2\Rightarrow f(x)=\frac{1}{x}(\ln x+2)$

$f(2)=1\Rightarrow 1=\frac{1}{2}(\ln 2+k)\Rightarrow k=2-\ln 2\Rightarrow f(x)=\frac{1}{x}(\ln x+2-\ln 2)$

2. **(a)** $\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{e}^{y-x}=1\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^{x-y}=\mathrm{e}^{x}\mathrm{e}^{-y}\Rightarrow\mathrm{e}^{y}\mathrm{d}y=\mathrm{e}^{x}\mathrm{d}x$

Integrate both sides: $\mathrm{e}^y=\mathrm{e}^x+C\Rightarrow y=\ln\left(\mathrm{e}^x+C\right)$

(b) Separate variables first

$$\frac{y\mathrm{d}y}{\mathrm{d}x}=y^2x+x=x\left(y^2+1\right)\Rightarrow\frac{y\mathrm{d}y}{y^2+1}=x\mathrm{d}x$$

$$\Rightarrow\frac{1}{2}\ln\left(y^2+1\right)=\frac{x^2}{2}+c\Rightarrow\ln\left(y^2+1\right)=x^2+2c$$

$$\Rightarrow y^2+1=\mathrm{e}^{x^2+2c}=\mathrm{e}^{x^2}v^{2c}=C\mathrm{e}^{x^2}\Rightarrow y^2=C\mathrm{e}^{x^2}-1$$

(c) $e^{x-y}dy = xdx \Rightarrow e^{x}e^{-y}dy = xdx \Rightarrow e^{-y}dy = e^{-x}xdx$

Left-hand side is direct evaluation while right-hand side needs integration by parts:

$-e^{-y} = -e^{-x}(x+1)+c \Rightarrow e^{-y} = e^{-x}(x+1)+C$

$\Rightarrow -y = \ln\left(e^{-x}(x+1)+C\right) \Rightarrow y = -\ln\left(e^{-x}(x+1)+C\right)$

$\Rightarrow y = \ln\left(\dfrac{e^{x}}{(x+1)+Ce^{x}}\right) = x - \ln\left(x+1+Ce^{x}\right)$

(d) $\dfrac{dy}{dx} = xy^2 - x - y^2 + 1 = x\left(y^2-1\right) - \left(y^2-1\right) = \left(y^2-1\right)(x-1)$

$\Rightarrow \dfrac{dy}{y^2-1} = (x-1)dx$, and using partial fractions in integrating LHS

$\dfrac{1}{2}\left(\ln(y-1) - \ln(y+1)\right) = \dfrac{(x-1)^2}{2} + c \Rightarrow \ln\left(\dfrac{y-1}{y+1}\right) = (x-1)^2 + 2c$

$\dfrac{y-1}{y+1} = e^{(x-1)^2+2c} = Ce^{(x-1)^2}$

(e) $(xy\ln x)\dfrac{dy}{dx} = (y+1)^2 \Rightarrow \dfrac{ydy}{(y+1)^2} = \dfrac{dx}{x\ln x}$

LHS can be integrated with substituting $u = y + 1$ or by partial fractions.

RHS can be integrated with substituting $u = \ln x$.

$\ln(y+1) + \dfrac{1}{y+1} = \ln(\ln x) + c$

$\Rightarrow \ln(y+1) + y\ln(y+1) + 1 = (y+1)\left(\ln(\ln x) + c\right)$

(f) $\dfrac{dy}{dx} = \dfrac{1+2y^2}{y\sin^2 x} \Rightarrow \dfrac{ydy}{1+2y^2} = \dfrac{dx}{\sin^2 x}$

LHS can be integrated with a substitution $u = 1+2y^2$, and RHS a standard cosecant^2x integral.

$\displaystyle\int \frac{ydy}{1+2y^2} = \int \frac{dx}{\sin^2 x} \Rightarrow \frac{1}{4}\ln\left(1+2y^2\right) = -\cot x + c \Rightarrow \ln\left(1+2y^2\right) = -4\cot x + 4c$

$\Rightarrow 1+2y^2 = e^{-4\cot x+4c} \Rightarrow 2y^2 = e^{-4\cot x}e^{4c} - 1 \Rightarrow y^2 = \dfrac{e^{4c}}{2}e^{-4\cot x} - \dfrac{1}{2} = Ce^{-4\cot x} - \dfrac{1}{2}$

(g) $(1+\tan y)\dfrac{\mathrm{d}y}{\mathrm{d}x}=x^2+1 \Rightarrow (1+\tan y)\,\mathrm{d}y=(x^2+1)$

LHS is a direct trigonometric integral and RHS is a polynomial

$y-\ln|\cos y|=\dfrac{1}{3}x^3+x+c$

(h) $\dfrac{\mathrm{d}y}{\mathrm{d}t}\dfrac{te^t}{y\sqrt{y^2+1}} \Rightarrow y\sqrt{y^2+1}\mathrm{d}y=te^t\mathrm{d}t$

LHS integral can be evaluated with the substitution $u=y^2+1$, while RHS with integration by parts

$\int y\sqrt{y^2+1}\,\mathrm{d}y=\int te^t\mathrm{d}t \Rightarrow \dfrac{1}{3}(y^2+1)^{\frac{3}{2}}=\mathrm{e}^t(t-1)+c$

$\Rightarrow \sqrt{(y^2+1)^3}=3\mathrm{e}^t(t-1)+C$

(i) $y\sec\theta\,\mathrm{d}y=\mathrm{e}^y\sin^2\theta\,\mathrm{d}\theta \Rightarrow y\mathrm{e}^{-y}\,\mathrm{d}y=\sin^2\theta\cos\theta\,\mathrm{d}\theta$

LHS integral can be evaluated with integration by parts while RHS with the substitution $u=\sin\theta$,

$-\mathrm{e}^{-y}(y+1)=\dfrac{\sin^3\theta}{3}+c \Rightarrow \mathrm{e}^{-y}(y+1)=-\dfrac{\sin^3\theta}{3}+c$

(j) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x(1+y^2) \Rightarrow \dfrac{\mathrm{d}y}{1+y^2}=\mathrm{e}^x\mathrm{d}x \Rightarrow \arctan y=\mathrm{e}^x+c \Rightarrow y=\tan(\mathrm{e}^x+c)$

3. (a) $x^{-3}\mathrm{d}y=4y\mathrm{d}x \Rightarrow \dfrac{\mathrm{d}y}{y}=4x^3\mathrm{d}x \Rightarrow \ln|y|=x^4+c$, and with initial condition

$\ln|3|=c \Rightarrow \ln|y|=x^4+\ln 3 \Rightarrow y=3\mathrm{e}^{x^4}$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=xy \Rightarrow \dfrac{\mathrm{d}y}{y}=x\mathrm{d}x \Rightarrow \ln|y|=\dfrac{x^2}{2}+c$, and with initial condition

$c=\ln|1|=0 \Rightarrow y=\mathrm{e}^{\frac{x^2}{2}}$

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x}-xy^2=0 \Rightarrow \dfrac{\mathrm{d}y}{y^2}=x\mathrm{d}x \Rightarrow -\dfrac{1}{y}=\dfrac{x^2}{2}+c$, and with initial condition

$-\dfrac{1}{2}=\dfrac{1}{2}+c \Rightarrow c=-1 \Rightarrow -\dfrac{1}{y}=\dfrac{x^2}{2}-1 \Rightarrow y=\dfrac{2}{2-x^2}$

(d) $\frac{\mathrm{d}y}{\mathrm{d}x} - y^2 = 0 \Rightarrow \frac{\mathrm{d}y}{y^2} = \mathrm{d}x \Rightarrow -\frac{1}{y} = x + c$

and with initial condition:

$-\frac{1}{1} = 2 + c \Rightarrow c = -3 \Rightarrow -\frac{1}{y} = x - 3 \Rightarrow y = \frac{1}{3 - x}$

(e) $\frac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^{y} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{e}^{y}} = \mathrm{d}x \Rightarrow -\mathrm{e}^{-y} = x + c,$

and with initial conditions:

$-e^{-1} = 0 + c \Rightarrow c = \frac{-1}{e} \Rightarrow e^{-y} = \frac{1}{e} - x \Rightarrow -y = \ln\left(\frac{1 - ex}{e}\right) \Rightarrow y = \ln\left(\frac{e}{1 - ex}\right)$

(f) $\frac{\mathrm{d}y}{\mathrm{d}x} = y^{-2}x + y^{-2} \Rightarrow y^2\mathrm{d}y = (x+1)\mathrm{d}x \Rightarrow \frac{y^3}{3} = \frac{(x+1)^2}{2} + c,$

and with initial conditions:

$\frac{1}{3} = \frac{1^2}{2} + c \Rightarrow c = -\frac{1}{6} \Rightarrow \frac{y^3}{3} = \frac{(x+1)^2}{2} - \frac{1}{6} \Rightarrow y^3 = \frac{3}{2}(x+1)^2 - \frac{1}{2}$

(g) $x\mathrm{d}y - y^2\mathrm{d}x = -\mathrm{d}y \Rightarrow \frac{\mathrm{d}y}{y^2} = \frac{\mathrm{d}x}{x+1} \Rightarrow -\frac{1}{y} = \ln|x+1| + c,$

and with initial conditions:

$-\frac{1}{1} = \ln|1| + c \Rightarrow c = -1 \Rightarrow y = \frac{1}{1 - \ln|x+1|}$

(h) $y^2\mathrm{d}y - x\mathrm{d}x = \mathrm{d}x - \mathrm{d}y \Rightarrow (y^2 + 1)\mathrm{d}y = (x+1)\mathrm{d}x \Rightarrow \frac{y^3}{3} + y = \frac{(x+1)^2}{2} + c$

$\frac{3^3}{3} + 3 = \frac{(1)^2}{2} + c \Rightarrow c = \frac{23}{2} \Rightarrow 2y^3 + 6y = 3x^2 + 6x + 72$

(i) $y\frac{\mathrm{d}y}{\mathrm{d}x} = xy^2 + x \Rightarrow \frac{y\mathrm{d}y}{y^2 + 1} = x\mathrm{d}x \Rightarrow \frac{1}{2}\ln(y^2 + 1) = \frac{1}{2}x^2 + c,$

with initial conditions:

$\frac{1}{2}\ln(1) = c \Rightarrow c = 0 \Rightarrow \ln(y^2 + 1) = x^2 \Rightarrow y^2 = \mathrm{e}^{x^2} - 1$

(j) $\frac{dy}{dx}=\frac{xy-y}{y+1}\Rightarrow\left(1+\frac{1}{y}\right)dy=(x-1)dx\Rightarrow y+\ln|y|=\frac{x^2}{2}-x+c$,

$1+\ln|1|=2-2+c\Rightarrow c=1\Rightarrow y+\ln|y|=\frac{x^2}{2}-x+1$

(k) Separate variables and integrate:

$\frac{dy}{dx}=x\sqrt{\frac{1-y^2}{1-x^2}}\Rightarrow\frac{dy}{\sqrt{1-y^2}}=\frac{xdx}{\sqrt{1-x^2}}\Rightarrow\arcsin y=-\sqrt{1-x^2}+c$, with initial conditions, $\arcsin 0=-\sqrt{1-0}+c\Rightarrow c=1\Rightarrow\arcsin y=1-\sqrt{1-x^2}$

(l) $\frac{dy}{dx}(1+e^x)=e^{x-y}\Rightarrow e^y dy=\frac{e^x dx}{1+e^x}\Rightarrow e^y=\ln(1+e^x)+c$, with initial conditions,

$e^0=\ln(1+e^1)+c\Rightarrow c=1-\ln(1+e)\Rightarrow e^y=\ln(1+e^x)+1-\ln(1+e)$

$e^y=\ln\left(\frac{e(1+e^x)}{1+e}\right)\Rightarrow y=\ln\left(\ln\left(\frac{e(1+e^x)}{1+e}\right)\right)$

(m) $(y+1)dy=(x^2y-y)dx\Rightarrow\left(1+\frac{1}{y}\right)dy=(x^2-1)dx\Rightarrow y+\ln|y|=\frac{x^3}{3}-x+c$,

$1+\ln|1|=9-3+c\Rightarrow c=-5\Rightarrow y+\ln|y|=\frac{x^3}{3}-x-5$

(n) Rearrange and separate variables

$(1+e^{-x})\sin y\,dy=-\cos y\,dx\Rightarrow\frac{-\sin y\,dy}{\cos y}=\frac{dx}{1+e^{-x}}=\frac{e^x dx}{1+e^x}$

$$\Rightarrow\int\frac{-\sin y\,dy}{\cos y}=\int\frac{e^x dx}{1+e^x}\Rightarrow\ln|\cos y|=\ln(1+e^x)+c$$

With initial conditions, $\ln\left|\cos\frac{\pi}{4}\right|=\ln(1+e^0)+c\Rightarrow\ln 1=\ln 2+c\Rightarrow c=-\ln 2$

Thus,

$$\ln|\cos y|=\ln(1+e^x)-\ln 2$$
$$=\ln\left(\frac{1+e^x}{2}\right)\Rightarrow\cos y$$
$$=e^{\frac{1+e^x}{2}}$$
$$\Rightarrow y=\arccos\left(\frac{1+e^x}{2}\right)$$

(o) $x\dfrac{dy}{dx} - y = 2x^2 y$

$\Rightarrow x dy = \left(2x^2+1\right) y dx$

$\Rightarrow \dfrac{dy}{y} = \left(2x + \dfrac{1}{x}\right) dx$

$\Rightarrow \ln|y| = x^2 + \ln|x| + c$

$\ln|1| = 1^2 + \ln|1| + c$

$\Rightarrow c = -1$

$\Rightarrow \ln|y| = x^2 + \ln|x| - 1$

$\Rightarrow |y| = e^{x^2+\ln|x|-1} = |x| e^{x^2-1}$

(p) $xy dx + e^{-x^2}\left(y^2-1\right) dy = 0$

$\Rightarrow x e^{x^2} dx = \left(\dfrac{1}{y} - y\right) dy$

$\Rightarrow \dfrac{e^{x^2}}{2} + c = \ln|y| - \dfrac{y^2}{2}$

$\dfrac{e^0}{2} + c = \ln|1| - \dfrac{1}{2}$

$\Rightarrow c = -1$

$\Rightarrow \dfrac{e^{x^2}}{2} - 1 = \ln|y| - \dfrac{y^2}{2}$

$\Rightarrow 2\ln|y| - y^2 = e^{x^2} - 2$

(q) Rewrite in terms of *dy* and *dx*: $x\cos x\, dx = \left(2y + e^{3y}\right) dy$

Variables are already separated. LHS can be integrated by parts.

$\int x\cos x\, dx = \int\left(2y + e^{3y}\right) dy \Rightarrow y^2 + \dfrac{1}{3}e^{3y} = \cos x + x\sin x + c$

$\Rightarrow 3y^2 + e^{3y} = 3(\cos x + x\sin x) + C$

With initial conditions: $0 + e^0 = 3(\cos 0 + 0) + C \Rightarrow C = -2$

Therefore, the solution is: $3y^2 + e^{3y} = 3(\cos x + x\sin x) - 2$

(r) Separate variables and integrate

$\int dy = \int\left(e^x - 2x\right) dx \Rightarrow y = e^x - x^2 + c$, and with initial conditions:

$3 = e^0 - 0 + c \Rightarrow c = 2 \Rightarrow y = e^x - x^2 + 2$

4. **(a)** $2x\frac{dy}{dx} = x + y \Rightarrow 2x\,dy = (x+y)\,dx$

This is a homogeneous DE.

We use the substitution $y = vx \Rightarrow dy = v\,dx + x\,dv$, and then separate variables:

$$2x\,dy = (x+y)\,dx$$
$$\Rightarrow 2x(v\,dx + x\,dv) = (x + vx)\,dx$$
$$\Rightarrow (xv - x)\,dx = -2x^2\,dv$$
$$\Rightarrow \frac{dx}{x} = \frac{2\,dv}{1-v}$$
$$\Rightarrow \ln|x| = -2\ln|1-v| + c$$
$$\Rightarrow \ln|x| = \ln\frac{1}{\left|1-\frac{y}{x}\right|^2} + c$$
$$\Rightarrow \ln|x| = \ln\frac{x^2}{|y-x|^2} + c$$
$$\Rightarrow \ln|x| = \ln|x-y|^2 + c$$
$$\Rightarrow |x| = e^{\ln|x-y|^2 + c} = e^c|x-y|^2 = C(x-y)^2$$

(b) This is also Homogeneous. We use the same substitution as before.

$$(x+y)\,dy = (x-y)\,dx$$
$$\Rightarrow (x+vx)(v\,dx + x\,dv) = (x - vx)\,dx$$
$$\Rightarrow x^2(1+v)\,dv = (x - vx - vx - xv^2)\,dx = x(1 - 2v - v^2)\,dx$$
$$\Rightarrow \frac{dx}{x} = \frac{(1+v)\,dv}{1-2v-v^2}$$
$$\Rightarrow \ln|x| + c = -\frac{1}{2}\ln|1 - 2v - v^2|$$

Substitute the value of v back and the result can be simplified to:

$|y^2 + 2xy - x^2| = C$ (remember that $\ln x^2 = 2\ln x$)

(c) $\left(x^2-y^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x}=xy \Rightarrow \left(x^2-y^2\right)\mathrm{d}y=xy\mathrm{d}x;\ y=vx \Rightarrow \mathrm{d}y=v\mathrm{d}x+x\mathrm{d}v$

$$\left(x^2-v^2x^2\right)\left(v\mathrm{d}x+x\mathrm{d}v\right)=xvx\mathrm{d}x$$

$$\Rightarrow \left(x^3-v^2x^3\right)\mathrm{d}v=v^3x^2\mathrm{d}x$$

$$\Rightarrow \left(\frac{1}{v^3}-\frac{1}{v}\right)\mathrm{d}v=\frac{\mathrm{d}x}{x}$$

$$\Rightarrow \ln|x|=\frac{-1}{2v^2}-\ln|v|+c$$

$$\Rightarrow \ln|x|=-\frac{x^2}{2y^2}-\ln\left|\frac{y}{x}\right|+c$$

$$\ln|y|=-\frac{x^2}{2y^2}+c \Rightarrow y=\mathrm{e}^{-\frac{x^2}{2y^2}+c}=C\mathrm{e}^{-\frac{x^2}{2y^2}}$$

(d) $x\mathrm{d}y-\left(2x\mathrm{e}^{-\frac{y}{x}}+y\right)\mathrm{d}x=0 \Rightarrow x\left(v\mathrm{d}x+x\mathrm{d}v\right)-\left(2x\mathrm{e}^{-v}+vx\right)\mathrm{d}x=0$

$$x\left(v\mathrm{d}x+x\mathrm{d}v\right)-\left(2x\mathrm{e}^{-v}+vx\right)\mathrm{d}x=0 \Rightarrow x^2\mathrm{d}v=2x\mathrm{e}^{-v}\mathrm{d}x$$

$$\Rightarrow \mathrm{e}^{v}\mathrm{d}v=\frac{2}{x}\mathrm{d}x \Rightarrow \mathrm{e}^{\frac{y}{x}}=2\ln|x|+c$$

With $y(1)=0$ $\mathrm{e}^{\frac{y}{x}}=2\ln|x|+1$

(e) $\left(x\sec\dfrac{y}{x}+y\right)\mathrm{d}x=x\mathrm{d}y \Rightarrow \left(x\sec v+vx\right)\mathrm{d}x=x\left(v\mathrm{d}x+x\mathrm{d}v\right)$

$$\Rightarrow x\sec v\mathrm{d}x=x^2\mathrm{d}v$$

$$\Rightarrow \frac{\mathrm{d}x}{x}=\cos v\mathrm{d}v$$

$$\Rightarrow \ln|x|=\sin\left(\frac{y}{x}\right)+c$$

$$(1,0) \Rightarrow c=0 \Rightarrow x=\mathrm{e}^{\sin\left(\frac{y}{x}\right)}$$

(f) $\left(x^2+y^2\right)\mathrm{d}x=\left(xy-x^2\right)\mathrm{d}y \Rightarrow \left(x^2+v^2x^2\right)\mathrm{d}x=\left(x^2v-x^2\right)\left(v\mathrm{d}x+x\mathrm{d}v\right)$

$\Rightarrow x^2\left(1+v\right)\mathrm{d}x=x^3\left(v-1\right)\mathrm{d}v \Rightarrow \frac{\mathrm{d}x}{x}=\left(1-\frac{2}{v+1}\right)\mathrm{d}v$

$\Rightarrow \ln|x|+c=\frac{y}{x}-2\ln\left|\frac{y}{x}+1\right|$

This can be simplified to different forms which also partly depends where we place the constant term, c. Possible forms for the answer:

$\ln\left|\frac{\left(x+y\right)^2}{cx}\right|=\frac{y}{x}$ or $\left(x+y\right)^2=cx\mathrm{e}^{\frac{y}{x}}$

(g) $y\mathrm{d}x=\left(x+\sqrt{xy}\right)\mathrm{d}y \Rightarrow vx\mathrm{d}x=\left(x+x\sqrt{v}\right)\left(v\mathrm{d}x+x\mathrm{d}v\right)$

$vx\mathrm{d}x=\left(x+x\sqrt{v}\right)\left(v\mathrm{d}x+x\mathrm{d}v\right) \Rightarrow -xv\sqrt{v}\mathrm{d}x=x^2\left(1+\sqrt{v}\right)\mathrm{d}v$

$\Rightarrow -\frac{\mathrm{d}x}{x}=\left(v^{-\frac{3}{2}}+\frac{1}{v}\right)\mathrm{d}v \Rightarrow -\ln|x|+c=-2\sqrt{\frac{x}{y}}+\ln\left|\frac{y}{x}\right| \Rightarrow 2\sqrt{\frac{x}{y}}=\ln|y|-c$

$\Rightarrow 4x=y\left(\ln|y|-c\right)^2$

5. **(a)** $\frac{\mathrm{d}T}{\mathrm{d}t}=m\left(T-21\right) \Rightarrow \frac{\mathrm{d}T}{T-21}=m\mathrm{d}t \Rightarrow \ln|T-21|=mt+c$

$\Rightarrow |T-21|=\mathrm{e}^{mt+c}$

With T in a Kettle is at least room temperature, $T-21\geq 0,$ and thus,

$|T-21|=T-21=\mathrm{e}^{mt+c} \Rightarrow T=\mathrm{e}^{mt}\mathrm{e}^{c}+21=C\mathrm{e}^{mt}+21$

(b) **(i)** $T=C\mathrm{e}^{mt}+21 \Rightarrow T\left(0\right)=99 \Rightarrow 99=C+21 \Rightarrow C=78$

$T\left(15\right)=69 \Rightarrow 69=78e^{15m}+21 \Rightarrow m=\frac{1}{15}\ln\frac{8}{13}$

6. **(a)** The growth of the tiger population can be modelled by

$$\frac{\mathrm{d}p}{\mathrm{d}t} = kp\left(1 - \frac{p}{200}\right), \quad 25 \le p \le 200, \text{ where } t \text{ is the number of years.}$$

This logistic DE can be solved using the model given in the text.

(review pages 848–850)

$$p = \frac{200}{1 + b\mathrm{e}^{-kt}}.$$

Since $p(0) = 25$, then $25 = \frac{200}{1 + be^{-0}} \Rightarrow 1 + b = 8 \Rightarrow b = 7 \Rightarrow p = \frac{200}{1 + 7e^{-kt}}$

We also know that $p(2) = 39$, then

$$39 = \frac{200}{1 + 7\mathrm{e}^{-2k}} \Rightarrow 7\mathrm{e}^{-2k} = 4.12821 \Rightarrow \mathrm{e}^{-2k} = 0.589744$$

$$\Rightarrow k = -\frac{\ln 0.589744}{2} \approx 0.2640 \Rightarrow p = \frac{200}{1 + 7\mathrm{e}^{-0.2640t}}$$

(b) $p(5) = \frac{200}{1 + 7\mathrm{e}^{-0.2640 \times 5}} \approx 69.69 \approx 70 \text{ tigers}$

(c) $100 = \frac{200}{1 + 7\mathrm{e}^{-0.2640t}} \Rightarrow 1 + 7\mathrm{e}^{-0.2640t} = 2 \Rightarrow \mathrm{e}^{-0.2640t} = \frac{1}{7} \Rightarrow t = \frac{\ln\left(\frac{1}{7}\right)}{-0.2640} \approx 7.37 \text{ years}$

7. **(a)** This situation corresponds to $t = 0$, $P(0) = \frac{2100}{1 + 29\mathrm{e}^{0}} = 70$

(b) Carrying capacity, as clear from the model is 2100, thus,

$$1050 = \frac{2100}{1 + 29\mathrm{e}^{-0.75t}} \Rightarrow 1 + 29\mathrm{e}^{-0.75t} = 2 \Rightarrow \mathrm{e}^{-0.75t} = \frac{1}{29} \Rightarrow t = \frac{\ln\left(\frac{1}{29}\right)}{-0.75} \approx 4.49 \text{ years}$$

(c) The logistic model is of the form $P(t) = \frac{L}{1 + be^{-kt}}$ and thus, $k = 0.75$, and the logistic DE corresponding to this is of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{L}\right) \Rightarrow \frac{\mathrm{d}P}{\mathrm{d}t} = 0.75P\left(1 - \frac{P}{2100}\right)$$

8. **(a)** This is a typical growth model

$$\frac{dP}{dt} = kP \Rightarrow \frac{dP}{P} = k\,dt$$

$$\Rightarrow \ln P = kt + c$$

$$\Rightarrow P = e^{kt+c} = e^c e^{kt}$$

$$P(0) = 500 = e^c \Rightarrow P = 500e^{kt}$$

$$P(2) = 2000 = 500e^{2k}$$

$$\Rightarrow k = \frac{\ln 4}{2} = \ln 2$$

$$\Rightarrow P = 500e^{t\ln 2}$$

$$P(12) = 500e^{12\times\ln 2} = 20\,48\,000$$

(b) This is the time it takes the bacteria to become 1000.

$$1000 = 500e^{t\ln 2} \Rightarrow t = 1 \text{ hour.}$$

9. A logistic model for population growth as described in the text is of the form

$$\frac{dP}{dt} = rP(L-P), P(0) = P_0 \Rightarrow P(t) = \frac{LP_0}{P_0 + (L-P_0)e^{-kt}}$$

We will use the numbers in 1000 of people.

Thus, considering the year 2000 as $t = 0$,
$P(0) = 20, P(10) = 50.870,$ and $P(15) = 78.680.$

(a) With the given initial conditions, we can find all parameters in the model

$$P(t) = \frac{20L}{20 + (L-20)e^{-kt}}$$

$$P(10) = \frac{20L}{20 + (L-20)e^{-10k}} = 50.870 \Rightarrow 20L = 50.870\left(20 + (L-20)e^{-10k}\right)$$

$$\Rightarrow e^{-10k} = \frac{20L - 1017.4}{50.87(L-20)}$$

$$P(15) = \frac{20L}{20 + (L-20)e^{-15k}} = 78.680 \Rightarrow 20L = 78.680\left(20 + (L-20)e^{-15k}\right)$$

$$\Rightarrow e^{-15k} = \frac{20L - 1573.6}{78.68(L-20)}$$

Use a GDC/Software: $k = 0.1$ and $L = 500.37$

$$P(t) = \frac{10007.4}{20 + 480.37e^{-0.1t}}$$

Alternatively, we can use the original Logistic model $P = \frac{L}{1+b\mathrm{e}^{-kt}}$ and using the initial conditions find the parameters, L, b, and k: $P = \frac{500.37}{1+24.02\mathrm{e}^{-0.1t}}$

(b) 500370. This is a result found in **(a)**.

Alternatively:

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{10007.4}{20+480.37e^{-0.1t}} = \frac{10007.4}{20} = 500.37$$

or $\lim_{t\to\infty} P = \lim_{t\to\infty} \frac{500.37}{1+24.02e^{-0.1t}} = 500.37$

(c) $P(30) = \frac{10007.4}{20+480.37\mathrm{e}^{-3}} \approx 227.87$ i.e., 227 870 inhabitants.

10. This is a basic growth model

$$\frac{\mathrm{d}P}{\mathrm{d}t} = kP \Rightarrow \frac{\mathrm{d}P}{P} = k\mathrm{d}t \Rightarrow P = C\mathrm{e}^{kt}$$

Since at $t = 0$, there were 10, then $P = C\mathrm{e}^{kt} \Rightarrow 10 = C\mathrm{e}^{0} \Rightarrow C = 10 \Rightarrow P(t) = 10\mathrm{e}^{kt}$

Doubling time is 3 hours, thus $20 = 10\mathrm{e}^{3k} \Rightarrow k = \frac{\ln 2}{3} \approx 0.231$

After 24 hours, the population of bacteria will be

$P(24) = 10\mathrm{e}^{24k} \approx 2560$ (answers may differ according to the accuracy used.

11. Again, a basic growth model

$P = C\mathrm{e}^{kt} = 2000\mathrm{e}^{kt}$

With doubling time of 4 hours

$$4000 = 2000e^{4k} \Rightarrow k = \frac{\ln 2}{4}$$

$$1\,000\,000 = 2000\mathrm{e}^{\left(\frac{\ln 2}{4}\right)t} \Rightarrow t = \frac{4\ln 500}{\ln 2} \approx 35.86 \text{ hours}$$

12. Let $P(t) = \dfrac{L}{1+b\mathrm{e}^{-rt}}$ be the logistic model, where L, b, and r are to be determined.

Now from the initial conditions

$$t = 0: 500 = \frac{L}{1+b} \Rightarrow L = 500(1+b)$$

$$t = 5: 800 = \frac{L}{1+b\mathrm{e}^{-5r}} \Rightarrow 8 = \frac{5(1+b)}{1+b\mathrm{e}^{-5r}}$$

$$t = 10: 1000 = \frac{L}{1+b\mathrm{e}^{-10r}} \Rightarrow 10 = \frac{5(1+b)}{1+b\mathrm{e}^{-10r}}$$

The last 2 equations make a system with 2 variables.

We either use software/GDC at this point and find b and r, or we do some more simplification.

In the second equation, we can solve for e^{-5r}: $\mathrm{e}^{-5r} = \dfrac{5b-3}{8b}$, then noticing that $\mathrm{e}^{-10r} = (\mathrm{e}^{-5r})^2$ we substitute the value found into the third equation and simplify:

$7b^2 - 2b - 9 = 0 \Rightarrow b \approx 1.2857$ with back substitution we find $L = 1142.85 \approx 1143$ and $r \approx 0.2197$ leading us to the model for this population: $P(t) = \dfrac{1142.85}{1+1.2857\mathrm{e}^{-0.2197t}}$

Therefore, 2020 represents $P(15) \approx 1091$.

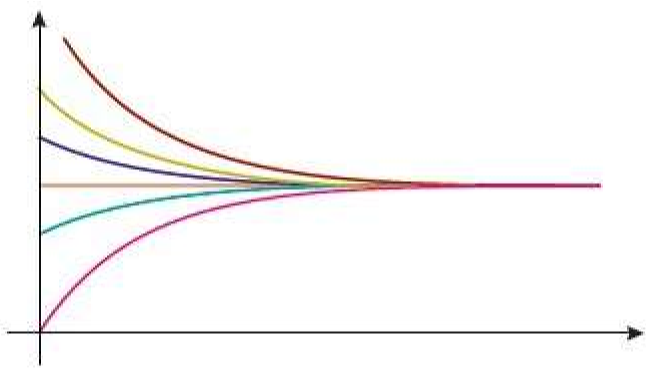
(Answers will differ according to approximations made in intermediate steps).

13. **(a)** This is a variables separable DE

Note that we replaced the arbitrary constants for convenience.

(b) Use your GDC/software to produce graphs for some values of the parameters included.

Here is a sample:



(c) On your graphs change the parameter values as requested.

Here is a summary of observations:

i. equilibrium is lower and is approached more rapidly

ii. equilibrium is higher

iii. equilibrium stays the same but is approached more rapidly

14. **(a)** Newton's law of cooling: $\frac{dT}{dt} = -k(T-S)$

This is a variables separable DE as we have seen.

$$\frac{dT}{dt} = -k(T-S) \Rightarrow \frac{dT}{T-S} = -k dt \Rightarrow \ln|T-S| = -kt + c$$

$$\Rightarrow |T-S| = e^{-kt+c} \Rightarrow T - S = Ae^{-kt}$$

Applying initial conditions

$$T = S + (T_0 - S)e^{-kt} = -12 + 33e^{-0.15t}$$

(b) $-12 + 33e^{-0.15t} = 0 \Rightarrow t = \frac{\ln\left(\frac{12}{33}\right)}{-0.15} \approx 6.7$ hours.

Exercise 20.2

1. The decay model described can be written as

$\frac{dQ}{dt} = -rQ \Rightarrow \frac{dQ}{Q} = -r dt \Rightarrow Q = Ae^{-rt}$, where Q is the quantity remaining and r is the rate of decay.

With initial conditions, we have $A = 10$, thus $Q = 10e^{-rt}$

Now, if 1620 the half-life, then

$$5 = 10e^{-1620r} \Rightarrow r = \frac{\ln 2}{1620} \Rightarrow Q = 10e^{-\frac{\ln 2}{1620}t}$$

After 25 years: $Q = 10e^{-\frac{\ln 2}{1620}\times 25} \approx 9.89$ grams.

2. **(a)** This is Newton's law.

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -r(T-20) \Rightarrow \frac{\mathrm{d}T}{T-20} = -r\mathrm{d}t \Rightarrow \ln|T-20| = -rt + c$$

$$\Rightarrow T - 20 = A\mathrm{e}^{-rt}$$

With initial condition that $T = 70$ when the tea is poured

$70 - 20 = A\mathrm{e}^{0} \Rightarrow A = 50 \Rightarrow T = 50\mathrm{e}^{-rt} + 20$, and with $T = 50$ at $t = 10$, we have

$$T = 20 + 50\mathrm{e}^{-\frac{t}{10}\ln\frac{5}{3}}$$

(b) $21 = 20 + 50\mathrm{e}^{-\frac{t}{10}\ln\frac{5}{3}} \Rightarrow \mathrm{e}^{-\frac{t}{10}\ln\frac{5}{3}} = \frac{1}{50} \Rightarrow t = \frac{10\ln 50}{\ln 5 - \ln 3} \approx 76.6$ minutes

3. Since we have the deceleration in m s^{-2}, we change the speed in m s^{-1}.

$$v_0 = 120 \text{ kmh}^{-1} = \frac{120 \times 1000}{3600} \approx 33.33 \text{ ms}^{-1}$$

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -10 \Rightarrow \mathrm{d}v = -10\mathrm{d}t \Rightarrow v = -10t + c$$

With initial conditions, $v = -10t + 33.33$

Let s be distance travelled:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v = -10t + 33.33 \Rightarrow s(t) = -5t^2 + 33.33t + c, \text{ but } s(0) = 0$$

$$s(t) = -5t^2 + 33.33t$$

The car comes to a stop when $v = 0$, thus, $0 = -10t + 33.33 \Rightarrow t = 3.33$ min

Therefore, the distance travelled will be

$$s(3.33) = -5(3.33)^2 + 33.33(3.33) \approx 55.56 \text{ m}$$

4. $$\frac{\mathrm{d}v}{\mathrm{d}t} = -20 \Rightarrow \mathrm{d}v = -20\mathrm{d}t \Rightarrow v = -20t + c$$

At time $t = 0$, the velocity was $v_0 \Rightarrow c = v_0 \Rightarrow v = -20t + v_0$

Let the distance travelled be s, then

$$\frac{\mathrm{d}s}{\mathrm{d}t} = v = -20t + v_0 \Rightarrow s(t) = -10t^2 + v_0 t + c, \text{but } s(0) = 0$$

$$s(t) = -10t^2 + v_0 t$$

After time t_1 the car comes to a stop. At this time, we have

$0 = -20t_1 + v_0$, and the distance travelled is 75 m

$s(t_1) = -10t_1^2 + v_0 t_1 = 75$

Substitute $v_0 = 20t_1$ into the distance equation

$-10t_1^2 + 20t_1^2 = 75 \Rightarrow t_1 = \sqrt{7.5}$

Therefore, after applying brakes for $\sqrt{7.5}$ minutes the car's speed is zero.

$v_0 = 20t_1 = 20\sqrt{7.5}\ \text{ms}^{-1}$. This can be changed into km/h

$$\frac{20\sqrt{7.5} \times 3600}{1000} \approx 197.18\ \text{km/h}$$

5. **(a)** The increase in population per year is $\dfrac{2250000 \times 365}{10^9} = 0.082125$ billion persons.

Thus, the annual growth rate is

$$\frac{dP}{dt} = kP \Rightarrow \text{ when } t = 0, k = \frac{\left(\frac{dP}{dt}\right)}{P} = \frac{0.08218}{7.6} = 0.01081$$

(b) $P(t) = P_0 e^{rt} = 7.6\,e^{0.01081t} = 7.6\,e^{0.01081(81)} = 18.64$ billion.

(c) Assuming the same growth rate and some estimated resources as in 2017

$$60 = 7.6\ e^{0.01081t} \Rightarrow t = \frac{\ln 7.895}{0.01081} \approx 191 \text{ years. That is in the year 2208.}$$

6. This is a case of exponential decay

$$\frac{dN}{dt} = -kN \Rightarrow N = N_0\,e^{-kt} = N_0\,e^{-0.0001216t} \text{ years}$$

If we start with 1 g of the material, 0.65 g will remain

$$0.65 = 1 \times e^{-0.0001216t} \Rightarrow t = \frac{-\ln 0.65}{0.0001216} \approx 3543$$

7. The basic Kirchhoff's law for the charge is given as

$$R\frac{\mathrm{d}Q}{\mathrm{d}t}+\frac{1}{C}Q=E(t)\Rightarrow 5\frac{\mathrm{d}Q}{\mathrm{d}t}+\frac{1}{0.05}Q=12\Rightarrow\frac{\mathrm{d}Q}{\mathrm{d}t}+4Q=12$$

Using the given hint in integration:

$$\mathrm{e}^{4t}\frac{\mathrm{d}Q}{\mathrm{d}t}+4\mathrm{e}^{4t}Q=12\mathrm{e}^{4t}\Rightarrow \mathrm{d}\left(\mathrm{e}^{4t}Q\right)=12\mathrm{e}^{4t}\mathrm{d}t\Rightarrow \mathrm{e}^{4t}Q=3\mathrm{e}^{4t}+c$$

With $Q(0) = 0,\ 0=3+c\Rightarrow c=-3\Rightarrow Q=3-3\mathrm{e}^{-4t}$

$$I=\frac{\mathrm{d}Q}{\mathrm{d}t}=12\mathrm{e}^{-4t}$$

8. **(a)** A basic exponential growth model and using the fact that the number increased from 20000 to 100000 in 10 minutes:

$$N(t)=N_0\mathrm{e}^{rt}\Rightarrow 100000=20000\mathrm{e}^{10t}\Rightarrow r=\frac{\ln 5}{10}$$

Therefore, the general model starting at the beginning of the experiment is

$$N(t)=10000\mathrm{e}^{\frac{\ln 5}{10}t}$$

(b) After 20 minutes, there were $N(20)=10000\mathrm{e}^{\frac{\ln 5}{10}\times 20}=250000$

(c) after t_1 there were 2000, which can be modelled by

$$20000=10000\mathrm{e}^{\frac{\ln 5}{10}t_1}\Rightarrow \ln 2=\frac{\ln 5}{10}t_1\Rightarrow t_1=\frac{10\ln 2}{\ln 5}\approx 4.3\text{ minutes.}$$

9. This is Newton's law again. Gaining rather than losing heat.

$$\frac{\mathrm{d}T}{\mathrm{d}t}=k(200-T)\Rightarrow T=200-C\mathrm{e}^{-kt}$$

Initial conditions:

$$t=0:10=200-C\mathrm{e}^{0}\Rightarrow C=190$$

$$t=75:52=200-190\mathrm{e}^{-75k}\Rightarrow \mathrm{e}^{-75k}=\frac{148}{190}\Rightarrow k=0.00333$$

Therefore, $T=200-190e^{-0.00333t}$

Using this model, we can estimate the time for the roast to reach $70°$.

$$70=200-190\mathrm{e}^{-0.00333t}\Rightarrow 190\mathrm{e}^{-0.00333t}=130\Rightarrow t\approx 114\text{ minutes}$$

10. Assuming 1990 to be year zero. The population growth model is (in 1000s) $P(t) = 25e^{rt}$. Using the data from 2000, we have

$30 = 25e^{10r} \Rightarrow r = 0.01823 \Rightarrow P(t) = 25e^{0.01823t}$, and for the year 2030

$P(40) = 25e^{0.01823\times 40} \approx 51.840$, about 52000 people.

11. Let A be the amount per kg.

$\frac{dA}{dt} = -kA \Rightarrow A(t) = A_0 e^{-kt}$, if half-life is 5 hours, then,

$\frac{A_0}{2} = A_0 e^{-5k} \Rightarrow k = 0.13863$

A 50-kg dog will need at least A(1) = 50(45)=2250 mg in its blood for an hour.

$A(1) = 2250 = A_0 e^{-0.13863\times 1} \Rightarrow A_0 = 2585$ mg to anesthetise the dog properly.

12. Numbers are in 1000s. The rate of increase of the number who have heard the rumour is proportional to the number who have not yet heard it can be modelled by

$\frac{dN}{dt} = k(100 - N),\ N(0) = 0$

The solution of this DE, will give us the number of people who have heard the rumor at time t.

$\frac{dN}{dt} = k(100 - N) \Rightarrow \frac{dN}{100 - N} = k\,dt \Rightarrow \ln(100 - N) = -kt + c$

$\Rightarrow 100 - N = e^{-kt+c} \Rightarrow N = 100 - e^{-kt+c} = 100 - Ae^{-kt}$

With initial conditions

$t = 0 : 0 = 100 - Ae^0 \Rightarrow A = 100$

$t = 7 : 10 = 100 - 100e^{-7k} \Rightarrow k = -\frac{\ln 0.9}{7} = 0.01505$

Thus, $N(t) = 100\left(1 - e^{-0.01505t}\right)$

Half of the city is 50000, thus, $50 = 100\left(1 - e^{-0.01505t}\right) \Rightarrow t \approx 46.05$

13. **(a)** The model we use for light intensity as described is $\frac{\mathrm{d}I}{\mathrm{d}x} = -kI$, which must be familiar by now :

$$\frac{\mathrm{d}I}{\mathrm{d}x} = -kI \Rightarrow \ln|I| = -kx + c \Rightarrow I = A\mathrm{e}^{-kx}$$

where x is the depth below the surface of water. With initial conditions:

$$x = 0 : 108 = A\mathrm{e}^{0} \Rightarrow A = 108$$

$$x = 25 : 10.8 = 108\mathrm{e}^{-25k} \Rightarrow k = -\frac{\ln 0.1}{25} = 0.0921 \Rightarrow I(x) = 108\mathrm{e}^{-0.0921x}$$

(b) $8.1 = 108e^{-0.0921x} \Rightarrow x = -\frac{\ln\frac{8.1}{108}}{0.0921} \approx 28.125\,\mathrm{m}$

14. **(a)** The net force on a large falling object is the difference between its weight, i.e., due to gravity, mg and drag force, which, as described is proportional to the square of its velocity. This is summarised in the following equation:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - kv^2.$$

(b) The limiting velocity will be stable: not increasing nor decreasing:

$$\frac{\mathrm{d}v}{\mathrm{d}t} = 0 \Rightarrow m\frac{\mathrm{d}v}{\mathrm{d}t} = mg - kv^2 = 0 \Rightarrow v = \sqrt{\frac{mg}{k}}$$

(c) $v = \sqrt{\frac{mg}{k}} \Rightarrow 49 = \sqrt{\frac{10\times 9.8}{k}} \Rightarrow k = \frac{10\times 9.8}{49^2} = \frac{2}{49} \approx 0.04082$

15. **(a)** We make use of the basic model suggested in the problem and replace I with $\frac{\mathrm{d}Q}{\mathrm{d}t}$

$$RI + \frac{Q}{C} = V \text{ and } I = \frac{\mathrm{d}Q}{\mathrm{d}t} \Rightarrow R\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{C} = V$$

$$\Rightarrow \frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{RC} = \frac{V}{R}$$

Next, multiply with $\mathrm{e}^{\frac{t}{RC}}$ as the hint suggests

$$\mathrm{e}^{\frac{t}{RC}}\left(\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{Q}{RC} = \frac{V}{R}\right) \Rightarrow d\left(\mathrm{e}^{\frac{t}{RC}}Q\right) = \frac{V}{R}\mathrm{e}^{\frac{t}{RC}}\mathrm{d}t \Rightarrow \mathrm{e}^{\frac{t}{RC}}Q = \int\frac{V}{R}\mathrm{e}^{\frac{t}{RC}}\mathrm{d}t$$

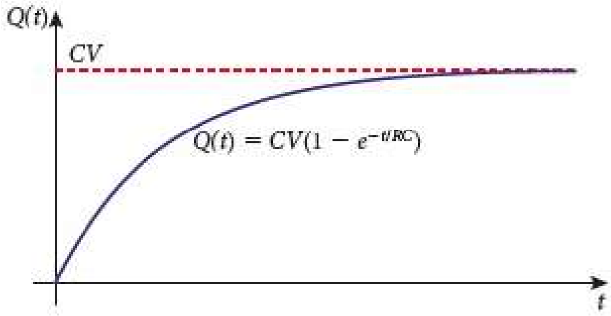
$$\int\frac{V}{R}\mathrm{e}^{\frac{t}{RC}}\mathrm{d}t = RC\frac{V}{R}\mathrm{e}^{\frac{t}{RC}} = CV\mathrm{e}^{\frac{t}{RC}} \Rightarrow \mathrm{e}^{\frac{t}{RC}}Q = CV\mathrm{e}^{\frac{t}{RC}} + A$$

$$\Rightarrow Q = CV + A\mathrm{e}^{\frac{t}{RC}}$$

Where A is an arbitrary integration constant.

With initial conditions

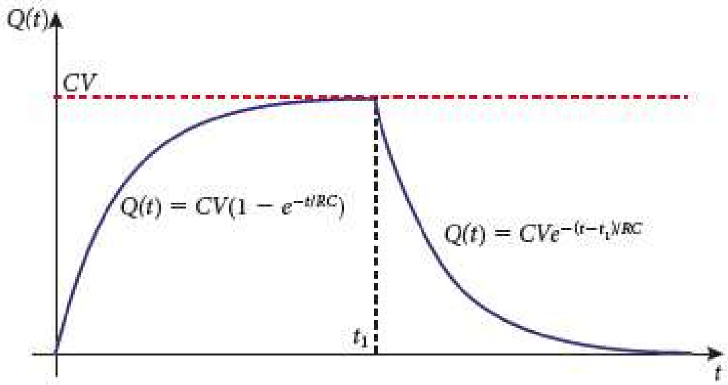
$$0 = CV + Ae^0 \Rightarrow A = -CV \Rightarrow Q = CV - CVe^{\frac{-t}{RC}} = CV\left(1 - e^{\frac{-t}{RC}}\right)$$



(b) $Q_L = \lim_{t\to\infty} CV\left(1 - e^{\frac{-t}{RC}}\right) = CV(1-0) = CV$

(c) When the battery is removed, $R\frac{dQ}{dt} + \frac{Q}{R} = 0$ which can be solved to give $Q(t) = Ke^{\frac{-t}{RC}}$, and initial conditions yield $K = CVe^{\frac{t_1}{RC}}$

Substitute this into the result and we have $Q(t) = CVe^{\frac{-(t-t_1)}{RC}}$



When there is no battery, the capacity will go down till it is completely exhausted.

16. **(a)** The rate of change of salt in the tank can be represented by $\frac{dq}{dt}$

Every minute, s litres of mixture flows in, each containing 0.25 kg of salt, thus we are adding $0.25s$ litres of salt per minute. Now, each minute, we are draining s litres of new mixture out, containing q litres of salt each, and hence the rate of salt out is $\frac{sq}{300}$. Remember that the amount of mixture is fixed at 300 litres since the rate in and out are the same.

$$\frac{dq}{dt} = 0.25s - \frac{sq}{300}$$

(b) Rearrange terms and multiply both sides with $e^{\frac{st}{300}}$ as hinted

$$e^{\frac{st}{300}}\left(\frac{dq}{dt}+\frac{sq}{300}=0.25s\right)\Rightarrow e^{\frac{st}{300}}\frac{dq}{dt}+e^{\frac{st}{300}}\frac{sq}{300}=e^{\frac{st}{300}}0.25s$$

$$\Rightarrow d\left(qe^{\frac{st}{300}}\right)=e^{\frac{st}{300}}0.25sdt\Rightarrow qe^{\frac{st}{300}}=\int e^{\frac{st}{300}}0.25sdt=0.25s\times\frac{300}{s}e^{\frac{st}{300}}+C$$

$$\Rightarrow qe^{\frac{st}{300}}=75e^{\frac{st}{300}}+C\Rightarrow q=75+ce^{\frac{st}{300}}$$

With initial conditions

$$q_0=75+Ce^0\Rightarrow C=q_0-75\Rightarrow q(t)=75+\left(q_0-75\right)e^{\frac{st}{300}}$$

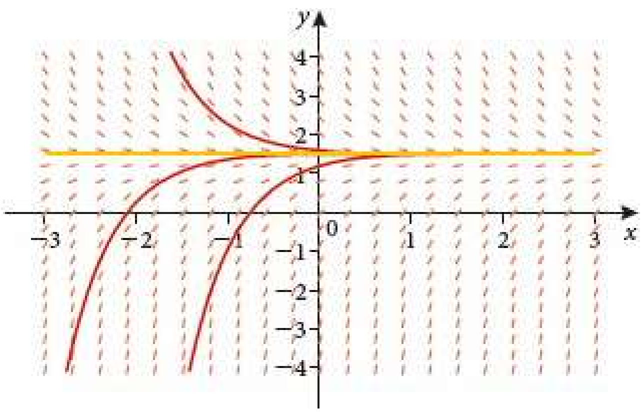
(c) In the long run, the amount of salt in the tank will be q_L :

$$q_L=\lim_{t\to\infty}q(t)=\lim_{t\to\infty}\left(75+\left(q_0-75\right)e^{-st/300}\right)=75+0=75$$

Exercise 20.3

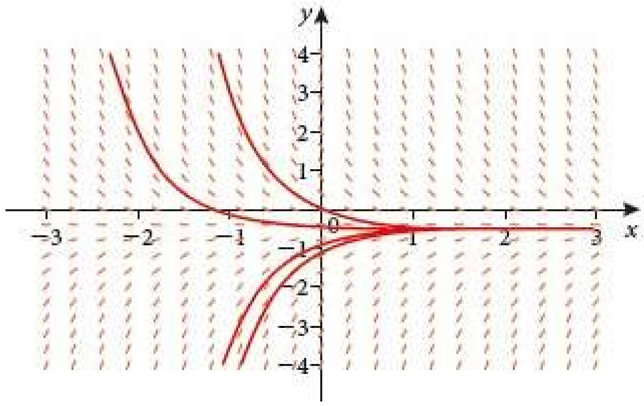
Note: All direction (slope) fields have been produced by a software program. Your GDC may have this ability, or you may use available software for such tasks.

1. **(a)** A direction field with a few curves are shown in diagram



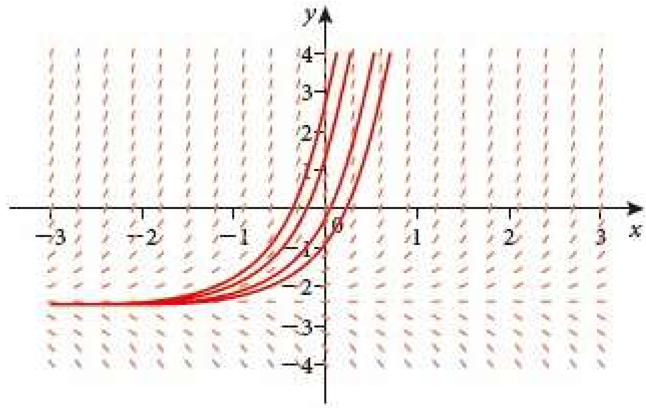
Solutions tend to $\frac{3}{2}$ as $x\to\infty$

(b) A direction field with a few curves are shown in diagram



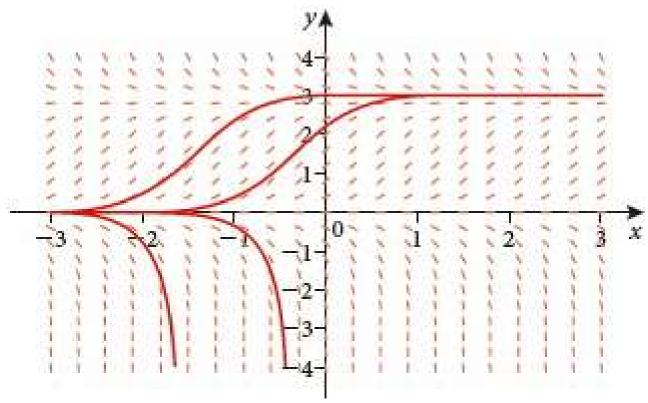
Solutions tend to $-\frac{1}{2}$ as $x \to \infty$

(c) A direction field with a few curves are shown in diagram



Solutions tend to approach $-\frac{5}{2}$ when $x \to -\infty$ and diverge to $\pm\infty$ as $x \to \infty$

(d) A direction field with a few curves are shown in diagram



Solutions tend to 3 if initial y-values are positive and diverge to $-\infty$ if initial y-values are negative. When $x \to -\infty$ all solutions approach 0.

2. $y' = y + xy,\ y(0) = 1 \Rightarrow F(x,y) = y + xy, x_0 = 0, y_0 = 1, h = 0.1$

Thus the recursive formula for y_n is

$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (0.1)(y_n + x_n y_n)$

Arranging the process in a table is an efficient organisation of the work involved

n	x_n	y_n	y_{n+1}
0	0	1	$1 + 0.1\times 1 = 1.1$
1	0.1	1.1	$1.1 + 0.1(1.1 + 0.1\times 1.1) = 1.221$
2	0.2	1.221	1.36752
3	0.3	1.36752	1.5453
4	0.4	1.5453	**1.7616**

3. **(a)** $\frac{dy}{dx} = 6x^2 - 3x^2 y,\quad y(0) = 3 \Rightarrow F(x,y) = 6x^2 - 3x^2 y, x_0 = 0, y_0 = 3$

Thus, the recursive formula for y_n is

$y_{n+1} = y_n + hF(x_n, y_n) = y_n + h(6x_n^2 - 3x_n^2 y_n)$

Since there is a large number of iterations required given the small sizes of the steps, it is best if we set up a spreadsheet to do the calculations.

$h \rightarrow$	0.1		0.01		0.001	
n	x(n)	y(n)	x(n)	y(n)	x(n)	y(n)
0	0	3	0	3	0	3
1	0.1	3	0.01	3	0.001	3
↓	↓	↓	↓	↓	↓	↓
10	1	**2.392794**	0.1	2.999145	0.01	2.999999
11			0.11	2.998846	0.011	2.999999
↓			↓	↓	↓	↓
100			1	**2.370111**	0.1	2.999015
101					0.101	2.998985
↓					↓	↓
1000					1	**2.3681**

(b) We substitute the suggested solution into the DE

$y = e^{-x^3} + 2 \Rightarrow \frac{dy}{dx} = -3x^2 e^{-x^3} = -3x^2(y-2) = 6x^2 - 3x^2 y$

(c) Exact value: $y = e^{-x^3} + 2 = e^{-1} + 2 \approx 2.367879$

Error when $h = 0.1$: $2.367879 - 2.392794 = -0.024915$

Error when $h = 0.01$: $2.367879 - 2.370111 = -0.002232$

Error when $h = 0.001$: $2.367879 - 2.3681 = -0.000221$

It appears as if the error is divided by 10 (approximately).

4. $\frac{dy}{dx} = 3x - 2y + 1, \quad y(1) = 2 \Rightarrow F(x, y) = 3x - 2y + 1, x_0 = 1, y_0 = 2$

Thus, the recursive formula for y_n is

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (0.5)(3x_n - 2y_n + 1)$$

$$y_{n+1} = y_n + (0.5)(3x_n - 2y_n + 1)$$
$$y_{0+1} = y_0 + (0.5)(3x_0 - 2y_0 + 1) = 2 + 0.5(3 - 2\times 2 + 1) = 2$$
$$y_{1+1} = y_1 + (0.5)(3x_1 - 2y_1 + 1) = 2 + 0.5(3\times 1.5 - 2\times 2 + 1) = 2.75$$
$$y_{2+1} = y_2 + (0.5)(3x_2 - 2y_2 + 1) = 2.75 + 0.5(3\times 2 - 2\times 2.75 + 1) = 3.5$$
$$y_{3+1} = y_3 + (0.5)(3x_3 - 2y_3 + 1) = 3.5 + 0.5(3\times 2.5 - 2\times 3.5 + 1) = 4.25$$

5. $\frac{dy}{dx} = x + y^2, \quad y(0) = 0 \Rightarrow F(x, y) = x + y^2, x_0 = 0, y_0 = 0$

Thus, the recursive formula for y_n is

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (0.2)(x_n + y_n^2)$$

We will use a spreadsheet for the iterations.

n	x(n)	y(n)
0	0	0
1	0.2	0
2	0.4	0.04
3	0.6	0.12032
4	0.8	0.243215
5	1	**0.415046**

6. $\frac{dy}{dx} = x^2 + y^2, \quad y(0) = 1 \Rightarrow F(x, y) = x^2 + y^2, x_0 = 0, y_0 = 1$

Thus, the recursive formula for y_n is

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (0.1)(x_n^2 + y_n^2)$$

Spreadsheet output for **Q6** is given below

n	x(n)	y(n)
0	0	1
1	0.1	1.1
2	0.2	1.222
3	0.3	1.375328
4	0.4	1.573481
5	0.5	**1.837066**

7. **(a)** $\frac{dy}{dt} = y(3-ty), \quad y(0) = 0.5 \Rightarrow F(t, y) = y(3-ty), t_0 = 0, y_0 = 0.5$

Thus, the recursive formula for y_n is

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (h)y_n(3 - t_n y_n)$$

Spreadsheet output below:

$h \rightarrow$	0.1		0.05		0.01	
n	x(n)	y(n)	x(n)	y(n)	x(n)	y(n)
0	0	0.5	0	0.5	0	0.5
1	0.1	0.65	0.05	0.575	0.01	0.515
↓	↓	↓	↓	↓	↓	↓
5	0.5	1.703081	0.25	0.989081	0.05	0.57933
6	0.6	2.06898	0.3	1.125214	0.06	0.596542
↓	↓	↓	↓	↓	↓	↓
10	1	3.06605	0.5	1.795475	0.1	0.670188
↓	↓	↓	↓	↓	↓	↓
15	1.5	2.440297	0.75	2.686308	0.15	0.773706
↓	↓	↓	↓	↓	↓	↓
30	3	1.119248	1.5	2.432919	0.3	1.168512
↓	↓	↓	↓	↓	↓	↓
50			2.5	1.377951	0.5	1.877339
↓	↓	↓	↓	↓	↓	↓
60			3	1.121911	0.6	2.261424
↓	↓	↓	↓	↓	↓	↓
150					1.5	2.426722
↓	↓	↓	↓	↓	↓	↓
300					3	1.124112

(b) $y' = 5 - 3\sqrt{y}, \quad y(0) = 2 \Rightarrow F(x, y) = 5 - 3\sqrt{y}, x_0 = 0, y_0 = 2$

Thus, the recursive formula for y_n is

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (h)\left(5 - 3\sqrt{y_n}\right)$$

Spreadsheet output below:

$h \rightarrow$	0.1		0.05		0.01	
n	x(n)	y(n)	x(n)	y(n)	x(n)	y(n)
0	0	2	0	2	0	2
1	0.1	2.075736	0.05	2.037868	0.01	2.007574
↓	↓	↓	↓	↓	↓	↓
5	0.5	2.307998	0.25	2.170531	0.05	2.037075
↓	↓	↓	↓	↓	↓	↓
10	1	2.490062	0.5	2.301666	0.1	2.072241
↓	↓	↓	↓	↓	↓	↓
15	1.5	2.600226	0.75	2.403337	0.15	2.105611
↓	↓	↓	↓	↓	↓	↓
30	3	2.73521	1.5	2.593517	0.3	2.195958
↓	↓	↓	↓	↓	↓	↓
50			2.5	2.705188	0.5	2.296863
↓	↓	↓	↓	↓	↓	↓
60			3	2.73209	0.6	2.340239
↓	↓	↓	↓	↓	↓	↓
150					1.5	2.588297
↓	↓	↓	↓	↓	↓	↓
300					3	2.729585

(c) $\frac{dy}{dt} = \frac{4-ty}{1+y^2}, y(0) = -2 \Rightarrow F(t,y) = \frac{4-ty}{1+y^2}, t_0 = 0, y_0 = -2$

Thus, the recursive formula for y_n is

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + (h)\left(\frac{4-t_n y_n}{1+y_n^2}\right)$$

Spreadsheet output below:

$h \rightarrow$	0.1		0.05		0.01	
n	t(n)	y(n)	t(n)	y(n)	t(n)	y(n)
0	0	-2	0	-2	0	-2
1	0.1	-1.92	0.05	-1.96	0.01	-1.992
↓	↓	↓	↓	↓	↓	↓
5	0.5	-1.48849	0.25	-1.775	0.05	-1.95907
↓	↓	↓	↓	↓	↓	↓
10	1	-0.41234	0.5	-1.46909	0.1	-1.91572
↓	↓	↓	↓	↓	↓	↓
15	1.5	1.046866	0.75	-1.02416	0.15	-1.86977
↓	↓	↓	↓	↓	↓	↓
30	3	1.51971	1.5	1.053515	0.3	-1.71402
↓	↓	↓	↓	↓	↓	↓
50			2.5	1.530002	0.5	-1.45212
↓	↓	↓	↓	↓	↓	↓
60			3	1.50549	0.6	-1.28842
↓	↓	↓	↓	↓	↓	↓
150					1.5	1.059414
↓	↓	↓	↓	↓	↓	↓
300					3	1.494898

8. To find the exact value of the solution to the DE $\frac{\mathrm{d}y}{\mathrm{d}x} - y = \cos x$ we need to multiply with e^{-x}. Now, multiply and simplify:

$$\mathrm{e}^{-x}\frac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^{-x}y = \mathrm{e}^{-x}\cos x \Rightarrow \mathrm{d}\left(\mathrm{e}^{-x}y\right) = \mathrm{e}^{-x}\cos x\,\mathrm{d}x \Rightarrow \mathrm{e}^{-x}y = \int \mathrm{e}^{-x}\cos x\,\mathrm{d}x$$

RHS can be evaluated using integration by parts:

$\mathrm{e}^{-x}y = \frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x) + c$, and with initial conditions $0 = \frac{1}{2}(0-1) + c \Rightarrow c = \frac{1}{2}$

The exact particular solution is

$$\mathrm{e}^{-x}y = \frac{\mathrm{e}^{-x}}{2}(\sin x - \cos x) + \frac{1}{2} \Rightarrow y = \frac{1}{2}\left(\sin x - \cos x + \mathrm{e}^{x}\right)$$

Finding the exact values is simple substitution of the given numbers into the solution formula found earlier.

For example, $y(0) = \frac{1}{2}\left(\sin 0 - \cos 0 + e^{0}\right) = 0$. The rest of the row is similarly done.

Using $h = 0.1$ or 0.2 to approximate the solution using Euler's method:

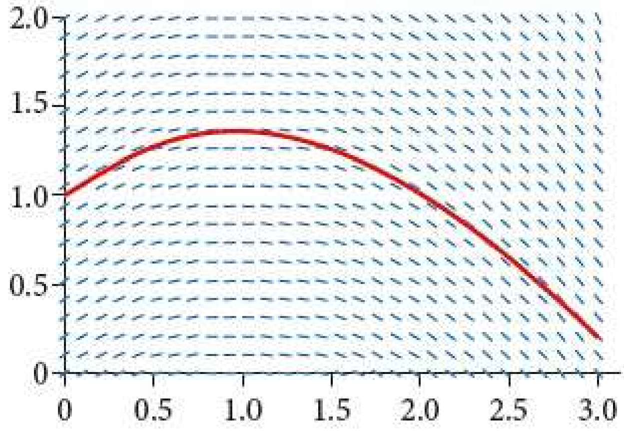
We first rewrite the equation if the form $y' = F(x, y)$

$\frac{\mathrm{d}y}{\mathrm{d}x} - y = \cos x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = y + \cos x$, and then do the iterations using

$y_{n+1} = y_n + hF(x_n, y_n) = y_n + h(y_n + \cos x_n)$. 15 iterations are needed, and thus, it is more efficient to use a spreadsheet for the calculations

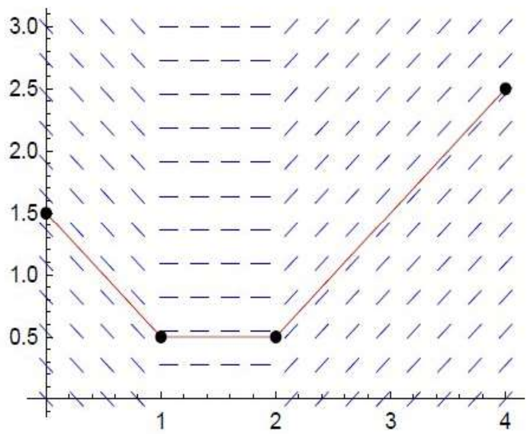
Exact		0.2		0.1	
x(n)	y(n)	x(n)	y(n)	x(n)	y(n)
0	0	0	0	0	0
0.2	0.220003	0.2	0.2	0.1	0.1
0.4	0.480091	0.4	0.436013	0.2	0.2095
0.6	0.780713	0.6	0.707428	0.3	0.328457
0.8	1.123095	0.8	1.013981	0.4	0.456836
1	1.509725	1	1.356118	0.5	0.594626
				0.6	0.741847
				0.7	0.898565
				0.8	1.064906
				0.9	1.241067
				1	1.427335

9. **(a)** Answers may differ in this question as the diagram is too small as given.



(b) $y(3)$ is somewhere between 0 and 0.5.

10. **(a)** Slope field is shown in diagram



(b) $y(4) = 2.5$

11. **(a)** We need to look at a few points such as (1, 1) where $y' = xy \Rightarrow y' = 1$, which may indicate I, or III. However, looking at (2, 1) where $y' = xy \Rightarrow y' = 2$ we can eliminate I since the gradient is negative. III is correct as you notice the tangent is steep enough to justify a gradient of 2.

(b) A similar approach to **(a)**. $y' = x^2 + y^2; (0.5, 1) \Rightarrow y' = 1.25$ or $(2, 1) \Rightarrow y' = 5$

indicates that IV is the only viable choice. I and II are not appropriate since the gradient can also be negative which cannot be the case here.

(c) Choosing (1.5, 1.5), $y' = 1.5^2 - 1.5^2 = 0$ can only be satisfied by II.

(d) Even though this is the only choice left but testing a few points can confirm the choice.

Take the point (2, 2) for example, $y' = 2 - 2^2 = -2$ which is clearly satisfied only by I.

12. An approach similar to question 12 will give us the following:

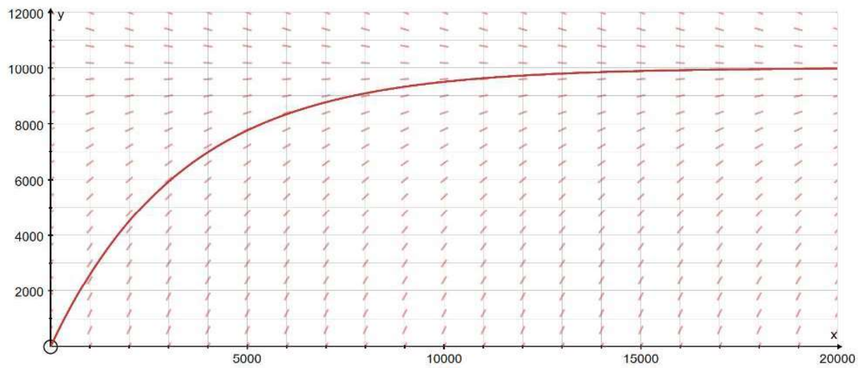
(a) ii **(b)** iv **(c)** i **(d)** iii

13. **(a)** The rate of increase of pollutant per hour is 300×0.01. Draining 300 litres out with pollutant concentration of $\frac{q}{1000000}$ will give us the following model:

$$\frac{dq}{dt}=300\times 0.01-300\times\frac{q}{1000000}=300\left(0.01-10^{-6}q\right);$$

q in grams and t in hours, $q(0) = 0$

(b) Using a direction field may help answer this question.



Apparently, the long term trend stabilises at 10000.

Since all solutions converge to that value, this limiting amount does not depend on the initial amounts.

(c) $$\frac{dq}{dt}=300\left(0.01-10^{-6}q\right)\Rightarrow 10^{6}\frac{dq}{10^{4}-q}=300\,dt\Rightarrow 10^{4}\frac{dq}{10^{4}-q}=3dt$$

$$\Rightarrow -10^{4}\ln\left|10^{4}-q\right|=3t+C$$

With initial conditions

$$-10^{4}\ln\left|10^{4}-0\right|=0+C\Rightarrow C=-10^{4}\ln 10^{4}\Rightarrow -10^{4}\ln\left|10^{4}-q\right|=3t-10^{4}\ln 10^{4}$$

$$\Rightarrow \ln\left|10^{4}-q\right|=\ln 10^{4}-3\times 10^{-4}t\Rightarrow q-10^{4}=-e^{\ln 10^{4}-3\times 10^{-4}t}$$

$$\Rightarrow q=10^{4}-10^{4}e^{-3\times 10^{-4}t}=10^{4}\left(1-e^{-3\times 10^{-4}t}\right)$$

(i) After one year the amount of pollutant will be approximately

$$10^{4}\left(1-e^{-3\times 10^{-4}\times 365\times 24}\right)\approx 9278\text{ g}$$

(ii) $$\lim_{t\to\infty} q\left(10^{4}\left(1-e^{-0.0003t}\right)\right)=10^{4}\text{ g}$$

(d) Since no pollutant is added then, we are only reducing the pollutant

$$\frac{\mathrm{d}q}{\mathrm{d}t} = 300\left(-10^{-6}q\right); \quad q(0) = 9278$$

(e) With separation of variables, the solution with initial conditions will be

$q(t) = 9278\left(\mathrm{e}^{-0.0003t}\right)$.

After one year, we will have $q(365\times 24) = 9278\left(\mathrm{e}^{-0.0003\times 365\times 24}\right) \approx 670$ g

(f) We need to solve the equation

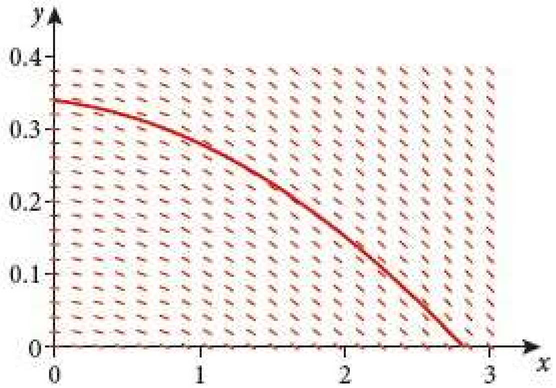
$$10 = 9278\left(\mathrm{e}^{-0.0003t}\right) \Rightarrow t = \frac{\ln\left(\frac{10}{9278}\right)}{-0.0003} \approx 22776 \text{ hours } \approx 2.6 \text{ years}$$

14. **(a)** $\frac{\mathrm{d}V}{\mathrm{d}t} = -kA$ per description. However, the surface area and volume of a sphere are not independent of each other. We will express the area, A, in terms the volume V.

$$A = 4\pi r^2, V = \frac{4}{3}\pi r^3 \Rightarrow A = \sqrt[3]{36\pi}V^{\frac{2}{3}}$$

Hence, $\frac{\mathrm{d}V}{\mathrm{d}t} = -kA = -kV^{\frac{2}{3}}$, where $k = c\sqrt[3]{36\pi}$ and c is a constant of proportionality that depends on several factors like temperature and friction.

(b) Here is a direction field

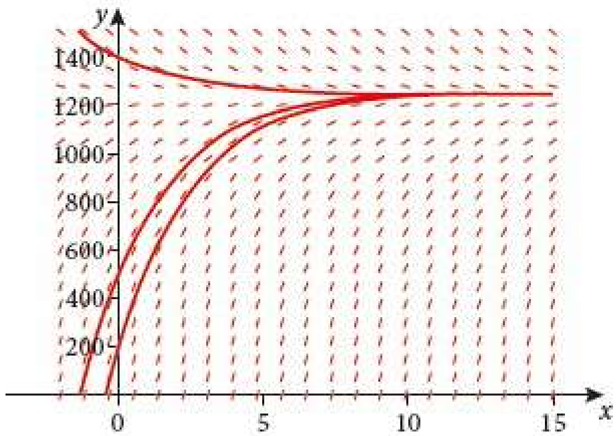


A solution corresponding to a volume of 0.034 is sketched and the t-intercept (x-intercept on this graph) is noted as approximately 2.8 minutes.

15. **(a)** The amount of drug entering the body every hour is 5×100 mg and leaves at a rate of $0.4q$ where q is the amount of drug present in the blood stream. This situation can be modelled by

$$\frac{dq}{dt} = 500 - 0.4q;\ q \text{ in mg, } t \text{ in h.}$$

(b) We use a slope field for this



The amount will stabilise at 1250 mg.

Exercise 20.4

As suggested in the book, this section's exercises are best done with a CAS or a GDC with special programs for phase portraits. We will demonstrate finding eigenvalues and eigenvectors in a few cases, but the rest is left for you since it is repetitive. Also refer to this chapter or to chapter 7 for more examples. Phase portraits were produced using open source software found around the internet.

1. **(a)** The first step is to find the eigenvalues.

$$A - \lambda I = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix}$$

In particular we need to determine where the value(s) of λ for which the determinant of this matrix is zero.

$$\det(A - \lambda I) = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4$$

Thus, the characteristic polynomial is $\lambda^2 - 3\lambda - 4$

Now, finding the zeros of this polynomial will give us the eigenvalues

$$\lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0$$

$$\Rightarrow \lambda_1 = -1, \lambda_2 = 4.$$

Each of these eigenvalues will generate an eigenvector.

For $\lambda_1 = -1$:

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\vec{v} = \vec{0},$$

Let $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\Rightarrow \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 3a + 3b = 0 \\ 2a + 2b = 0 \end{cases}$

The last system can be easily reduced by subtraction $\begin{cases} a + b = 0 \\ 0 = 0 \end{cases}$

$$a + b = 0 \Rightarrow b = -a \Rightarrow \vec{v} = \begin{pmatrix} a \\ -a \end{pmatrix} = a\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

For $\lambda_2 = 4$:

$$(A - \lambda I)\vec{v} = \vec{0} \Rightarrow \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\vec{v} = \vec{0},$$

Let $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, $\Rightarrow \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -2a + 3b = 0 \\ 2a - 3b = 0 \end{cases}$

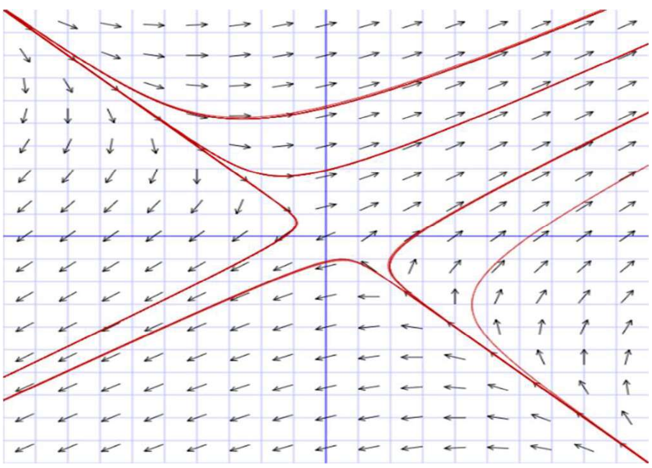
The last system can be easily reduced by subtraction $\begin{cases} -2a + 3b = 0 \\ 0 = 0 \end{cases}$

$$-2a + 3b = 0 \Rightarrow b = \tfrac{2}{3}a \Rightarrow \vec{v} = \begin{pmatrix} a \\ \tfrac{2}{3}a \end{pmatrix} = a\begin{pmatrix} 1 \\ \tfrac{2}{3} \end{pmatrix} = k\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Therefore, a general solution is $\begin{cases} x = 3C_1e^{4t} + C_2e^{-t} \\ y = 2C_1e^{4t} - C_2e^{-t} \end{cases}$

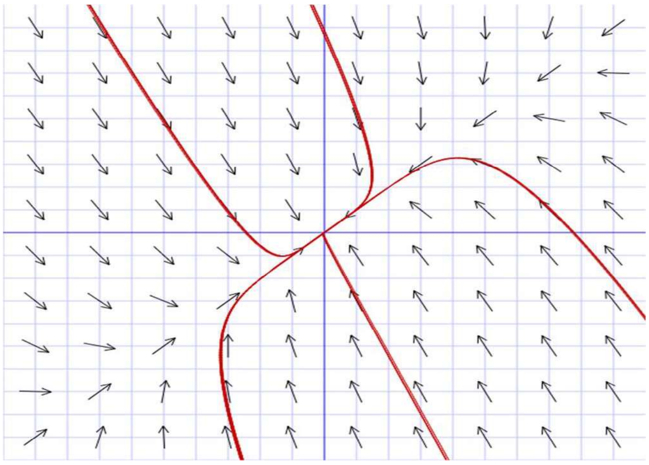
Since eigenvalues are of opposite signs, Trajectories approach equilibrium and then move away. Saddle point. Unstable.

Here is a sample phase portrait with a few trajectories



(b) Eigenvalues: –4, –25, eigenvectors $\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\-5\end{pmatrix}$

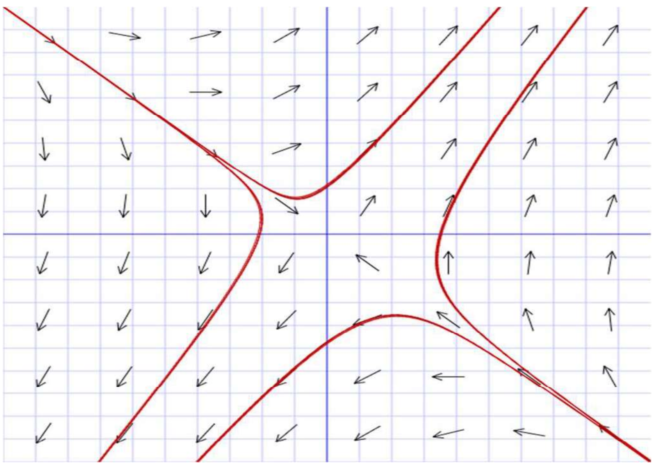
General solution: $\begin{cases} x = C_1e^{-4t} + 2C_2e^{-25t} \\ y = C_1e^{-4t} - 5C_2e^{-25t} \end{cases}$



Since eigenvalues are both negative, we have a stable system. Trajectories move towards the equilibrium point which is called stable node in this case.

(c) Eigenvalues: –2, 6, eigenvectors $\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}3\\5\end{pmatrix}$

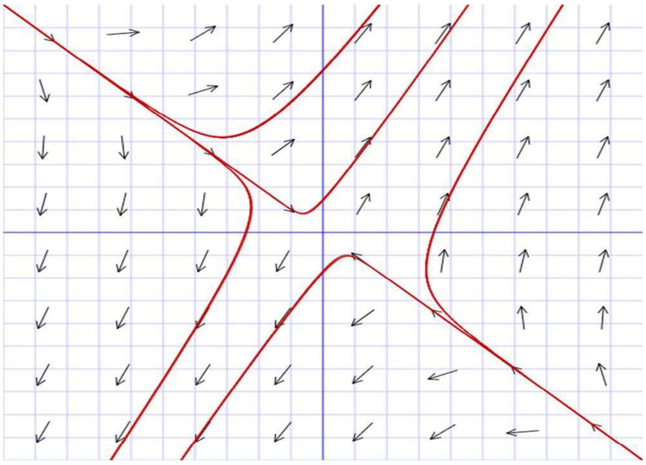
General solution: $\begin{cases} x = C_1e^{-2t} + 3C_2e^{6t} \\ y = -C_1e^{-2t} + 5C_2e^{6t} \end{cases}$



Since eigenvalues are of opposite signs, Trajectories approach equilibrium and then move away. Saddle point. Unstable.

(d) Eigenvalues: 5, –1, eigenvectors $\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}$

General solution: $\begin{cases} x = C_1 e^{5t} - C_2 e^{-t} \\ y = 2C_1 e^{5t} + C_2 e^{-t} \end{cases}$

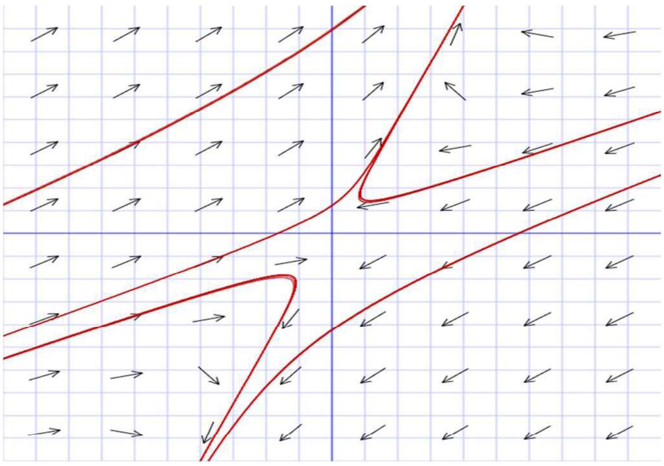


Trajectories approach equilibrium and then move away.

Saddle point. Unstable.

(e) Eigenvalues: –3, 1, eigenvectors $\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \end{pmatrix}$

General solution: $\begin{cases} x = 2C_1 e^{-3t} + 2C_2 e^{t} \\ y = C_1 e^{-3t} + 5C_2 e^{t} \end{cases}$

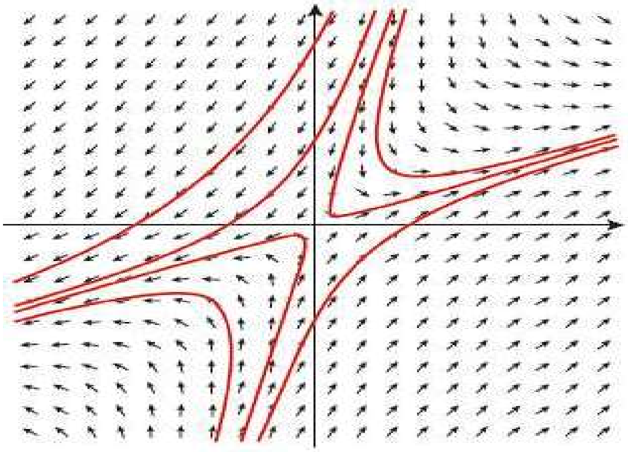


Again, trajectories approach equilibrium and then move away.

Saddle point. Unstable.

(f) Eigenvalues: 8, –10, eigenvectors $\begin{pmatrix}5\\2\end{pmatrix},\begin{pmatrix}1\\4\end{pmatrix}$

General solution: $\begin{cases}x = 5C_1\mathrm{e}^{8t} + C_2\mathrm{e}^{-10t}\\ y = 2C_1\mathrm{e}^{8t} + 4C_2\mathrm{e}^{-10t}\end{cases}$

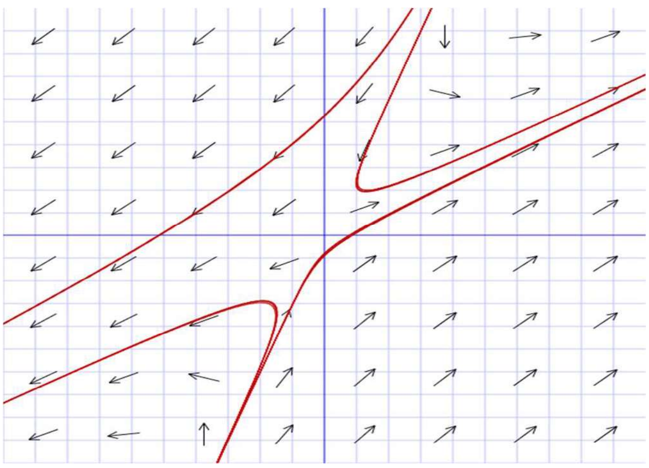


Trajectories approach equilibrium and then move away.

Saddle point. Unstable.

(g) Eigenvalues: –2, 5, eigenvectors $\begin{pmatrix}1\\3\end{pmatrix},\begin{pmatrix}1\\ \frac{2}{3}\end{pmatrix}$

General solution: $\begin{cases}x = C_1\mathrm{e}^{-2t} + C_2\mathrm{e}^{5t}\\ y = 3C_1\mathrm{e}^{-2t} + \frac{2}{3}C_2\mathrm{e}^{5t}\end{cases}$

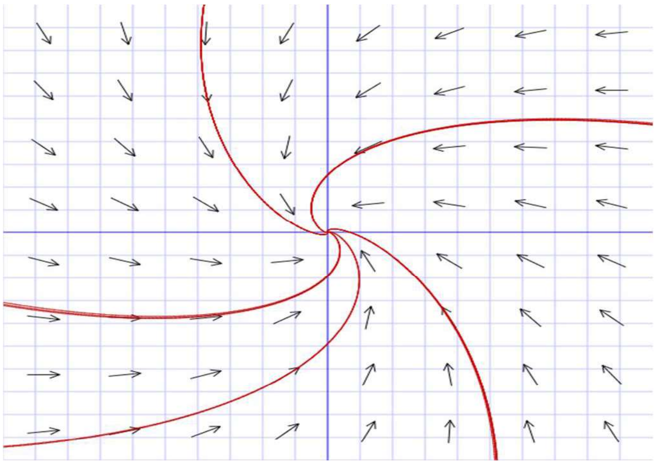


Trajectories approach equilibrium and then move away.

Saddle point. Unstable.

(h) Eigenvalues: –1, –4, eigenvectors $\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}, \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$

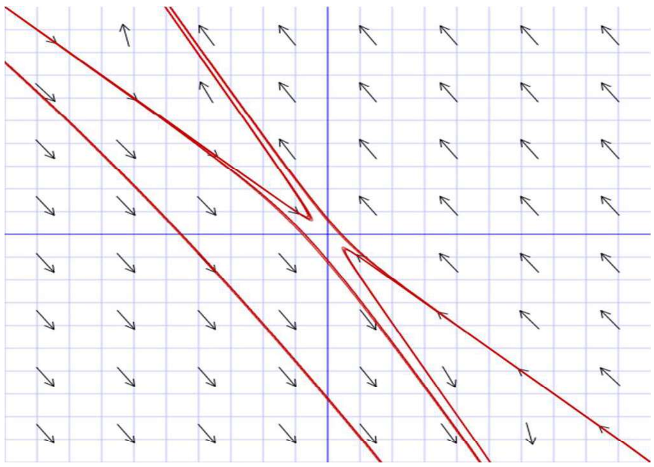
General solution: $\begin{cases} x = C_1 e^{-t} - \sqrt{2} C_2 e^{-4t} \\ y = \sqrt{2} C_1 e^{-t} + C_2 e^{-4t} \end{cases}$



With both eigenvalues negative, the curves move towards the equilibrium. Stable system with the origin as a **node**.

(i) Eigenvalues: –1, 2, eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

General solution: $\begin{cases} x = -C_1 e^{-t} - C_2 e^{2t} \\ y = C_1 e^{-t} + 2C_2 e^{2t} \end{cases}$



Trajectories approach equilibrium and then move away.

Saddle point. Unstable.

2. **(a)** The first step is to find the eigenvalues.

$$A-\lambda I=\begin{pmatrix}6 & -1\\ 5 & 4\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}6-\lambda & -1\\ 5 & 4-\lambda\end{pmatrix}$$

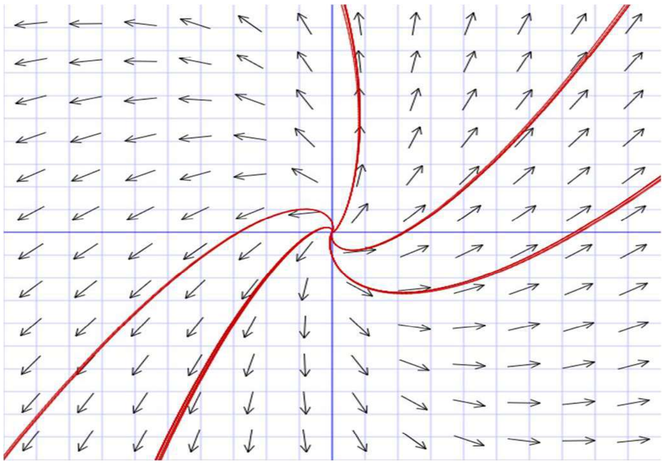
In particular we need to determine where the value(s) of λ for which the determinant of this matrix is zero.

$$\det(A-\lambda I)=(6-\lambda)(4-\lambda)+5=\lambda^2-10\lambda+29$$

Thus, the characteristic polynomial is $\lambda^2-10\lambda+29$. Now, finding the zeros of this polynomial will give us the eigenvalues

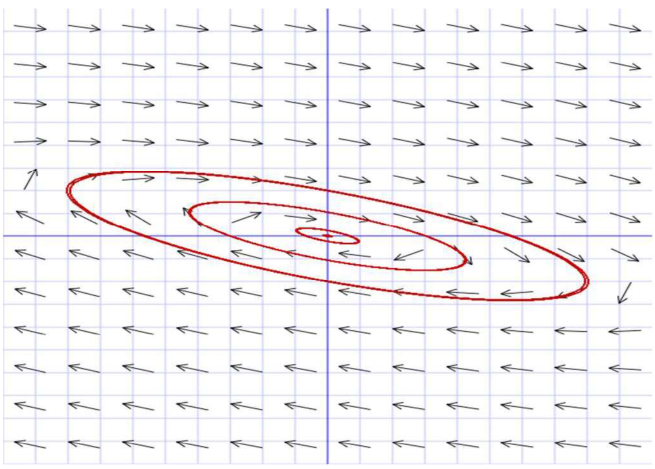
$$\lambda^2-10\lambda+29=0$$

$$\Rightarrow \lambda_1=5-2i, \lambda_2=5+2i.$$



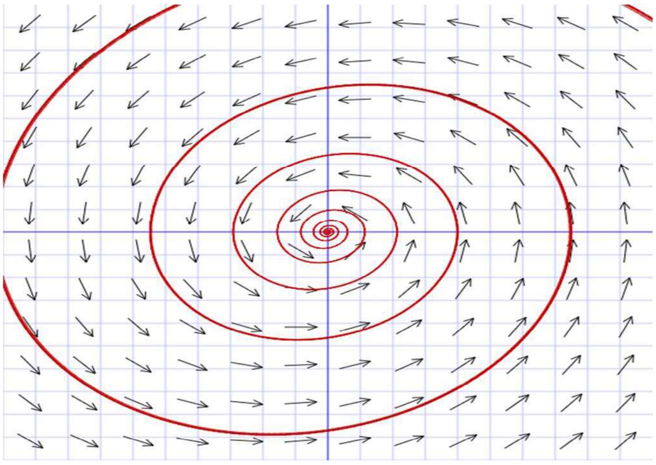
Eigenvalues are complex and the real part is positive, the trajectories move away from the origin in a spiral manner, the origin is an ***unstable focus***, also known as a ***spiral source***.

(b) Eigenvalues: $\pm$ 2i.



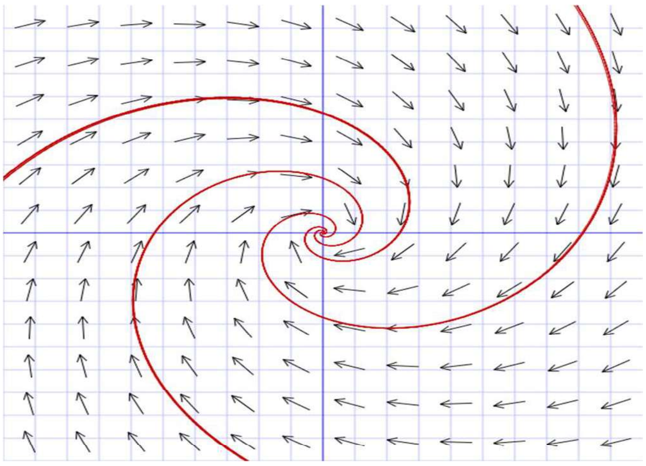
Eigenvalues are purely imaginary; the origin is a ***stable equilibrium point*** (but not asymptotically stable). It is also called a ***centre***.

(c) Eigenvalues: $-0.1 \pm \mathrm{i}$



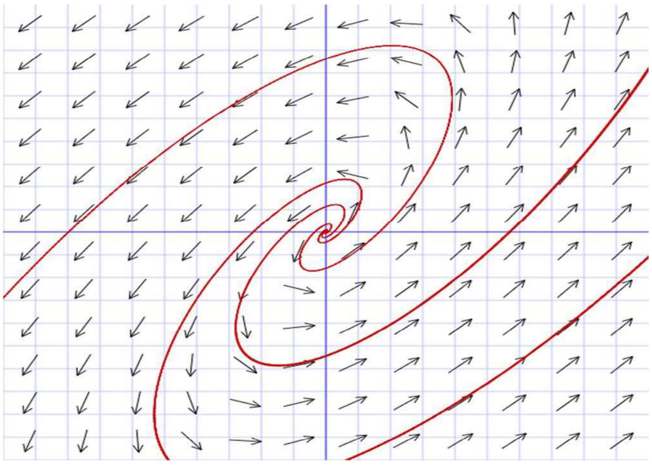
Eigenvalues are complex and the real part is negative, the trajectories approach the origin in a spiral manner.The origin is an ***asymptotically stable focus***, also known as a ***spiral sink***.

(d) Eigenvalues: $-\frac{1}{2} \pm i$



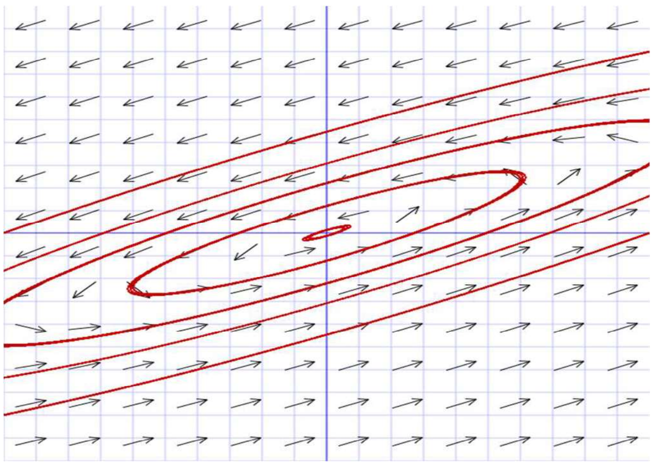
Eigenvalues are complex and the real part is negative, the trajectories approach the origin in a spiral manner. The origin is an ***asymptotically stable focus***, also known as a ***spiral sink***.

(e) Eigenvalues: $1 \pm \mathrm{i}$



Eigenvalues are complex and the real part is positive, the trajectories move away from the origin in a spiral manner, the origin is an ***unstable focus***, also known as a ***spiral source***.

(f) Eigenvalues: $\pm \mathrm{i}$

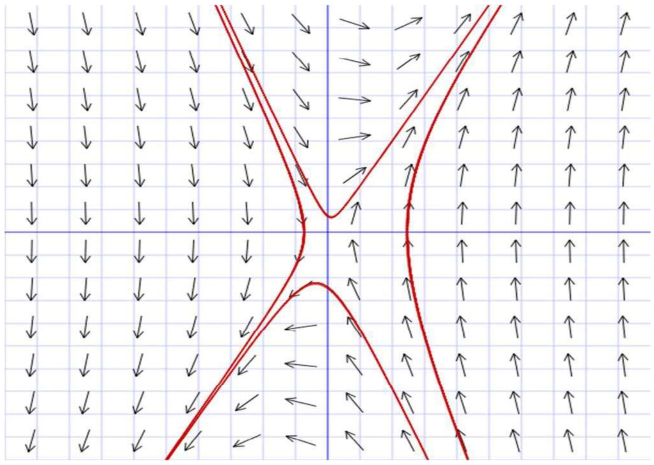


Eigenvalues are purely imaginary; the origin is a ***stable equilibrium point*** (but not asymptotically stable). It is also called a ***centre.***

3. **(a)** Eigenvalues: –3, 2, eigenvectors $\begin{pmatrix}1\\-3\end{pmatrix},\begin{pmatrix}1\\2\end{pmatrix}$

General solution: $\begin{cases}x = C_1\mathrm{e}^{-3t} + C_2\mathrm{e}^{2t}\\ y = -3C_1\mathrm{e}^{-3t} + 2C_2\mathrm{e}^{2t}\end{cases}$

Particular solution: $\begin{cases}3 = C_1 + C_2\\ 1 = -3C_1 + 2C_2\end{cases} \Rightarrow \begin{cases}C_1 = 1\\ C_2 = 2\end{cases} \Rightarrow \begin{cases}x = \mathrm{e}^{-3t} + 2\mathrm{e}^{2t}\\ y = -3\mathrm{e}^{-3t} + 4\mathrm{e}^{2t}\end{cases}$



Trajectories approach equilibrium and then move away.

Saddle point. Unstable

Euler: $\begin{cases}x_{n+1} = x_n + h(y_n)\\ y_{n+1} = y_n + h(6x_n - y_n)\end{cases}$

$\Rightarrow \begin{cases}x_{0+1} = x_0 + h(y_0)\\ y_{0+1} = y_0 + h(6x_0 - y_0)\end{cases} \Rightarrow \begin{cases}x_1 = 3 + 0.1(1) = 3.1\\ y_1 = 1 + 0.1(18-1) = 2.7\end{cases}$

$\Rightarrow \begin{cases}x_2 = x_1 + h(y_1)\\ y_2 = y_1 + h(6x_1 - y_1)\end{cases} \Rightarrow \begin{cases}x_2 = 3.1 + 0.1(2.7)\\ y_2 = 2.7 + 0.1(6\times 3.1 - 2.7)\end{cases} \Rightarrow \begin{cases}x_2 = 3.37\\ y_2 = 4.29\end{cases}$

Exact: $\begin{cases}x = \mathrm{e}^{-3(0.2)} + 2\mathrm{e}^{2(0.2)} \approx 3.53\\ y = -3\mathrm{e}^{-3(0.2)} + 4\mathrm{e}^{2(0.2)} \approx 4.32\end{cases} \Rightarrow \begin{cases}4.5\%\text{ error}\\ 0.7\%\text{ error}\end{cases}$

(b) The first step is to find the eigenvalues.

$$A-\lambda I=\begin{pmatrix}0 & -1\\ 10 & -7\end{pmatrix}-\lambda\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}=\begin{pmatrix}-\lambda & -1\\ 10 & -7-\lambda\end{pmatrix}$$

In particular we need to determine where the value(s) of λ for which the determinant of this matrix is zero.

$$\det(A-\lambda I)=(-\lambda)(-7-\lambda)+10=\lambda^2+7\lambda+10$$

Thus, the characteristic polynomial is $\lambda^2+7\lambda+10$. Now, finding the zeros of this polynomial will give us the eigenvalues

$$\lambda^2+7\lambda+10=(\lambda+2)(\lambda+5)=0$$

$$\Rightarrow \lambda_1=-2, \lambda_2=-5$$

Each of these eigenvalues will generate an eigenvector.

For $\lambda_1=-2$:

$$(A-\lambda I)\vec{v}=\vec{0}\Rightarrow\begin{pmatrix}2 & -1\\ 10 & -5\end{pmatrix}\vec{v}=\vec{0},$$

Let $\vec{v}=\begin{pmatrix}a\\ b\end{pmatrix}$, $\Rightarrow\begin{pmatrix}2 & -1\\ 10 & -5\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}\Rightarrow\begin{cases}5a-b=0\\ 10a-2b=0\end{cases}$

The last system can be easily reduced by subtraction $\begin{cases}5a-b=0\\ 0=0\end{cases}$

$$5a-b=0\Rightarrow b=5a\Rightarrow\vec{v}=\begin{pmatrix}a\\ 5a\end{pmatrix}=a\begin{pmatrix}1\\ 5\end{pmatrix}$$

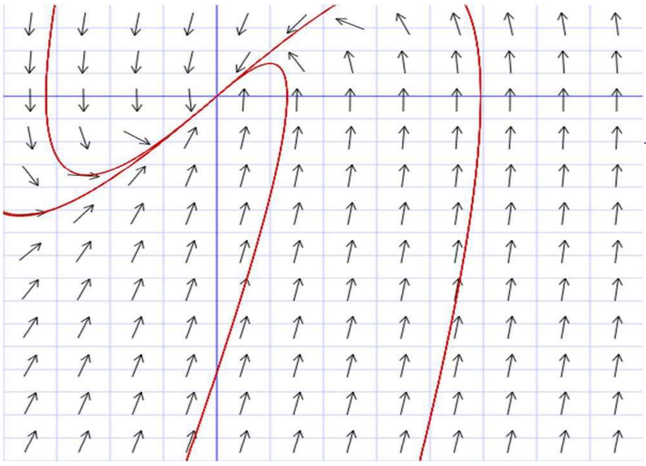
For $\lambda_2=-5$:

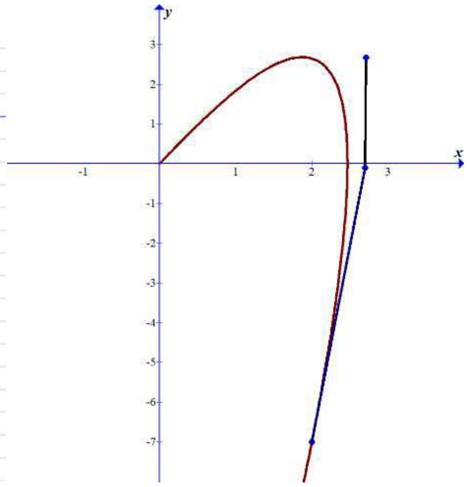
$$(A-\lambda I)\vec{v}=\vec{0}\Rightarrow\begin{pmatrix}5 & -1\\ 10 & -2\end{pmatrix}\vec{v}=\vec{0},$$

Similar steps to earlier will yield $\vec{v}=\begin{pmatrix}a\\ 5a\end{pmatrix}=a\begin{pmatrix}1\\ 5\end{pmatrix}$

Therefore, a general solution is: $\begin{cases}x=C_1e^{-2t}+C_2e^{-5t}\\ y=2C_1e^{-2t}+5C_2e^{-5t}\end{cases}$

Particular solution: $\begin{cases}2=C_1+C_2\\ -7=2C_1+5C_2\end{cases}\Rightarrow\begin{cases}C_1=\dfrac{17}{3}\\ C_2=-\dfrac{11}{3}\end{cases}\Rightarrow\begin{cases}x=\dfrac{17}{3}e^{-2t}-\dfrac{11}{3}e^{-5t}\\ y=\dfrac{34}{3}e^{-2t}-\dfrac{55}{3}e^{-5t}\end{cases}$





With both eigenvalues negative, the curves move towards the equilibrium.
Stable system.

Euler: $\begin{cases} x_{n+1} = x_n + h(-y_n) \\ y_{n+1} = y_n + h(10x_n - 7y_n) \end{cases}$

$$\Rightarrow \begin{cases} x_1 = 2 + 0.1(7) = 2.7 \\ y_1 = -7 + 0.1(10\times 2 - 7\times(-7)) = -0.1 \end{cases} \Rightarrow \begin{cases} x_2 = 2.7 + 0.1(0.1) \\ y_2 = -0.1 + 0.1(10\times 2.7 - 7\times(-0.1)) \end{cases}$$

$$\Rightarrow \begin{cases} x_2 = 2.71 \\ y_2 = 2.67 \end{cases}$$

Exact: $\begin{cases} x = \dfrac{17}{3}\mathrm{e}^{-2(0.2)} - \dfrac{11}{3}\mathrm{e}^{-5(0.2)} \approx 2.45 \\ y = \dfrac{34}{3}\mathrm{e}^{-2(0.2)} - \dfrac{55}{3}\mathrm{e}^{-5(0.2)} \approx 0.85 \end{cases}$

Note here that the estimate for y is off. This is due to the fact that the gradients of trajectory functions are very large, and the functions are convex at this stage which will cause this discrepancy (see diagram above). In such cases, Euler's method will not be a good estimate. There are other methods available but are not included in the present syllabus.

(c) Eigenvalues: –1, 4, eigenvectors $\begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

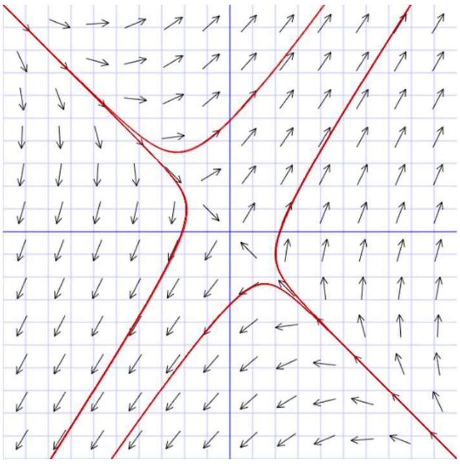
General solution: $\begin{cases} x = -C_1\mathrm{e}^{-t} + 2C_2\mathrm{e}^{4t} \\ y = C_1\mathrm{e}^{-t} + 3C_2\mathrm{e}^{4t} \end{cases}$

Particular solution: $\begin{cases} x = \dfrac{8}{5}\mathrm{e}^{-t} - \dfrac{8}{5}\mathrm{e}^{4t} \\ y = -\dfrac{8}{5}\mathrm{e}^{-t} - \dfrac{12}{5}\mathrm{e}^{4t} \end{cases}$

Trajectories approach equilibrium and then move away. Saddle point. Unstable.

Euler: $\begin{cases} x_{n+1} = x_n + h(x_n + 2y_n) \\ y_{n+1} = y_n + h(3x_n + 2y_n) \end{cases} \Rightarrow \begin{cases} x_2 = -1.84 \\ y_2 = -6 \end{cases}$

Exact: $\begin{cases} x = \dfrac{8}{5}e^{-(0.2)} - \dfrac{8}{5}e^{4(0.2)} \approx -2.251 \\ y = -\dfrac{8}{5}e^{-(0.2)} - \dfrac{12}{5}e^{4(0.2)} \approx -6.651 \end{cases}$



(d) Eigenvalues: –1, –6, eigenvectors $\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

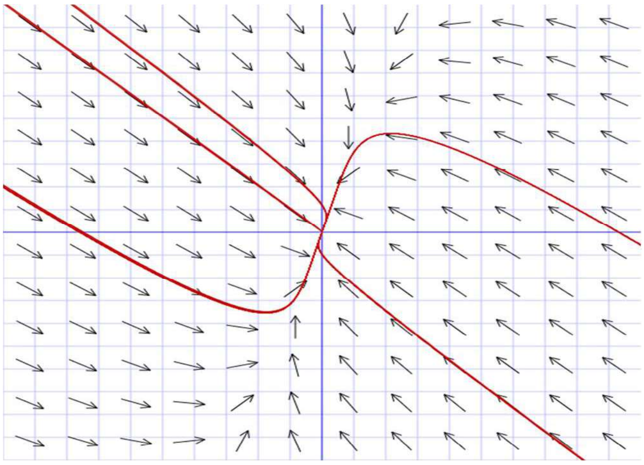
General solution: $\begin{cases} x = C_1e^{-t} - C_2e^{-6t} \\ y = 4C_1e^{-t} + C_2e^{-6t} \end{cases}$

Particular solution: $\begin{cases} x = \dfrac{3}{5}e^{-t} + \dfrac{2}{5}e^{-6t} \\ y = \dfrac{12}{5}e^{-t} - \dfrac{2}{5}e^{-6t} \end{cases}$

With both eigenvalues negative, the curves move towards the equilibrium. Stable system with the origin as a **node**.

Euler: $\begin{cases} x_{n+1} = x_n + h(-5x_n + y_n) \\ y_{n+1} = y_n + h(4x_n - 2y_n) \end{cases} \Rightarrow \begin{cases} x_2 = 0.55 \\ y_2 = 1.88 \end{cases}$

Exact: $\begin{cases} x = \dfrac{3}{5}e^{-(0.2)} + \dfrac{2}{5}e^{-6(0.2)} \approx 0.61 \\ y = \dfrac{12}{5}e^{-(0.2)} - \dfrac{2}{5}e^{-6(0.2)} \approx 1.84 \end{cases}$



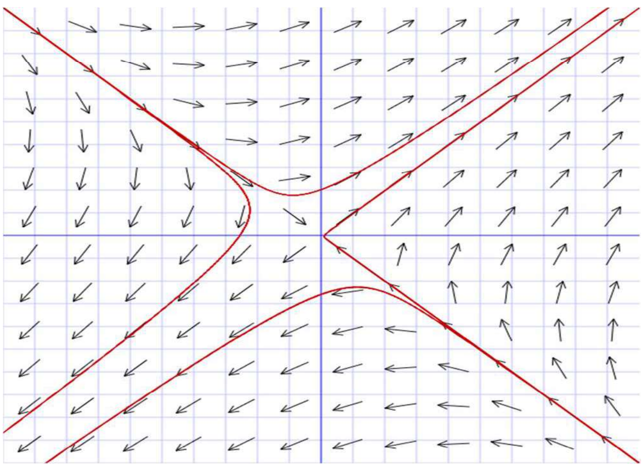
(e) Eigenvalues: –2, 6, eigenvectors $\begin{pmatrix}-1\\1\end{pmatrix},\begin{pmatrix}1\\1\end{pmatrix}$

General solution: $\begin{cases}x=-C_1\mathrm{e}^{-2t}+C_2\mathrm{e}^{6t}\\y=C_1\mathrm{e}^{-2t}+C_2\mathrm{e}^{6t}\end{cases}$

Particular solution: $\begin{cases}x=3e^{-2t}+2e^{6t}\\y=-3e^{-2t}+2e^{6t}\end{cases}$

Euler: $\begin{cases}x_{n+1}=x_n+h(2x_n+4y_n)\\y_{n+1}=y_n+h(4x_n+2y_n)\end{cases}\Rightarrow\begin{cases}x_2=6.4\\y_2=3.2\end{cases}$

Exact: $\begin{cases}x=3\mathrm{e}^{-2(0.2)}+2\mathrm{e}^{6(0.2)}\approx 8.65\\y=-3\mathrm{e}^{-2(0.2)}+2\mathrm{e}^{6(0.2)}\approx 4.63\end{cases}$



Trajectories approach equilibrium and then move away.

Saddle point. Unstable.

4. We will demonstrate details of the solution in two cases. The rest are similar and for convenience, we leave details out.

(a) We use the substitution $u' = y$, which implies that $u'' = y'$, thus the original differential equation becomes

$y' + 4y + 3u = 0$,

and thus,

$y' = -3u - 4y$

Now we have the system

$u' = y$

$y' = -3u - 4y$

Which can be written in matrix form as

$$\vec{Y}' = \begin{pmatrix} u' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} u \\ y \end{pmatrix} = M\vec{Y}$$

To find the eigenvalues we need to set $\det(M - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -3 & -\lambda - 4 \end{vmatrix} = 0$

Thus, the eigenvalues are $\lambda_1 = -1, \lambda_2 = -3$

The respective eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

The general solution is then

$$\vec{Y} = \begin{pmatrix} u \\ y \end{pmatrix} = C_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{-3t}\begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

With initial conditions

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} u \\ u' \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 3 \end{pmatrix} \Rightarrow C_1 = 4, C_2 = 2$$

Since we need y to solve the initial differential equation, then the first row of the solution is the value we need

$u = 4e^{-t} - 2e^{-3t}$

Euler's iterations: We will use a step of 0.1 for convenience as the exercise does not specify the step size.

$$\begin{cases} u_{n+1} = u_n + h(u'_n) \\ y_{n+1} = y_n + h(-3u_n - 4y_n) \end{cases} \Rightarrow \begin{cases} u_1 = 2 + 0.1 \times 2 = 2.2 \\ y_1 = 2 + 0.1(-3 \times 2 - 4 \times 2) = 0.6 \end{cases}$$

$$\{u_2 = u_1 + h(u'_1) = u_1 + h(y_1) = 2.2 + 0.1 \times 0.6 = 2.26$$

(b) We use the substitution $y' = u$, which implies that $y'' = u'$, thus the original differential equation becomes

$u' - u - 2y = 0$

and thus,

$u' = 2y + u$

Now we have the system

$y' = u$
$u' = 2y + u$

Which can be written in matrix form as

$$\overrightarrow{U'} = \begin{pmatrix} y' \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix}$$

To find the eigenvalues we need to set $\det(M - kI) = \begin{vmatrix} -k & 1 \\ 2 & 1-k \end{vmatrix} = 0$

Thus, the eigenvalues are $k_1 = 2, k_2 = -1$

The respective eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

The general solution is then

$$\vec{U} = \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} = C_1 e^{2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

With initial conditions

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} 3 \\ 8 \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow C_1 = \frac{11}{3}, C_2 = -\frac{2}{3}$$

Since we need y to solve the initial differential equation, then the first row of the solution is the value we need

$$y = \frac{11}{3}\mathrm{e}^{2t} - \frac{2}{3}\mathrm{e}^{-t}$$

Euler's iterations: We will use a step of 0.1 for convenience as the exercise does not specify the step size.

$$\begin{cases} y_{n+1} = y_n + h(y'_n) \\ u_{n+1} = u_n + h(2y_n + u_n) \end{cases} \Rightarrow \begin{cases} y_1 = 3 + 0.1 \times 8 = 3.8 \\ u_1 = 8 + 0.1(2 \times 3 + 8) = 9.4 \end{cases}$$

$$\{y_2 = y_1 + h(y'_1) = y_1 + h(2y_1 + u_1) = 3.8 + 0.1(2 \times 3.8 + 9.4) = 5.5$$

(c) We use the substitution $y' = u$, which implies that $y'' = u'$, thus the original differential equation becomes

$$2u' - 5u + 2y = 0$$

and thus,

$$u' = -y + \frac{5}{2}u$$

Now we have the system

$$y' = u$$

$$u' = -y + \frac{5}{2}u$$

Which can be written in matrix form as

$$\overline{U'} = \begin{pmatrix} y' \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \frac{5}{2} \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix}$$

To find the eigenvalues we need to set $\det\left(M - kI\right) = \begin{vmatrix} -k & 1 \\ 2 & 1-k \end{vmatrix} = 0$

Thus, the eigenvalues are $k_1 = \frac{1}{2},\ k_2 = 2$

The respective eigenvectors are

$$\vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

The general solution is then

$$\vec{U} = \begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} = C_1 e^{\frac{t}{2}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

With initial conditions

$$\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} -2 \\ -\frac{7}{4} \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow C_1 = -\frac{3}{4}, C_2 = -\frac{1}{2}$$

Since we need y to solve the initial differential equation, then the first row of the solution is the value we need

$$y = -\frac{3}{2}e^{\frac{t}{2}} - \frac{1}{2}e^{2t}$$

Euler's iterations: We will use a step of 0.1 for convenience as the exercise does not specify the step size.

$$\begin{cases} y_{n+1} = y_n + h(y'_n) \\ u_{n+1} = u_n + h\left(-y_n + \frac{5}{2}u_n\right) \end{cases} \Rightarrow \begin{cases} y_1 = -2 + 0.1 \times -\frac{7}{4} = -2.175 \\ u_1 = -\frac{7}{4} + 0.1\left(2 + \frac{5}{2}\left(-\frac{7}{4}\right)\right) = -1.9875 \end{cases}$$

$$\{y_2 = y_1 + h(y'_1) = y_1 + h(u_1) = -2.175 + 0.1(-1.9875) = -2.374$$

(d) With substitution $u' = v; v' = -2u - 3v;$ we transform the equation into a system

$\overrightarrow{V'} = \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$, which leads to eigenvalues $k_1 = -1, k_2 = -2$

and respective eigenvectors $\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$

and with initial conditions, the solution is $u = -7e^{-t} + 6e^{-2t}$

Euler's iterations: performing 2 iterations we get $u(0.2) = -1.83$

(e) With substitution $y' = u; u' = -5y - 6u,$ we transform the equation into a system

$\overrightarrow{U'} = \begin{pmatrix} y' \\ u' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -5 & -6 \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix}$, which leads to eigenvalues $k_1 = -1, k_2 = -5$

and with initial conditions, the solution is $y = \frac{7}{2}e^{-t} - \frac{1}{2}e^{-5t}$

Euler's iterations: performing 2 iterations we get $y(0.2) = 2.71$

5. **(a)** $\frac{x}{100}$ grams per litre and $\frac{y}{200}$ grams per litre, respectively.

(b) Tank 1 will have 10 *l*/min coming in from Tank 2 with concentration of $\frac{y}{200}$

It will be losing 30 *l*/min to Tank 2, thus,

Tank 1: $\frac{dx}{dt} = r_{in} - r_{out} = 10\frac{y}{200} - 30\frac{x}{100} = -\frac{3x}{10} + \frac{y}{20}$

Tank 2 will have 30 *l*/min from Tank 1, losing 10 *l*/min to Tank 1, and losing 20 *l*/min to the outside, thus,

Tank 2: $\frac{dy}{dt} = r_{in} - r_{out} = 30\frac{x}{100} - 10\frac{y}{200} - 20\frac{y}{200} = \frac{3x}{10} - \frac{3y}{20}$

(c) The system can now be modelled by

$$X' = \begin{pmatrix} -\frac{3}{10} & \frac{1}{20} \\ \frac{3}{10} & -\frac{3}{20} \end{pmatrix} X$$

Eigenvalues: $\lambda_1 = \frac{-9-\sqrt{33}}{40}$; $\lambda_2 = \frac{-9+\sqrt{33}}{40}$

Eigenvectors: $\begin{pmatrix} 1 \\ \frac{3-\sqrt{33}}{2} \end{pmatrix}$; $\begin{pmatrix} 1 \\ \frac{3+\sqrt{33}}{2} \end{pmatrix}$

General solution: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{3-\sqrt{33}}{2} \end{pmatrix} e^{\lambda_1 t} + c_2 \begin{pmatrix} 1 \\ \frac{3+\sqrt{33}}{2} \end{pmatrix} e^{\lambda_2 t}$

With initial conditions

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = 25\left(1-\frac{1}{\sqrt{33}}\right) \begin{pmatrix} 1 \\ \frac{3-\sqrt{33}}{2} \end{pmatrix} e^{\lambda_1 t} + 25\left(1+\frac{1}{\sqrt{33}}\right) \begin{pmatrix} 1 \\ \frac{3+\sqrt{33}}{2} \end{pmatrix} e^{\lambda_2 t}$$

Chapter 20 practice questions

1. Separate the variables and integrate both sides

$$\frac{dy}{dx} = \frac{xy}{\sqrt{1+x^2}} \Rightarrow \frac{dy}{y} = \frac{x\,dx}{\sqrt{1+x^2}} \Rightarrow \int \frac{dy}{y} = \int \frac{x\,dx}{\sqrt{1+x^2}}$$

$$\Rightarrow \ln|y| = \sqrt{1+x^2} + c$$

With initial condition

$$\ln|1| = \sqrt{1+0} + c \Rightarrow c = -1 \Rightarrow \ln|y| = \sqrt{1+x^2} - 1$$

$$\Rightarrow y = e^{\sqrt{1+x^2}-1} = \frac{e^{\sqrt{1+x^2}}}{e}$$

2. $$\frac{dy}{dx} = \sin x \cos^2 y \Rightarrow \sec^2 y\,dy = \sin x\,dx \Rightarrow \tan y = -\cos x + C$$

With initial condition

$$\tan\frac{\pi}{4} = -\cos\frac{\pi}{2} + C \Rightarrow C = 1 \Rightarrow \tan y = 1 - \cos x \Rightarrow y = \arctan(1-\cos x)$$

3. This is also a variables separable DE.

$$x\frac{dy}{dx} = y(3-y) \Rightarrow \frac{dy}{y(3-y)} = \frac{dx}{x}$$

LHS can be simplified using partial fractions and then integrated

$$\frac{dy}{y(3-y)} = \frac{dx}{x} \Rightarrow \frac{dy}{3y} - \frac{dy}{3(y-3)} = \frac{dx}{x} \Rightarrow \frac{1}{3}\left(\ln|y| - \ln|y-3|\right) = \ln|x| + c$$

$$\Rightarrow \ln\left|\frac{y}{y-3}\right| = \ln\left|x^3\right| + 3c \Rightarrow \left|\frac{y}{y-3}\right| = K\left|x^3\right|$$

With initial condition

$$\frac{y}{y-3} = Kx^3 \Rightarrow \frac{2}{-1} = 8K \Rightarrow K = -\frac{1}{4}$$

$$\frac{y}{y-3} = -\frac{1}{4}x^3 \Rightarrow -4y = x^3(y-3) \Rightarrow y = \frac{3x^3}{x^3+4}$$

4. If $y = Cx^{\ln\sqrt{x}}$, then

$$\ln y = \ln\left(Cx^{\ln\sqrt{x}}\right) = \ln C + \ln\sqrt{x}\ln x = \ln C + \frac{1}{2}\ln x \cdot \ln x = \ln C + \frac{1}{2}(\ln x)^2$$

$$\Rightarrow \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\times 2\times \ln x \times \frac{1}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = y \times \frac{\ln x}{x}$$

$$\Rightarrow x\frac{\mathrm{d}y}{\mathrm{d}x} = x \times y \times \frac{\ln x}{x} = y\ln x$$

5. $$\frac{\mathrm{d}Q}{\mathrm{d}t} = kQ \Rightarrow \frac{\mathrm{d}Q}{Q} = k\mathrm{d}t \Rightarrow \ln Q = kt + c \Rightarrow Q = Ae^{kt}$$

At time $t = 0$, there will be the original amount, Q_0, and thus the model is $Q = Q_0\mathrm{e}^{kt}$.

The half-life is 1620 means that

$$\frac{1}{2}Q_0 = Q_0e^{1620k} \Rightarrow 1620k = -\ln 2 \Rightarrow k \approx -1.00042787$$

Remains of 10 grams after 25 years is $Q \approx 10\mathrm{e}^{-0.000427867\times 25} = 9.89$ grams

6. **(a)** $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{y} \Rightarrow y\mathrm{d}y = 2x\mathrm{d}x \Rightarrow \frac{y^2}{2} = x^2 + c \Rightarrow y^2 = 2x^2 + k$$

(b) $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x^2} \Rightarrow \frac{\mathrm{d}y}{y^2} = \frac{\mathrm{d}x}{x^2} \Rightarrow -\frac{1}{y} = -\frac{1}{x} + c \Rightarrow y = \frac{1}{\frac{1}{x} - c} = \frac{x}{1 - cx}$$

(c) $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x} = y^2 - y \Rightarrow \dfrac{\mathrm{d}y}{y^2 - y} = \dfrac{\mathrm{d}x}{x^2}$. LHS can be evaluated with partial fractions

$$\frac{1}{y(y-1)} = \frac{1}{y-1} - \frac{1}{y} \Rightarrow \int\frac{\mathrm{d}y}{y^2 - y} = \ln|y-1| - \ln|y| = \int\frac{\mathrm{d}x}{x^2} = -\frac{1}{x} + c$$

$$\ln\left|\frac{y-1}{y}\right| = -\frac{1}{x} + c \Rightarrow \frac{y-1}{y} = \mathrm{e}^{-\frac{1}{x}+c} = A\mathrm{e}^{-\frac{1}{x}}$$

(d) $$x\frac{\mathrm{d}y}{\mathrm{d}x} = \tan y \Rightarrow \frac{\cos y\mathrm{d}y}{\sin y} = \frac{\mathrm{d}x}{x} \Rightarrow \ln|\sin y| = \ln|x| + c \Rightarrow \sin y = \mathrm{e}^{\ln|x|+c} = C|x|$$

$$\Rightarrow y = \arcsin Cx$$

(e) $$\frac{\mathrm{d}y}{\mathrm{d}x} = xy \Rightarrow \frac{\mathrm{d}y}{y} = x\mathrm{d}x \Rightarrow \ln|y| = \frac{x^2}{2} + c \Rightarrow y = \mathrm{e}^{\frac{x^2}{2}+c} = A\mathrm{e}^{\frac{x^2}{2}}$$

(f) $$\sqrt{x^2+1}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} \Rightarrow y\mathrm{d}y = \frac{x\mathrm{d}x}{\sqrt{x^2+1}} \Rightarrow y^2 = 2\sqrt{x^2+1} + c$$

(g) $\frac{dy}{dx} = \frac{y^2-1}{e^x} \Rightarrow \frac{dy}{y^2-1} = e^{-x}dx \Rightarrow \frac{dy}{2(y+1)} + \frac{dy}{2(y-1)} = e^{-x}dx$

(we used partial fractions for the LHS)

$$\frac{dy}{2(y+1)} + \frac{dy}{2(y-1)} = e^{-x}dx \Rightarrow \frac{1}{2}\ln\left(y^2-1\right) = -e^{-x} + c \Rightarrow \ln\sqrt{y^2-1} = c - e^{-x}$$

(h) $\ln y\frac{dy}{dx} = 1 \Rightarrow \ln y\, dy = dx$

Use by parts on the LHS

$y\ln|y| - y = x + c$

7. Factor numerator and denominator and separate variable:

$$\frac{dy}{dx} = \frac{xy+y}{xy+x} = \frac{y(x+1)}{x(y+1)} \Rightarrow \frac{y+1}{y}dy = \frac{x+1}{x}dx$$

$$\int\frac{y+1}{y}dy = \int\frac{x+1}{x}dx \Rightarrow y + \ln|y| = x + \ln|x| + c$$

$$\Rightarrow e^y e^{\ln y} = e^x e^{\ln x} e^c \Rightarrow ye^y = Axe^x$$

8. $y\frac{dy}{dx} = \cos x \Rightarrow y dy = \cos x\, dx \Rightarrow \int y dy = \int \cos x\, dx$

$$\Rightarrow \frac{y^2}{2} = \sin x + c \Rightarrow y^2 = 2\sin x + C \Rightarrow y = \pm\sqrt{2\sin x + C}$$

The constant C cannot be completely arbitrary because $2\sin x + C \geq 0$. If $C < -2$, then $2\sin x + C$ will be negative for all values of x. If $-2 \leq C \leq 2$, then $2\sin x + C$ will be positive for some values of x.

9. In order to be able to find the limiting value of the population, we first solve the DE.

$\frac{dp}{dt} = 5p - 2p^2 \Rightarrow \frac{dp}{p(5-2p)} = dt$, LHS evaluated using partial fractions

$$\int\frac{dp}{p(5-2p)} = \int dt \Rightarrow \frac{1}{5}\ln\left|\frac{p}{2p-5}\right| = t + c \Rightarrow \ln\left|\frac{p}{2p-5}\right| = 5t + K$$
,

$$\Rightarrow \frac{p}{2p-5} = e^{5t+K} = Ae^{5t}$$

then, with initial condition

$$\frac{p}{2p-5} = Ae^{5t} \Rightarrow \frac{4}{3} = A \Rightarrow \frac{p}{2p-5} = \frac{4}{3}e^{5t}$$

Solving for p, we have $p = \frac{20e^{5t}}{8e^{5t}-3}$

(a) $\lim_{t\to\infty} p = \lim_{t\to\infty} \frac{20e^{5t}}{8e^{5t}-3} = \frac{20}{8} = \frac{5}{2}$

(b) with new initial condition

$$\frac{p}{2p-5} = Ae^{5t} \Rightarrow \frac{0.5}{-4} = A \Rightarrow \frac{p}{2p-5} = -\frac{1}{8}e^{5t}$$

Solving for p, we have

$p = \frac{5e^{5t}}{8+2e^{5t}}$, and

$$\lim_{t\to\infty} p = \lim_{t\to\infty} \frac{5e^{5t}}{8+2e^{5t}} = \frac{5}{2}$$

(c) Regardless of the initial value of the population, as time increases, the population stabilises at 2 500.

10. $(1+x^2)\frac{dy}{dx} = 1+y^2 \Rightarrow \frac{dy}{1+y^2} = \frac{dx}{1+x^2} \Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} \Rightarrow \arctan y = \arctan x + c$

$y(2) = 3 \Rightarrow \arctan 3 = \arctan 2 + c \Rightarrow c = \arctan 3 - \arctan 2$

$\Rightarrow \arctan y - \arctan x = \arctan 3 - \arctan 2$

$\Rightarrow \tan(\arctan y - \arctan x) = \tan(\arctan 3 - \arctan 2) \Rightarrow \frac{y-x}{1+xy} = \frac{1}{1+6}$

$7y - 7x = 1 + xy \Rightarrow y(7-x) = 1 + 7x \Rightarrow y = \frac{1+7x}{7-x}$

11. **(a)** $\frac{1}{x^2-x-2} = \frac{1}{(x-2)(x+1)} \Rightarrow \frac{1}{(x-2)(x+1)} \equiv \frac{A}{x-2} + \frac{B}{x+1}$

$\Rightarrow 1 \equiv A(x+1) + B(x-2) \Rightarrow A = \frac{1}{3},\ B = -\frac{1}{3}$

$\Rightarrow \frac{1}{x^2-x-2} \equiv \frac{1}{3(x-2)} - \frac{1}{3(x+1)}$

(b) $\frac{dy}{dx} = \frac{y^2}{x^2-x-2} \Rightarrow \frac{dy}{y^2} = \frac{dx}{x^2-x-2} \Rightarrow \int \frac{dy}{y^2} = \int \frac{dx}{3(x-2)} - \int \frac{dx}{3(x+1)}$

$\Rightarrow -\frac{1}{y} = \frac{1}{3}\ln(x-2) - \frac{1}{3}\ln(x+1) \Rightarrow \frac{1}{y} = \frac{1}{3}\ln(x+1) - \frac{1}{3}\ln(x-2) = \frac{1}{3}\ln\frac{x+1}{x-2}$

$\Rightarrow \frac{3}{y} = \ln\frac{x+1}{x-2} + c$

With initial values

$$\Rightarrow \frac{3}{1} = \ln\frac{5+1}{5-2} + c \Rightarrow c = 3 - \ln 2$$

$$\Rightarrow \frac{3}{y} = \ln\frac{x+1}{x-2} + 3 - \ln 2 \Rightarrow \frac{3-3y}{y} + \ln 2 = \ln\frac{x+1}{x-2}$$

$$\Rightarrow \mathrm{e}^{\frac{3-3y}{y}+\ln 2} = 2\mathrm{e}^{\frac{3-3y}{y}} = \frac{x+1}{x-2}$$

12. **(a)** $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x+2y}{3y-2x} \Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+2\frac{y}{x}}{3\frac{y}{x}-2} \Rightarrow v + x\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{1+2v}{3v-2}$

(b) Multiply and collect like terms

$$3v^2 - 2v + x(3v-2)\frac{\mathrm{d}v}{\mathrm{d}x} = 1 + 2v \Rightarrow -\frac{3v-2}{3v^2-4v-1}\mathrm{d}v = \frac{\mathrm{d}x}{x}$$

$$\Rightarrow -\frac{1}{2}\int\frac{6v-4}{3v^2-4v-1}\mathrm{d}v = \int\frac{\mathrm{d}x}{x} \Rightarrow \ln\left|3v^2-4v-1\right| = -2\ln|x| + c$$

$$\Rightarrow 3v^2 - 4v - 1 = -\frac{1}{x^2} + c \Rightarrow 3\frac{y^2}{x^2} - 4\frac{y}{x} - 1 = -\frac{1}{x^2} + c$$

$$\Rightarrow 3y^2 - 4xy - x^2 = -1 + cx^2$$

With initial value

$$0 - 0 - 1 = -1 + c \Rightarrow c = 0 \Rightarrow 3y^2 - 4xy - x^2 + 1 = 0$$

13. Substitute $y = vx$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$

$$y^2 - x^2 + xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow v^2 - 1 + v\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) = 0$$

$$\Rightarrow v^2 - 1 + v^2 + vx\frac{\mathrm{d}v}{\mathrm{d}x} = 0 \Rightarrow \int\frac{v\mathrm{d}v}{1-2v^2} = \int\frac{\mathrm{d}x}{x}$$

$$\Rightarrow -\frac{1}{4}\ln\left|2v^2-1\right| = \ln|x| + c \Rightarrow \ln\left|\frac{1}{2v^2-1}\right| = \ln x^4 + 4c$$

$$\Rightarrow \frac{1}{2\frac{y}{x^2}^2 - 1} = \mathrm{e}^{4c}x^4 \Rightarrow \frac{x^2}{2y^2 - x^2} = Ax^4 \Rightarrow 1 = A\left(2x^2y^2 - x^4\right)$$

$C = 2x^2y^2 - x^4$, where $C = \frac{1}{A}$.

14. **(a)** $\frac{dy}{dx} = \frac{y^2+y}{x} \Rightarrow x dy = (y^2+y)dx \Rightarrow \int \frac{dy}{y^2+y} = \int \frac{dx}{x}$

LHS can be evaluated using partial fractions

$$\ln|y| - \ln|y+1| = \ln|x| + c \Rightarrow \left|\frac{y}{y+1}\right| = e^c|x| = C|x|,$$

(b) With initial value

$$\left|\frac{y}{y+1}\right| = C|x| \Rightarrow \frac{1}{2} = C \Rightarrow \left|\frac{y}{y+1}\right| = \frac{1}{2}|x|$$

(c)–(d) Use a spreadsheet for Euler's method calculations

$y_{n+1} = y_n + hF(x_n, y_n)$, and $\frac{dy}{dx} = F(x, y) = \frac{y^2+y}{x}$, we have

$$y_{n+1} = y_n + hF(x_n, y_n) = y_n + h\left(\frac{y_n^2+y_n}{x_n}\right).$$

For example, $y_{0+1} = y_0 + hF(x_0, y_0) = 1 + 0.2\left(\frac{1+1}{1}\right) = 1.4.$

x_n	*approx.* y_n	*exact* y_n	*% error*
1.2	1.400	1.5	$6.\overline{6}$
1.4	1.960	$2.\overline{3}$	16
1.6	2.789	4	30.3
1.8	4.110	9	54.3

15. $y_{n+1} = y_n + hF(x_n, y_n) \Rightarrow y_{n+1} = y_n + h(x_n y_n^2).$

If 5 steps are needed, then $h = 0.2$.

For example, $y_{0+1} = y_0 + hF(x_0, y_0) = 1 + 0.2(0) = 1$, and

$y_{1+1} = y_1 + hF(x_1, y_1) = 1 + 0.2(0.2 \times 1) = 1.04,$

x(n)	y(n)
0	1
0.2	1
0.4	1.04
0.6	1.126528
0.8	1.278816
1	1.540475

16. $y_{n+1} = y_n + hF(x_n, y_n) \Rightarrow y_{n+1} = y_n + 0.1\left(e^{x_n y_n}\right)$.

For example, $y_{0+1} = y_0 + 0.1F(x_0, y_0) = 1 + 0.1\left(e^{0\times 1}\right) = 1.1$,

x(n)	y(n)
0	1
0.1	1.1
0.2	1.211628
↓	↓
0.9	3.539766
1	5.958404

17. Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{x^2+3y^2}{2xy} \Rightarrow v + x\frac{dv}{dx} = \frac{1+3v^2}{2v} \Rightarrow \frac{2v\,dv}{1+v^2} = \frac{dx}{x}$$

$$\Rightarrow \ln\left(1+v^2\right) = \ln|x| + c \Rightarrow 1+v^2 = A|x|$$

$$\Rightarrow 1+\frac{y^2}{x^2} = Ax \Rightarrow y^2 = Ax^3 - x^2$$

18. $y_{n+1} = y_n + hF(x_n, y_n) \Rightarrow y_{n+1} = y_n + 0.1\left(x_n\sqrt{y_n}\right)$.

For example, $y_{0+1} = y_0 + 0.1F(x_0, y_0) = 4 + 0.1\left(1\times\sqrt{4}\right) = 4.2$,

x(n)	y(n)
1	4
1.1	4.2
1.2	4.425433
1.3	4.677873
1.4	4.959043
1.5	5.270807

19. **(a)** If $y = e^{-x} + x - 1$, then $\frac{dy}{dx} = -e^{-x} + 1$

Now substitute these values in the given DE.

$$\frac{dy}{dx} = x - y \Rightarrow -e^{-x} + 1 = x - \left(e^{-x} + x - 1\right) = -e^{-x} + 1$$

(b)–(c) With 5 steps, we will need $h = 0.2$; with 10 steps, we will need $h = 0.1$

$$y_{n+1} = y_n + hF(x_n, y_n) \Rightarrow y_{n+1} = y_n + h(x_n - y_n).$$

0.2		0.1	
x(n)	y(n)	x(n)	y(n)
0	0	0	0
0.2	0	0.1	0
0.4	0.04	0.2	0.01
0.6	0.112	0.3	0.029
0.8	0.2096	0.4	0.0561
1	0.32768	0.5	0.09049
		0.6	0.131441
		0.7	0.178297
		0.8	0.230467
		0.9	0.28742
		1	0.348678

(d) Actual value to 10 s.f. is $y(1) \approx 0.3678794412$; Note that with 10 steps the discrepancy is less than with 5 steps. Thus, using more steps (and a smaller step size gives a better approximation.

20. $$\frac{d\alpha}{dt} = -k(\alpha - 20) \Rightarrow \frac{d\alpha}{\alpha - 20} = -k\,dt \Rightarrow \ln(\alpha - 20) = -kt + c$$

$$\Rightarrow \alpha - 20 = Ae^{-kt} \Rightarrow \alpha = Ae^{-kt} + 20$$

With the 2 initial values (0, 70) and (10, 50), we can find the values of A and k.

$$(0,70): \quad \alpha = Ae^{-kt} + 20 \Rightarrow 70 = A + 20 \Rightarrow A = 50 \Rightarrow \alpha = 50e^{-kt} + 20$$

$$(10,50:) \quad \alpha = 50e^{-kt} + 20 \Rightarrow 50 = 50e^{-10k} + 20 \Rightarrow e^{-10k} = \frac{3}{5} \Rightarrow k = -\frac{1}{10}\ln\frac{3}{5}$$

$$\Rightarrow \alpha = 50e^{-\frac{1}{10}\ln\frac{3}{5}t} + 20$$

21. **(a)** Exponential decay is relative to the quantity present, that is

$$\frac{dN}{dt} = -kN \Rightarrow \ln N = -kt + c \Rightarrow N(t) = N_0 e^{-kt}, \text{ so,}$$

$$\frac{N_0}{2} = N_0 e^{-kt} \Rightarrow \frac{1}{2} = e^{-kt} \Rightarrow t = \frac{\ln 2}{k} = \frac{\ln 2}{0.0001216} \approx 5700 \text{ years}$$

(b) We need to find how long it will take such material to decay to one quarter of its value: $\frac{N_0}{4} = N_0 e^{-kt} \Rightarrow t = \frac{\ln 4}{0.0001216} \approx 11400 \text{ years}$

22. This is exponential decay where the model $N(t) = N_0 e^{-kt}$ is appropriate.

We need to find how long it takes a substance with 5.1×10^{10} atoms to decay to 4.3×10^{10} atoms. That is

$$4.3 \times 10^{10} = 5.1 \times 10^{10} e^{-0.0001216t} \Rightarrow t = \frac{\ln\left(\frac{5.1}{4.3}\right)}{0.0001216} \approx 1403 \text{ years.}$$

Given that the Roman empire existed more than 2000 years ago, this appears not to be genuine from the era.

23. **(a)** To achieve an approximation of y when $x = 1$, we will need 4 iterations with step size of 0.25. We will use a spreadsheet for the calculations

$$y_{n+1} = y_n + hF(x_n, y_n) \Rightarrow y_{n+1} = y_n + 0.25\left(x_n^2 + y_n^2\right).$$

$x(n)$	$y(n)$
0	2
0.25	3
0.5	5.265625
0.75	12.25983
1	49.97629

(b) Less than actual value; $\frac{dy}{dx} > 0$ so solution curve is curving upward; short segments from Euler's method to approximate solution curve will be below the actual solution curve.

24. **(a)** s g/L of salt in 4 L of solution are entering the tank.

Thus, the rate of salt going in is $s\frac{\text{g}}{\text{L}} \times 4\frac{\text{L}}{\text{min}} = 4s\frac{g}{\text{min}}$

There is Q g of salt in the solution in the tank, hence the concentration of salt is $\frac{Q}{120}\frac{\text{g}}{\text{L}}$. The rate of salt going out is therefore, $\frac{Q}{120}\frac{\text{g}}{\text{L}} \times 4\frac{\text{L}}{\text{min}} = \frac{4Q}{120}\frac{\text{g}}{\text{min}}$

So, the initial value problem is: $\frac{dQ}{dt} = 4s - \frac{4}{120}Q = 4s - \frac{1}{30}Q;\ Q(0) = 0$

(b) Rearrange and as suggested multiply both sides of the DE with $e^{\frac{t}{30}}$

$$e^{\frac{t}{30}}\cdot\frac{dQ}{dt}+e^{\frac{t}{30}}\cdot\frac{1}{30}Q=4s\cdot e^{\frac{t}{30}}\Rightarrow d\left(e^{\frac{t}{30}}\cdot Q\right)=4s\cdot e^{\frac{t}{30}}dt$$

$$\Rightarrow e^{\frac{t}{30}}\cdot Q=120s\,e^{\frac{t}{30}}+c$$

With initial conditions

$$e^{0}\cdot 0=120s\,e^{0}+c\Rightarrow c=-120s$$

$$\Rightarrow e^{\frac{t}{30}}\cdot Q=120s\,e^{\frac{t}{30}}-120s\Rightarrow Q=120s\left(1-e^{-\frac{t}{30}}\right)$$

(c) $$\lim_{t\to\infty}Q(t)=\lim_{t\to\infty}120s\left(1-e^{\frac{-t}{30}}\right)=120s$$

25. In the absence of other factors, we have

$$\frac{dP}{dt}=kP\Rightarrow\frac{dP}{P}=k\,dt\Rightarrow\ln P=kt+c\Rightarrow P=P_0e^{kt}$$

Given the fact that it doubles every week: $2P_0=P_0e^{k}\Rightarrow k=\ln 2$

Now, the initial value problem can be set up:

$$\frac{dP}{dt}=(\ln 2)P-140\,000;\quad P(0)=200000$$

Rearrange and as suggested multiply both sides of the DE with $e^{-(\ln 2)t}$.

$$e^{-(\ln 2)t}\cdot\frac{dP}{dt}-e^{-(\ln 2)t}\cdot(\ln 2)P=-140\,000e^{-(\ln 2)t}\Rightarrow d\left(e^{-(\ln 2)t}\cdot P\right)=-140\,000e^{-(\ln 2)t}dt$$

$$\Rightarrow e^{-(\ln 2)t}\cdot P=\int -140\,000e^{-(\ln 2)t}dt=\frac{140000}{\ln 2}e^{-(\ln 2)t}+c$$

$$\Rightarrow P=\frac{140000}{\ln 2}+ce^{(\ln 2)t}$$

With initial conditions

$$200000=\frac{140000}{\ln 2}+c\Rightarrow c=-1977.31$$

Therefore, the flies population is modelled by

$$P\approx 201977.31-1977.31e^{t\ln 2}$$
$$=201977.31-1977.31\times 2^{t};\ 0\le t\le 6.67\text{ weeks}$$

$0 \le t \le 6.67$ weeks is added to make sure that the number of flies is non-negative!